The forward smile in stochastic local volatility models

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Abstract

We introduce an approximation of forward start options in a multi-factor local-stochastic volatility model. We derive explicit expansion formulas for the so-called forward implied volatility which can be useful to price complex path-dependent options, as cliquets. The expansion involves only polynomials and can be computed without the need for numerical procedures or special functions. Recent results on the exploding behaviour of the forward smile in the Heston model are confirmed and generalized to a wider class of local-stochastic volatility models. We illustrate the effectiveness of the technique through some numerical tests.

Keywords: forward implied volatility, cliquet option, local volatility, stochastic volatility, analytical approximation

Key messages

- approximation for the forward implied volatility
- local stochastic volatility models
- explosion of the out-of-the-money forward smile

1 Introduction

In an arbitrage-free market, we consider the risk-neutral dynamics described by the d-dimensional Markov diffusion

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{1.1}$$

where W is a m-dimensional Brownian motion. The first component X^1 represents the log-price of an asset, while the other components of X represent a number of things, e.g., stochastic volatilities, economic indicators or functions of these quantities. We are interested in the forward start payoff

$$\left(e^{X^1_{t+\tau}-X^1_t} - e^k\right)^+ \tag{1.2}$$

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where t and τ are positive numbers representing the *forward (or reset) date* and the *forward maturity* respectively, and k is the log-strike.

Payoffs of the form (1.2) constitute the building block for the class of cliquet and ratchet options. A cliquet option is a path-dependent contract that is very sensitive to model risk, and specifically to forward skew assumptions. In a local volatility (LV) model, forward skews are typically flat: therefore the value of certain payoffs, as a digital cliquet, given by a LV model may be substantially lower than the price given by a stochastic volatility (SV) model (cf. [28]). Notwithstanding, it is well-known (cf. [11]) that SV models are generally unable to reproduce the term structure of the volatility skew and for this reason, practitioners still show interest in LV models: indeed a basic requirement for a model to be useful in practice is the ability to fit, at least approximately, the current implied volatility surface. What is more, [4] and [16] point out that also the Heston [12] and exponential Lévy models may get exposed to a large model risk when pricing cliquet options. Problems in pricing standard cliquet options are well documented in the literature (cf. [14]) and culminated in losses at a number of financial institutions.

These considerations favor models which combine local and stochastic volatility. It is nevertheless mandatory to have also a fast and accurate method for computing the prices and implied volatilities as a function of the model parameters, as it is required for model calibration. Indeed, this is the main drawback of LV models and, broadly speaking, of any more sophisticated model, possibly with variable coefficients, that do not have explicit solution (see, for instance, [5]). This requirement also explains the popularity of the Heston and other affine or jump models that admit quasi-closed form solution via Fourier inversion.

In this paper we derive an approximation of forward start options in a multi-factor local-stochastic volatility model. Our main result are explicit expansion formulas for the *forward implied volatility* in the large class of models described by (1.1): we emphasize that our analysis does not require the knowledge of the density nor the characteristic function of the underlying process. Moreover, although we consider a diffusive setting, our methodology can be extended to models with jumps and in that case it yields explicit approximations of the forward characteristic function.

In order to introduce the definition of *forward implied volatility*, we first recall that in the Black-Scholes (BS) model, the dynamics of the logarithm of the asset price are given by

$$dZ_t = -\frac{\sigma^2}{2}dt + \sigma dW_t, \qquad Z_0 = z,$$

where W is a real Brownian motion, σ is a positive parameter that represents the instantaneous volatility and we assume that the interest rates are zero. Then the no-arbitrage price at time zero of a European Call option with payoff $\left(e^{Z_\tau} - e^k\right)^+$ is given by the famous BS formula

$$\mathrm{Call}^{\mathrm{BS}}\left(z, \tau, k; \sigma\right) = e^z \mathcal{N}\left(d_+\right) - e^k \mathcal{N}\left(d_-\right), \qquad d_\pm = \frac{z-k}{\sigma\sqrt{\tau}} \pm \frac{\sigma\sqrt{\tau}}{2}, \tag{1.3}$$

where $\mathcal{N}$ is the cumulative distribution function of the standard normal distribution. By the stationarity of the increments, the no-arbitrage price at time zero of a forward-start Call option with payoff $\left(e^{Z_{t+\tau}-Z_t} - e^k\right)^+$, is equal to

$$E\left[\left(e^{Z_{t+\tau}-Z_t} - e^k\right)^+\right] = \mathrm{Call}^{\mathrm{BS}}\left(0, \tau, k; \sigma\right).$$

For a given market price $\mathrm{Call}^{\mathrm{obs}}(t,\tau,k)$ of the forward-start Call option, the *forward implied volatility* $\sigma_{t,\tau}(k)$ is then defined as the unique solution to

$$\mathrm{Call}^{\mathrm{obs}}(t,\tau,k) = \mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\sigma_{t,\tau}(k))$$

or, equivalently, as

$$\sigma_{t,\tau}(k) := \mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\cdot)^{-1}\left(\mathrm{Call}^{\mathrm{obs}}(t,\tau,k)\right),$$

whenever $\mathrm{Call}^{\mathrm{obs}}(t,\tau,k)$ belongs to the no-arbitrage interval $\left]\left(1-e^k\right)^+,1\right[$. It is clear that the forward implied volatility $\sigma_{t,\tau}(k)$ is a generalization of the usual *spot* implied volatility and the two coincide when $t=0$.

The literature on forward-start-based payoffs is sparse. Most of the authors (cf. [21], [17], [22], [6], [1], [26] and [29]) consider models for which the forward characteristic function of the underlying process is known. In that case, due to the tower property of conditional expectation, it is possible to derive semi-explicit solutions for Call options. An interesting exception is [8] where an analytical approximation of the forward smile in a generic one-dimensional LV model is proposed: the results of [8] are similar in spirit to our approach, even if of limited practical use due to the inability of LV models to reproduce forward skews.

Recently, [13] investigated the asymptotics of the forward implied volatility in the Heston model using large deviations techniques. One of the main findings of [13] is that, for out-of-the-money options (i.e. for $k \neq 0$) and for any positive reset date t, the Heston forward smile $\sigma_{t,\tau}(k)$ explodes as the forward maturity τ approaches zero. Here we confirm the results of [13] for the Heston model (see Section 4). Furthermore, our analytical approximations indicate that the singularity of the forward smile is a rather general feature of local-stochastic volatility models. In particular we show, also through numerical tests, that the forward smile for out-of-the-money options explodes in any local-stochastic volatility model, under rather general assumptions.

The starting point of our analysis is the perturbation technique recently introduced in [18] and [20]. In Section 2 we extend the results of [18] to forward contracts by means of a conditioning expectation argument. The first main result, Theorem 2.5, gives an expansion of forward-start option prices: the computational complexity of the expansion is comparable to the Black-Scholes formula. In Section 3 we derive a general expansion of the forward implied volatility: the expansion involves only polynomials and can be computed without the need for numerical procedures or special functions. In the last part of the paper, Section 4, we focus on two-dimensional models (i.e. $d = 2$ in (1.1)): in that case we get more compact expressions for the formulas of Theorem 2.5 and provide the first order expansion (4.6) of the forward volatility smile in a general local-stochastic volatility model. Eventually, we illustrate the flexibility and accuracy of our methodology by showing numerical tests under the CEV local volatility and the Heston stochastic volatility models.

2 Price expansions

2.1 Spot price expansion

In this section we briefly recall the polynomial approximation proposed in [18]. Under mild assumptions on the market dynamics (1.1) (see, for instance, [25]) and assuming that the interest rates are zero, the price of an option with payoff function φ is given by

$$E\left[\varphi(X_T)|X_t = x\right] = u(t,x), \qquad t < T,\ x \in \mathbb{R}^d, \tag{2.1}$$

where u solves the backward Cauchy problem

$$\begin{cases} (\partial_t + \mathcal{A})u(t,x) = 0, & t < T,\ x \in \mathbb{R}^d, \\ u(T,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.2}$$

In (2.2), $\mathcal{A}$ is the Kolmogorov operator of (1.1)

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{d} C_{ij}(t,x)\partial_{x_i x_j} + \sum_{i=1}^{d} \mu_i(t,x)\partial_{x_i}, \qquad C := \sigma\sigma^*, \tag{2.3}$$

which we rewrite also in compact form as follows:

$$\mathcal{A} = \sum_{|\alpha|\le 2} a_\alpha(t,x) D_x^\alpha. \tag{2.4}$$

In (2.4) we use the standard notations

$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d, \qquad |\alpha| = \sum_{i=1}^{d} \alpha_i, \qquad D_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}.$$

Now we briefly summarize the method proposed by [18] to construct an approximation of u in (2.1). Assume that $a_\alpha(t,\cdot) \in C^N(\mathbb{R}^d)$ for some $N \in \mathbb{N}$: then, for a fixed $\bar{x} \in \mathbb{R}^d$ and $n \le N$, let

$$a_{\alpha,n}(\cdot,x) := \sum_{|\beta|=n} \frac{D_x^\beta a_\alpha(\cdot,\bar{x})}{\beta!}(x-\bar{x})^\beta,$$

be the n-th order term of the Taylor expansion of a_α in the spatial variables around $\bar{x}$. Eventually, define

$$\mathcal{A}_{\bar{x},n} = \sum_{|\alpha|\le 2} a_{\alpha,n}(t,x) D_x^\alpha.$$

More generally we may assume that $\bar{x} = \bar{x}(t)$ is a function of time: we shall see that in some cases the choice $\bar{x}(t) \approx E\left[X_t\right]$ results in a highly accurate approximation for option prices and implied volatilities also for long maturities; for instance, we illustrate this for the Heston model in Section 4.2.

Coming back to the approximation of option prices, we have the following asymptotic expansion for u in (2.2):

$$u \approx u_{\bar{x},N} := \sum_{n=0}^{N} u_n$$

where the functions $(u_n)_{n=0,\dots,N}$ are defined recursively as solutions of the Cauchy problems

$$\begin{cases} (\partial_t + \mathcal{A}_{\bar{x},0})u_0(t,x) = 0, & t < T, \ x \in \mathbb{R}^d, \\ u_0(T,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.5}$$

and, for $n \geq 1$,

$$\begin{cases} (\partial_t + \mathcal{A}_{\bar{x},0})u_n(t,x) = \sum\limits_{k=1}^{n} \mathcal{A}_{\bar{x},k} u_{n-k}(t,x), & t < T, \ x \in \mathbb{R}^d, \\ u_0(T,x) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{2.6}$$

It is proved in [18] that the approximation $u_{\bar{x},N}$ can be computed explicitly and is asymptotically convergent to u as the time to maturity tends to zero. In fact, notice that the coefficients of $\mathcal{A}_{\bar{x},0}$ in (2.5)-(2.6) depend only on the time variable and more precisely we have

$$\mathcal{A}_{\bar{x},0} = \frac{1}{2} \sum_{i,j=1}^{d} \bar{C}_{ij}(t) \partial_{x_i x_j} + \sum_{i=1}^{d} \bar{\mu}_i(t) \partial_{x_i},$$

with $\bar{C} = (\sigma\sigma^*)(\cdot, \bar{x}(\cdot))$ and $\bar{\mu} = \mu(\cdot, \bar{x}(\cdot))$. Let

$$\mathbf{C}(t,T) = \int_t^T \bar{C}(s) ds, \qquad \mathbf{m}(t,T) = \int_t^T \bar{\mu}(s) ds. \tag{2.7}$$

If $\mathbf{C}(t,T)$ in (2.7) is positive definite then the d-dimensional Gaussian function

$$\Gamma_{\bar{x}}(t,x,T,y) = \frac{1}{\sqrt{(2\pi)^d \det \mathbf{C}(t,T)}} \exp\left(-\frac{1}{2} \langle \mathbf{C}^{-1}(t,T)(y - x - \mathbf{m}(t,T)), y - x - \mathbf{m}(t,T) \rangle \right) \tag{2.8}$$

is the fundamental solution of $\mathcal{A}_{\bar{x},0}$ and u_0 in (2.5) is equal to

$$u_0(t,x) = \int_{\mathbb{R}^d} \Gamma_{\bar{x}}(t,x,T,y) \varphi(y) dy. \tag{2.9}$$

The explicit expression of u_n, for $n \geq 1$, is given by [18]: we have

$$u_n(t,x) = \mathcal{L}_{\bar{x},n}(t,x,T) u_0(t,x), \qquad t < T, \ x \in \mathbb{R}^d, \ n \leq N, \tag{2.10}$$

where $\mathcal{L}_{\bar{x},0}(t,x,T) \equiv 1$ and $\mathcal{L}_{\bar{x},n}(t,x,T)$ is the differential operator defined as follows:

$$\mathcal{L}_{\bar{x},n}(t,x,T) := \sum_{h=1}^{n} \int_t^T ds_1 \int_{s_1}^T ds_2 \cdots \int_{s_{h-1}}^T ds_h \sum_{i \in I_{n,h}} \mathcal{G}_{i1}(t,x,s_1) \dots \mathcal{G}_{ih}(t,x,s_h) \tag{2.11}$$

where

$$I_{n,h} = \{ i = (i_1, \dots, i_h) \in \mathbb{N}^h | i_1 + \dots + i_h = n \}, \qquad 1 \leq h \leq n,$$

and $\mathcal{G}_n(t,x,s)$ is the differential operator

$$\mathcal{G}_n(t,x,s) = \sum_{|\alpha| \leq 2} a_{\alpha,n}(s, \mathcal{M}_{\bar{x}}(t,x,s)) D_x^\alpha, \qquad \mathcal{M}_{\bar{x}}(t,x,s) = x + \mathbf{m}(t,s) + \mathbf{C}(t,s) \nabla_x. \tag{2.12}$$

The proof of (2.10) is based on the following useful symmetry property of $\Gamma_{\bar{x},0}$, which will be used also for later computations:

$$y \Gamma_{\bar{x}}(t,x,T,y) = \mathcal{M}_{\bar{x}}(t,x,T) \Gamma_{\bar{x}}(t,x,T,y), \qquad t < T, \ x, y \in \mathbb{R}^d. \tag{2.13}$$

Remark 2.1. Operators $\mathcal{M}_{\bar{x}}(t,x,s)$ *commute when applied to* $\Gamma_{\bar{x}}(t,x,T,y)$ and, more generally, to any function u_0 as in (2.9). Therefore, since $x \mapsto a_{\alpha,n}(s,x)$ is a polynomial function, the operator $a_{\alpha,n}(s,\mathcal{M}_{\bar{x}}(t,x,s))$ in (2.12) is defined unambiguously, as a composition of operators, when acting on u_0.

Remark 2.2. From (2.11) it is not difficult to recognize that $\mathcal{L}_{\bar{x},n}$ is an operator of the form

$$\mathcal{L}_{\bar{x},n}(t,x,T) = \sum_{\substack{|\alpha|\leq n \\ |\beta|\leq 3n}} (x-\bar{x}(t))^{\alpha} f_{\bar{x},n,\alpha,\beta}(t,T) D_x^{\beta}, \tag{2.14}$$

where the coefficients $f_{\bar{x},n,\alpha,\beta}(t,T)$ are independent of x. For $n=0$ we simply have $f_{\bar{x},0,0,0}(t,T) \equiv 1$ since $\mathcal{L}_{\bar{x},0}(t,x,T) \equiv 1$ by definition.

Using a computer algebra program such as Wolfram's Mathematica it is straightforward to derive the explicit expression of $\mathcal{L}_{\bar{x},n}(t,x,T)$ for specific models. Mathematica codes for several popular models are freely available on the authors' websites. We also recall that the third order approximation of the spot implied volatility in the Heston model has been recently included in the QuantLib library.

We close this section by stating the following result proved in [18].

Theorem 2.3. *Assume that $\sigma\sigma^*$ is uniformly positive definite and $\varphi \in C^h(\mathbb{R}^d)$, $h \in \{0,1,2\}$, with bounded derivatives. Then we have*

$$|u(t,x) - u_{x,N}(t,x)| \leq C_N (T-t)^{\frac{N+1+h}{2}}, \qquad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{2.15}$$

where C_N is a positive constant that depends only on $N, h, T, \|\varphi\|_{C^h}$, $\|a_\alpha\|_{C^N}$ and the smallest eigenvalue of $\sigma\sigma^*$.

We remark explicitly that Theorem 2.3 is an asymptotic convergence result for small times and in general, the approximation does not converge as N goes to infinity. It is well known that divergent asymptotic series often yield good approximations if the first few terms are taken: we shall confirm this fact in Section 4 through several numerical tests. In [23] a local version of estimate (2.15) is proved under the more general assumption of *local non-degeneracy* of $\sigma\sigma^*$: this allows to include popular models such as the CEV and Heston models; similar results were obtained in [3] using Malliavin calculus techniques. Also the assumptions on φ can be relaxed to consider standard payoff functions and to get an approximation of the transition density Γ of X in the form

$$\Gamma(t,x,T,y) \approx \sum_{n=0}^{N} \mathcal{L}_{\bar{x},n}(t,x,T)\Gamma_{\bar{x}}(t,x,T,y), \tag{2.16}$$

with $\Gamma_{\bar{x}}$ as in (2.8). Eventually, in [24] and [19] the technique has been generalized to models with jumps: in that case it yields an explicit approximation of the characteristic function of the underlying process that can be combined with standard Fourier methods to compute option prices.

2.2 Forward price expansion

In this section we prove our main result about the approximation of prices of forward-start options. Let X^1 denote the first component of the multi-dimensional process X in (1.1): we assume that e^{X^1} is a positive martingale representing an asset price process. Then the price at time s of a forward-start Call option with forward-start date $t \geq s$, maturity $T \geq t$ and log-strike k is given by

$$u(s,x) = E\left[\left(e^{X_T^1 - X_t^1} - e^k\right)^+ \mid X_s = x\right]. \tag{2.17}$$

A natural way to construct an approximation of the forward-start price u, is to combine the asymptotic expansion (2.16) of Γ with the tower and Markov properties for conditional expectation:

$$\begin{aligned} u(s,x) &= E\left[E\left[\left(e^{X_T^1 - X_t^1} - e^k\right)^+ \mid X_t\right] \mid X_s = x\right] \\ &= \int_{\mathbb{R}^d} \Gamma(s,x,t,y) \int_{\mathbb{R}^d} \Gamma(t,y,T,z) \left(e^{z_1 - y_1} - e^k\right)^+ dzdy \approx \end{aligned}$$

(by expanding $\Gamma(s,x,t,y)$ and $\Gamma(t,y,T,z)$ as in (2.16), at orders N and M around $\bar{\bar{x}}$ and $\bar{x}$ respectively)

$$\approx \sum_{n=0}^{N} \sum_{m=0}^{M} \int_{\mathbb{R}^d} \mathcal{L}_{\bar{\bar{x}},n}(s,x,t) \Gamma_{\bar{\bar{x}}}(s,x,t,y) \int_{\mathbb{R}^d} \mathcal{L}_{\bar{x},m}(t,y,T) \Gamma_{\bar{x}}(t,y,T,z) \left(e^{z_1 - y_1} - e^k\right)^+ dzdy$$

where y_1, z_1 denote the first components of the vectors y, z respectively. Thus we obtain

$$u(s,x) \approx \sum_{n=0}^{N} \sum_{m=0}^{M} \mathcal{L}_{\bar{\bar{x}},n}(s,x,t) u_m(s,x), \tag{2.18}$$

where

$$u_m(s,x) = \int_{\mathbb{R}^d} \Gamma_{\bar{\bar{x}}}(s,x,t,y) \int_{\mathbb{R}^d} \mathcal{L}_{\bar{x},m}(t,y,T) \Gamma_{\bar{x}}(t,y,T,z) \left(e^{z_1 - y_1} - e^k\right)^+ dzdy. \tag{2.19}$$

In Theorem 2.5 below, we show that the function u_m admits an explicit representation. Moreover, in the particular case of two-dimensional models, an even more compact expression for u_m is derived in Section 4.

We now introduce some specific notations for the *time-dependent BS model*: the dynamics of the log-price are given by

$$dZ_t = -\frac{\sigma(t)^2}{2} dt + \sigma(t) dW_t,$$

where $t \mapsto \sigma(t)$ is a positive, square integrable function representing the instantaneous volatility. The no-arbitrage price at time t of a Call option with log-strike k and maturity T, as a function of $z = Z_t$ is given by the BS formula

$$\mathrm{Call}^{\mathrm{BS}}(z,t,T,k;\sigma) = e^z \mathcal{N}(d_+) - e^k \mathcal{N}(d_-), \tag{2.20}$$

where

$$d_{\pm} = \frac{z-k}{\Sigma(t,T)} \pm \frac{\Sigma(t,T)}{2}, \qquad \Sigma^2(t,T) = \int_t^T \sigma^2(s) ds. \tag{2.21}$$

The first and second derivatives of $\mathrm{Call}^{\mathrm{BS}}$ with respect to the *underlying price* are denoted by

$$\mathrm{Delta}^{\mathrm{BS}}(z,t,T,k;\sigma) = \mathcal{N}(d_+) \quad \text{and} \quad \mathrm{Gamma}^{\mathrm{BS}}(z,t,T,k;\sigma) = \frac{\exp\left(-z - \frac{d_+^2}{2}\right)}{\sqrt{2\pi}\,\Sigma(t,T)}. \tag{2.22}$$

In the following technical lemma, we put

$$\bar{\sigma}(t) = \sqrt{C_{11}(t, \bar{x}(t))} \tag{2.23}$$

where $C = C(t, x)$ is the diffusion matrix in (2.3).

Lemma 2.4. *Let $\Gamma_{\bar{x}}$ be the Gaussian function in* (2.8)*. We have*

$$\int_{\mathbb{R}^d} \Gamma_{\bar{x}}(t, 0, T, y) \left(e^{y_1} - e^k\right)^+ dy = \mathrm{Call}^{\mathrm{BS}}\left(0, t, T, k; \bar{\sigma}\right). \tag{2.24}$$

Moreover, for any $j \in \mathbb{N}$, we have

$$\partial_z^j \mathrm{Call}^{\mathrm{BS}}\left(0, t, T, k; \bar{\sigma}\right) = \mathrm{Delta}^{\mathrm{BS}}\left(0, t, T, k; \bar{\sigma}\right) + \mathrm{Gamma}^{\mathrm{BS}}\left(0, t, T, k; \bar{\sigma}\right) g_j\left(t, T, k; \bar{\sigma}\right) \tag{2.25}$$

where

$$g_j\left(t, T, k; \bar{\sigma}\right) = \sum_{i=0}^{j-2} \left(\sqrt{2}\bar{\Sigma}(t, T)\right)^{-i} \mathbf{H}_i\left(\frac{k + \frac{1}{2}\bar{\Sigma}^2(t, T)}{\sqrt{2}\bar{\Sigma}(t, T)}\right), \qquad \bar{\Sigma}^2(t, T) = \int_t^T \bar{\sigma}^2(s) ds, \tag{2.26}$$

and $\mathbf{H}_i(x) = (-1)^i \frac{\partial_x^i e^{-x^2}}{e^{-x^2}}$ is the i-th Hermite polynomial. In (2.26) *we adopt the convention $\sum_{i=0}^{-1} = 0$.*

Proof. It suffices to note that by the martingale condition on e^{X^1} we have $\mu_1 = -\frac{1}{2}C_{11}$ in (2.3) and therefore, recalling the notations (2.7), (2.21) and (2.23), we also have $\mathbf{m}_1 = -\frac{1}{2}\bar{\Sigma}^2$. Then formulas (2.24) and (2.25) follow by straightforward computations. □

Theorem 2.5. *Consider the expansion* (2.18)-(2.19) *of u in* (2.17) *at orders N, M, around the points $\bar{x}, \bar{\bar{x}}$. For any $m \geq 0$, the function u_m in* (2.19) *is equal to*

$$u_m(s, x) = \sum_{j=0}^{3m} F_{\beta^j, m}(s, x, t, T) \partial_z^j \mathrm{Call}^{\mathrm{BS}}\left(0, t, T, k; \bar{\sigma}\right) \tag{2.27}$$

where $\bar{\sigma}$ is as in (2.23) *and*

$$F_{\beta, m}(s, x, t, T) = \sum_{|\alpha| \leq m} f_{\bar{x}, m, \alpha, \beta}(t, T) \left(\mathcal{M}_{\bar{\bar{x}}}(s, x, t) - \bar{x}(t)\right)^\alpha, \tag{2.28}$$

with $f_{\bar{x}, m, \alpha, \beta}$ as in (2.14) *and $\beta^j = (j, 0, \dots, 0)$.*

Proof. We have

$$u_m(s, x) = \int_{\mathbb{R}^d} \Gamma_{\bar{\bar{x}}}(s, x, t, y) \int_{\mathbb{R}^d} \mathcal{L}_{\bar{x}, m}(t, y, T) \Gamma_{\bar{x}}(t, y, T, \xi) \left(e^{\xi_1 - y_1} - e^k\right)^+ d\xi dy =$$

(by (2.14))

$$= \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq 3m}} f_{\bar{x}, m, \alpha, \beta}(t, T) \int_{\mathbb{R}^d} \Gamma_{\bar{\bar{x}}}(s, x, t, y)(y - \bar{x}(t))^\alpha \int_{\mathbb{R}^d} D_y^\beta \Gamma_{\bar{x}}(t, y, T, \xi) \left(e^{\xi_1 - y_1} - e^k\right)^+ d\xi dy =$$

(by the symmetry property (2.13), with $F_{\beta,m}(s,x,t,T)$ as in (2.28))

$$= \sum_{|\beta|\le 3m} F_{\beta,m}(s,x,t,T)\int_{\mathbb{R}^d}\Gamma_{\bar{\bar{x}}}(s,x,t,y)\int_{\mathbb{R}^d} D_y^{\beta}\Gamma_{\bar{x}}(t,y,T,\xi)\left(e^{\xi_1-y_1}-e^k\right)^+ d\xi dy =$$

(by the symmetry property $\nabla_y\Gamma_{\bar{x}}(t,y,T,\xi) = -\nabla_\xi\Gamma_{\bar{x}}(t,y,T,\xi)$)

$$= \sum_{|\beta|\le 3m} F_{\beta,m}(s,x,t,T)\int_{\mathbb{R}^d}\Gamma_{\bar{\bar{x}}}(s,x,t,y)\int_{\mathbb{R}^d}(-1)^{|\beta|} D_\xi^{\beta}\Gamma_{\bar{x}}(t,y,T,\xi)\left(e^{\xi_1-y_1}-e^k\right)^+ d\xi dy =$$

(by the change of variable $\eta = \xi - y$)

$$= \sum_{|\beta|\le 3m} F_{\beta,m}(s,x,t,T)\int_{\mathbb{R}^d}\Gamma_{\bar{\bar{x}}}(s,x,t,y)\int_{\mathbb{R}^d}(-1)^{|\beta|} D_\eta^{\beta}\Gamma_{\bar{x}}(t,0,T,\eta)\left(e^{\eta_1}-e^k\right)^+ d\eta dy =$$

(since $\Gamma_{\bar{\bar{x}}}(s,x,t,y)$ is a density and integrates to one)

$$= \sum_{|\beta|\le 3m} F_{\beta,m}(s,x,t,T)\int_{\mathbb{R}^d}(-1)^{|\beta|} D_\eta^{\beta}\Gamma_{\bar{x}}(t,0,T,\eta)\left(e^{\eta_1}-e^k\right)^+ d\eta.$$

Now we set

$$I_\beta(t,T) = \int_{\mathbb{R}^d}(-1)^{|\beta|} D_\eta^{\beta}\Gamma_{\bar{x}}(t,0,T,\eta)\left(e^{\eta_1}-e^k\right)^+ d\eta.$$

By (2.24), we have $I_0(t,T) = \mathrm{Call}^{\mathrm{BS}}(0,t,T,k;\bar{\sigma})$. Next we use again the identity

$$\partial_{\eta_1}\Gamma_{\bar{x}}(t,0,T,\eta) = -\partial_{x_1}\Gamma_{\bar{x}}(t,x,T,\eta)|_{x=0}.$$

Then, for $\beta = \beta^j = (j,0,\dots,0)$, we find

$$I_{\beta^j}(t,T) = \partial_z^j \mathrm{Call}^{\mathrm{BS}}(z,t,T,k;\bar{\sigma})\,|_{z=0}.$$

Eventually, if $\beta_j > 0$ for some $j \in \{2,\dots,d\}$ then integrating by parts we find $I_\beta(t,T) = 0$. This concludes the proof. $\square$

Remark 2.6. By (2.27) and Remark 2.2, the approximation of order 0, i.e. the expansion in (2.18) with $M = N = 0$, simply reduces to

$$u(s,x) \approx \mathcal{L}_{\bar{\bar{x}},0}(s,x,t)u_0(s,x) \equiv \mathrm{Call}^{\mathrm{BS}}(0,t,T,k;\bar{\sigma}). \tag{2.29}$$

Thus the leading term of the approximation is the price of a forward-start Call option in a time-dependent BS model with $\bar{\sigma}$ as in (2.23): as such, it is independent of the initial value x and time s. Formula (2.29) shows that it is also independent of $\bar{\bar{x}}$.

3 Forward implied volatility

In this section we derive an expansion of the forward implied volatility under the multi-dimensional dynamics (1.1). For the rest of the section, we consider as fixed a log-strike k, a forward start date t and a forward maturity τ; moreover, for simplicity we set the initial time s equal to zero and denote by

$$\mathrm{Call}(x,t,\tau,k) := E\left[\left(e^{X^1_{t+\tau}-X^1_t}-e^k\right)^+ \mid X_0 = x\right] \tag{3.1}$$

the no-arbitrage price of the forward-start Call option.

First we prove a general result which allows to get an implied volatility expansion from a price expansion: we recall that the forward implied volatility $\sigma_{t,\tau}(k)$ is defined by the equation

$$\mathrm{Call}\,(x,t,\tau,k) = \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_{t,\tau}(k))\,,$$

where $\mathrm{Call}^{\mathrm{BS}}$ is the Black-Scholes price in (1.3). In the next statement ρ_u denotes the radius of convergence of the Taylor series of $\mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\cdot)^{-1}$ about u in the interval $\left]\,\left(1-e^k\right)^+,1\right[$.

Theorem 3.1. *Assume that for some positive σ_0 and some sequence $(v_n)_{n\geq 1}$ of real numbers, the forward-start Call price in* (3.1) *admits the expansion*

$$\mathrm{Call}\,(x,t,\tau,k) = \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0) + \sum_{n=1}^{\infty} v_n.$$

Consider the analytic function

$$u(\varepsilon) := \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0) + \sum_{n=1}^{\infty} \varepsilon^n v_n, \qquad \varepsilon \in [0,1], \tag{3.2}$$

and assume that $\max_{\varepsilon\in[0,1]} \left|u(\varepsilon) - \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0)\right| < \rho_{\mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\sigma_0)}$. *Then the forward implied volatility is given by*

$$\sigma_{t,\tau}(k) = \sigma_0 + \sum_{n=1}^{\infty} \sigma_n,$$

where the sequence $(\sigma_n)_{n\geq 1}$ is defined recursively by

$$\sigma_n = \frac{v_n}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0)} - \frac{1}{n!}\sum_{h=2}^{n} A_h(\sigma_0)\mathbf{B}_{n,h}\,(1!\,\sigma_1, 2!\,\sigma_2,\dots,(n-h+1)!\,\sigma_{n-h+1})$$

with

$$A_h(\sigma_0) = \frac{\partial_\sigma^h \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0)}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma_0)} \tag{3.3}$$

and $\mathbf{B}_{n,h}$ denotes the (n,h)-th partial Bell polynomial[1].

Proof. We note that $\sigma(\varepsilon) := \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\cdot)^{-1}\,(u(\varepsilon))$ is the composition of two analytic functions: it is therefore an analytic function of ε and admits an expansion about $\varepsilon = 0$ of the form

$$\sigma(\varepsilon) = \sigma_0 + \sum_{n=1}^{\infty} \varepsilon^n \sigma_n, \qquad \sigma_n = \frac{1}{n!}\partial_\varepsilon^n \sigma(\varepsilon)|_{\varepsilon=0}, \tag{3.4}$$

which is convergent for any $\varepsilon \in [0,1]$. By (3.2) we also have

$$v_n = \frac{1}{n!}\partial_\varepsilon^n \mathrm{Call}^{\mathrm{BS}}\,(0,\tau,k;\sigma(\varepsilon))\,|_{\varepsilon=0}. \tag{3.5}$$

[1] Partial Bell polynomials are implemented in Mathematica as $\mathrm{BellY}[\mathrm{n},\mathrm{h},\{\mathrm{x}_1,\dots,\mathrm{x}_{\mathrm{n-h+1}}\}]$.

We compute the n-th derivative of the composition of the two functions in (3.5) by applying the Bell polynomial version of the Faa di Bruno's formula, which can be found in [27] and [15]. We get

$$v_n = \frac{1}{n!}\sum_{h=1}^{n} \partial_\sigma^h \mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\sigma_0)\,\mathbf{B}_{n,h}\left(\partial_\varepsilon\sigma(\varepsilon), \partial_\varepsilon^2\sigma(\varepsilon),\dots,\partial_\varepsilon^{n-h+1}\sigma(\varepsilon)\right)|_{\varepsilon=0}. \tag{3.6}$$

Then the thesis follows by inserting (3.4) into (3.6) and solving for σ_n. □

The following lemma provides an iterative algorithm to compute the coefficients A_n in (3.3) in a simple and explicit way: in particular, it shows that each $A_n(\sigma)$ is a rational function of σ and no special functions appear in its expression.

Proposition 3.2. *We have*

$$A_n(\sigma) = \frac{P_n(\mathcal{J}_z)\mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\sigma)}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}(0,\tau,k;\sigma)}, \tag{3.7}$$

where $\mathcal{J}_z$ is the differential operator

$$\mathcal{J}_z = \tau(\partial_{zz} - \partial_z),$$

and P_n is the polynomial function of order n defined recursively by $P_0(y) = 1$, $P_1(y) = \sigma y$ and

$$P_n(y) = \sigma y P_{n-1}(y) + (n-1) y P_{n-2}(y), \qquad n \geq 2.$$

Consequently, $A_n(\sigma)$ is a rational function of σ and linear combination of Hermite polynomials.

Proof. First, we recall the classical relation among the Delta, Gamma and Vega of European options in the BS setting:

$$\partial_\sigma \mathrm{Call}^{\mathrm{BS}}(z,\tau,k;\sigma) = \sigma\mathcal{J}_z\mathrm{Call}^{\mathrm{BS}}(z,\tau,k;\sigma). \tag{3.8}$$

In order to prove (3.7), we show by induction on n that

$$\partial_\sigma^n \mathrm{Call}^{\mathrm{BS}}(z,\tau,k;\sigma) = P_n(\mathcal{J}_z)\mathrm{Call}^{\mathrm{BS}}(z,\tau,k;\sigma).$$

The result is trivially true for $n = 0$, and by the definition of P_1 and (3.8) is valid also for $n = 1$. Let us suppose the result is valid for $k \leq n$. Using the product rule for derivatives and omitting the arguments for simplicity, we have

$$\begin{aligned}
\partial_\sigma^{n+1}\mathrm{Call}^{\mathrm{BS}} &= \partial_\sigma^n\left(\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\right) = \partial_\sigma^n\left(\sigma\mathcal{J}_z\mathrm{Call}^{\mathrm{BS}}\right) \\
&= \sum_{h=0}^{n}\binom{n}{h}(\partial_\sigma^h\sigma)\left(\mathcal{J}_z\partial_\sigma^{n-h}\mathrm{Call}^{\mathrm{BS}}\right) = \left(\sigma\mathcal{J}_z\partial_\sigma^n + n\mathcal{J}_z\partial_\sigma^{n-1}\right)\mathrm{Call}^{\mathrm{BS}} \\
&= \left(\sigma\mathcal{J}_z P_n(\mathcal{J}) + n\mathcal{J}P_{n-1}(\mathcal{J}_z)\right)\mathrm{Call}^{\mathrm{BS}} = P_{n+1}(\mathcal{J}_z)\mathrm{Call}^{\mathrm{BS}}.
\end{aligned}$$

This proves (3.7). Next we show that $A_n(\sigma)$ is a sum of Hermite polynomials. We use the identities

$$\frac{\partial_z^n \exp\left(-\left(\frac{z-a}{b}\right)^2\right)}{\exp\left(-\left(\frac{z-a}{b}\right)^2\right)} = \frac{(-1)^n}{b^n}\mathbf{H}_n\left(\frac{z-a}{b}\right), \qquad n\in\mathbb{N},\ z,a\in\mathbb{R},\ b\in\mathbb{R}\setminus\{0\}, \tag{3.9}$$

and

$$\mathcal{J}_z\mathrm{Call}^{\mathrm{BS}}\left(z,\tau,k;\sigma\right)=\frac{e^k\sqrt{\tau}}{\sigma\sqrt{2\pi}}\exp\left(-\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}\right)^2\right). \tag{3.10}$$

Thus, by (3.8) and (3.10), we obtain

$$\frac{\mathcal{J}_z^{n+1}\mathrm{Call}^{\mathrm{BS}}\left(z,\tau,k;\sigma\right)}{\partial_\sigma\mathrm{Call}^{\mathrm{BS}}\left(z,\tau,k;\sigma\right)}=\frac{\mathcal{J}_z^{n}\mathcal{J}_z\mathrm{Call}^{\mathrm{BS}}\left(z,\tau,k;\sigma\right)}{\sigma\mathcal{J}_z\mathrm{Call}^{\mathrm{BS}}\left(z,\tau,k;\sigma\right)}=\frac{\mathcal{J}_z^n\exp\left(-\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}\right)^2\right)}{\sigma\exp\left(-\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2t}}\right)^2\right)}=$$

(by the binomial expansion of $(\partial_{zz}-\partial_z)^n$)

$$=\frac{\tau^n}{\sigma}\sum_{h=0}^{n}\binom{n}{h}(-1)^h\frac{\partial_z^{2n-h}\exp\left(-\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}\right)^2\right)}{\sigma\exp\left(-\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}\right)^2\right)}=$$

(by (3.9) with $a=k+\frac{\sigma^2\tau}{2}$ and $b=\sigma\sqrt{2\tau}$)

$$=\sum_{h=0}^{n}\binom{n}{h}\frac{\tau^{\frac{h}{2}}}{\sigma(\sigma\sqrt{2})^{2n-h}}\mathbf{H}_{2n-h}\left(\frac{z-k-\sigma^2\tau/2}{\sigma\sqrt{2\tau}}\right).$$

Combining this last expression for $\frac{\mathcal{J}^{n+1}u^{\mathrm{BS}}}{\partial_\sigma u^{\mathrm{BS}}}$ with (3.7), we conclude that $A_n(\sigma)$ can be expressed as a sum of Hermite polynomials. In particular, computing $A_n(\sigma)$ does not involve any special function nor integration. □

Now let us consider the forward-start Call option in (3.1) under the dynamics (1.1), with related Kolmogorov operator given by (2.3). We combine the forward price expansion (2.18)-(2.27) with Theorem 3.1: notice that Theorem 3.1 is applied in a formal way since the series (2.18) is not necessarily convergent. To this end, we choose an enumeration $(v_n)_{n\geq 0}$ of the terms of the expansion (2.18): for example, we might set

$$v_n(s,x)=\sum_{h+k=n}\mathcal{L}_{\bar{\bar{x}},h}(s,x,t)u_k(s,x),\qquad n\in\mathbb{N}_0.$$

Then, by Remark 2.6, we identify

$$\sigma_0=\sigma_{t,\tau,0}(k):=\sqrt{\frac{1}{\tau}\int_t^{t+\tau}C_{11}(s,\bar{x}(s))ds}, \tag{3.11}$$

$$\sigma_n=\sigma_{t,\tau,n}(k):=\frac{v_n}{\partial_\sigma\mathrm{Call}^{\mathrm{BS}}\left(0,\tau,k;\sigma_0\right)}-\frac{1}{n!}\sum_{h=2}^{n}A_h(\sigma_0)\mathbf{B}_{n,h}\left(\sigma_1,2!\,\sigma_2,\dots,(n-h+1)!\,\sigma_{n-h+1}\right), \tag{3.12}$$

where A_n is as in (3.7). This motivates the following definition.

Definition 3.3. Let $N\in\mathbb{N}_0$ and $(\sigma_{t,\tau,n}(k))_{n\leq N}$ as in (3.11)-(3.12). We say that

$$\bar{\sigma}_{t,\tau,N}(k):=\sum_{n=0}^{N}\sigma_{t,\tau,n}(k), \tag{3.13}$$

is an *N-th order asymptotic forward smile expansion.*

In Section 4 we examine in detail the two-dimensional case, i.e. $d = 2$ in (1.1), and compute explicitly the terms

$$\frac{v_n}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\left(0, \tau, k; \sigma_0\right)} \tag{3.14}$$

appearing in expansion (3.12).

Concerning the error estimates of the implied volatility expansion, recently [20] proved an asymptotic convergence result for short maturities in the *spot* case. Rigorous bounds for the forward volatility are only available in the one-dimensional case and up to the second order, for the expansion proposed by [8] that is very similar to (3.13) (actually, in the spot case they turn out to be equivalent). The analysis of the general multi-dimensional case is not a straightforward extension of the previous results and is object of ongoing research: as in the spot case, the derivation of explicit formulas for the asymptotic forward smile expansion is the first crucial step towards a formal proof of the asymptotic convergence of the expansion.

4 Local-stochastic volatility models

In this section we focus on two-dimensional models. Before presenting the numerical tests, we show that the formulas of Theorem 2.5 can be further simplified, leading to simple and effective approximations: in particular we give the first two terms of the forward volatility expansion in a general local-stochastic volatility model (see (4.6)). In the last part, we illustrate the flexibility and accuracy of our methodology by applying it to the CEV local volatility model and the Heston stochastic volatility model.

We consider the local-stochastic volatility model

$$\begin{aligned} dX_t &= -\frac{1}{2}\sigma(t, X_t, Y_t)^2 dt + \sigma(t, X_t, Y_t) dW_t, \\ dY_t &= \nu(t, X_t, Y_t) dt + \eta(t, X_t, Y_t) dB_t, \\ d\langle W, B\rangle_t &= \rho(t, X_t, Y_t) dt, \end{aligned} \tag{4.1}$$

where as usual X represents the underlying's log-price under a risk-neutral measure. The related Kolmogorov operator $\mathcal{A}$ is given by

$$\mathcal{A} = a_{2,0}(t,x,y)\partial_{xx} + a_{1,0}(t,x,y)\partial_x + a_{0,2}(t,x,y)\partial_{yy} + a_{0,1}(t,x,y)\partial_y + a_{1,1}(t,x,y)\partial_{x,y} + a_{0,0}(t,x,y)$$

where

$$a_{2,0} = \frac{\sigma^2}{2}, \quad a_{0,2} = \frac{\eta^2}{2}, \quad a_{1,1} = \rho\sigma\eta, \quad a_{1,0} = -\frac{\sigma^2}{2}, \quad a_{0,1} = \nu, \quad a_{0,0} = 0.$$

According to the notations of Section 2.1, for a fixed $(\bar{x}, \bar{y})$, we set

$$\bar{C}(t) = \begin{pmatrix} 2a_{2,0}(t, \bar{x}(t), \bar{y}(t)) & a_{1,1}(t, \bar{x}(t), \bar{y}(t)) \\ a_{1,1}(t, \bar{x}(t), \bar{y}(t)) & 2a_{0,2}(t, \bar{x}(t), \bar{y}(t)) \end{pmatrix}$$

and

$$\bar{\mu}(t) = \left(a_{1,0}(t, \bar{x}(t), \bar{y}(t)), a_{0,1}(t, \bar{x}(t), \bar{y}(t))\right).$$

Moreover we define

$$\mathbf{C}(t,T) = \int_t^T \bar{C}(s)ds, \qquad \mathbf{m}(t,T) = \int_t^T \bar{\mu}(s)ds,$$

and we denote by $\mathcal{L}_{\bar{x},\bar{y},n}(t,x,y,T)$ the operator formally defined by $\mathcal{L}_{\bar{x},\bar{y},0}(t,x,y,T) \equiv 1$ for $n=0$ and as in (2.11) for $n \in \mathbb{N}$, where now $\mathcal{M}_{\bar{x},\bar{y}}(t,x,y,T)$ has two components:

$$\mathcal{M}_{1,\bar{x},\bar{y}}(t,x,y,T) = x + \mathbf{m}_1(t,T) + \mathbf{C}_{11}(t,T)\partial_x + \mathbf{C}_{12}(t,T)\partial_y,$$
$$\mathcal{M}_{2,\bar{x},\bar{y}}(t,x,y,T) = y + \mathbf{m}_2(t,T) + \mathbf{C}_{21}(t,T)\partial_x + \mathbf{C}_{22}(t,T)\partial_y.$$

By Remark 2.2, $\mathcal{L}_{\bar{x},\bar{y},n}(t,x,y,T)$ can be written in the form

$$\mathcal{L}_{\bar{x},\bar{y},n}(t,x,y,T) = \sum_{\substack{\alpha_1+\alpha_2\leq n \\ \beta_1+\beta_2\leq 3n}} (x-\bar{x}(t))^{\alpha_1}(y-\bar{y}(t))^{\alpha_2} f_{\bar{x},\bar{y},n,\alpha_1,\alpha_2,\beta_1,\beta_2}(t,T)\partial_x^{\beta_1}\partial_y^{\beta_2}. \tag{4.2}$$

Now let

$$u(s,x,y) := E\left[\left(e^{X_T-X_t} - e^k\right)^+ \mid X_s = x, Y_s = y\right]$$

be the forward-start Call price at time s: by the results of Section 2.2, we have

$$u(s,x,y) \approx \sum_{n=0}^{N}\sum_{m=0}^{M} \mathcal{L}_{\bar{\bar{x}},\bar{\bar{y}},n}(s,x,y,t)u_m(s,x,y), \tag{4.3}$$

with u_m defined by (2.19). A more explicit expression for u_m is given by the following

Corollary 4.1. *For any $m \in \mathbb{N}$ the function u_m in (4.3) is equal to*

$$u_m(s,x,y) = \text{Gamma}^{\text{BS}}\left(0,t,T,k;\bar{\sigma}\right)\sum_{j=2}^{3m} g_j\left(t,T,k;\bar{\sigma}\right)F_{j,m}(s,x,y,t,T) \tag{4.4}$$

where Gamma^{BS} *is as in* (2.22), $\bar{\sigma}(t) = \sqrt{2a_{2,0}(t,\bar{x}(t),\bar{y}(t))}$ *and*

$$F_{j,m}(s,x,y,t,T) = \sum_{\alpha_1+\alpha_2\leq m} f_{\bar{x},\bar{y},m,\alpha_1,\alpha_2,j,0}(t,T)\left(\mathcal{M}_{1,\bar{\bar{x}},\bar{\bar{y}}}(s,x,y,t) - \bar{x}(t)\right)^{\alpha_1}\left(\mathcal{M}_{2,\bar{\bar{x}},\bar{\bar{y}}}(s,x,y,t) - \bar{y}(t)\right)^{\alpha_2},$$

with $f_{\bar{x},\bar{y},m,\alpha_1,\alpha_2,j,0}(t,T)$ *as in* (4.2) *and* $g_j\left(t,T,k;\bar{\sigma}\right)$ *as in* (2.26).

Proof. The thesis is a direct consequence of Lemma 2.4, Theorem 2.5 and the identities

$$f_{\bar{x},\bar{y},m,\alpha_1,\alpha_2,0,0}(t,T) = 0 \quad \text{and} \quad \sum_{j=1}^{3m} f_{\bar{x},\bar{y},m,\alpha_1,\alpha_2,j,0}(t,T) = 0. \tag{4.5}$$

The proof of (4.5) is based on a tedious but straightforward computation: the main ingredient is the identity $\partial_x^i\partial_y^j a_{1,0} = -\partial_x^i\partial_y^j a_{2,0}$ which follows from the martingale condition. □

Remark 4.2. Formula (4.4) leads to an extremely significant simplification in the computation of the forward implied volatility expansion (3.11)-(3.12). Indeed, (3.14) is a sum of terms of the form[2]

$$\frac{\mathcal{L}_{\bar{\bar{x}},\bar{\bar{y}},h}(s,x,y,t)u_m(s,x,y)}{\partial_\sigma \text{Call}^{\text{BS}}\left(0,\tau,k;\sigma_0\right)} \qquad (\text{with } \sigma_0 = \sigma_{t,\tau,0}(k) \text{ as in (3.11)})$$

[2]Recall that we denote by $\tau = T-t$ the forward maturity and therefore $\text{Call}^{\text{BS}}\left(z,\tau,k;\sigma\right) = \text{Call}^{\text{BS}}\left(z,t,T,k;\sigma\right)$ according to notations (1.3) and (2.20) for the constant and time-dependent BS models respectively.

or equivalently, by (4.4),

$$\frac{\mathrm{Gamma}^{\mathrm{BS}}\left(0,t,t+\tau,k;\bar{\sigma}\right)}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\left(0,\tau,k;\sigma_0\right)}\sum_{j=2}^{3m} g_j\left(t,T,k;\bar{\sigma}\right)\mathcal{L}_{\bar{\bar{x}},\bar{\bar{y}},h}(s,x,y,t)F_{j,m}(s,x,y,t,T),$$

where:

i) $g_j\left(t,T,k;\bar{\sigma}\right)$ is the rational function defined explicitly in (2.26);

ii) $\mathcal{L}_{\bar{\bar{x}},\bar{\bar{y}},h}(s,x,y,t)F_{j,m}(s,x,y,t,T)$ is a polynomial function;

iii) by identity (3.8), we simply have

$$\frac{\mathrm{Gamma}^{\mathrm{BS}}\left(0,t,t+\tau,k;\bar{\sigma}\right)}{\partial_\sigma \mathrm{Call}^{\mathrm{BS}}\left(0,\tau,k;\sigma_0\right)}=\frac{1}{\tau\sigma_0}.$$

Next we report the first two terms of the expansion of the forward implied volatility $\sigma_{t,\tau}(k)$, evaluated a time $s=0$, for the forward-start Call option with payoff $\left(e^{X_{t+\tau}-X_t}-e^k\right)^+$ in the local-stochastic volatility model (4.1) with time-homogeneous coefficients. By setting $\bar{x}=\bar{\bar{x}}=X_0=x$ and $\bar{y}=\bar{\bar{y}}=Y_0=y$, we get

$$\sigma_{t,\tau,0}(k)=\sqrt{2a_{2,0}(x,y)}, \tag{4.6}$$

$$\sigma_{t,\tau,1}(k)=\frac{\left(ka_{1,1}(x,y)+\tau a_{2,0}(x,y)\left(2a_{0,1}(x,y)+a_{1,1}(x,y)\right)\right)\partial_y a_{2,0}(x,y)+\left(k-2ta_{2,0}(x,y)\right)\partial_x\left(a_{2,0}(x,y)\right)^2}{4\sqrt{2}(a_{2,0}(x,y))^{\frac{3}{2}}}.$$

The general expression for $\sigma_{t,\tau,2}(k)$ is too long to be reported here: we only give the limit

$$\lim_{\tau\to 0^+}\tau\sigma_{t,\tau,2}(k)=\frac{t\left(k\partial_x a_{2,0}(x,y)\right)^2}{4\sqrt{2}(a_{2,0}(x,y))^{\frac{3}{2}}}, \tag{4.7}$$

which shows that $\sigma_{t,\tau,2}(k)=\mathrm{O}\left(\tau^{-1}\right)$ as τ tends to zero, whenever t,k and $\partial_x a_{2,0}$ are not null. The exploding behaviour of the forward smile for out-of-the-money options was proved by [13] in the case of the Heston stochastic volatility model. Formula (4.7) indicates that the singularity of the forward smile is a rather general feature of local-stochastic volatility models. This is also well documented by the numerical experiments of the following sections.

4.1 Tests in the CEV model

In the Constant Elasticity of Variance (CEV) local volatility model [9], the risk-neutral dynamics of the underlying S are given by

$$dS_t=\delta S_t^{\beta-1}S_t dW_t,$$

where δ is the positive volatility parameter and $\beta\in[0,1]$. Notice that it is not restrictive to assume that $S_0=1$, since we can always rescale the CEV equation by setting $Y_t=\frac{S_t}{S_0}$ to get

$$dY_t=\tilde{\delta}Y_t^{\beta}dW_t$$

where $\tilde{\delta} = \delta S_0^{\beta-1}$.

In log notation $X := \log S$, we have the dynamics

$$dX_t = -\frac{1}{2}\delta^2 e^{2(\beta-1)X_t} dt + \delta e^{(\beta-1)X_t} dW_t,$$

and the Kolmogorov operator of X is given by

$$\mathcal{A} = \frac{1}{2}\delta^2 e^{2(\beta-1)x}(\partial_{xx} - \partial_x).$$

Using Corollary 4.1 we compute the expansion of the forward implied volatility $\sigma_{t,\tau}(k)$, evaluated a time zero, for the forward-start Call option with payoff $\left(e^{X_{t+\tau}-X_t} - e^k\right)^+$. By setting $\bar{x} = \bar{\bar{x}} = X_0 = 0$, we get $\sigma_{t,\tau,0}(k) = \delta$ at order zero and

$$\sigma_{t,\tau,1}(k) = \frac{1}{2}(\beta - 1)\delta(k - t\delta^2)$$

at order one. Thus the first two terms of the expansion do not depend on the forward maturity τ. At orders two and three, we find

$$\sigma_{t,\tau,n}(k) = \sum_{j=-1}^{2} A_{n,j}(t)\tau^j, \qquad n = 2, 3,$$

where

$$\begin{aligned}
A_{2,-1}(t) &= \frac{1}{2}k^2 t(\beta-1)^2\delta, \\
A_{2,0}(t) &= \frac{1}{24}(\beta-1)^2(2k^2\delta - 6t(k-2)\delta^3 + 9t^2\delta^5) \\
A_{2,2}(t) &= \frac{1}{24}(\beta-1)^2\delta^3(1 - 3t\delta^2), \\
A_{2,3}(t) &= -\frac{1}{96}(\beta-1)^2\delta^5,
\end{aligned}$$

and

$$\begin{aligned}
A_{3,-1}(t) &= \frac{1}{4}k^2(1-\beta)^3\left(kt\delta + 2t^2\delta^3\right), \\
A_{3,0}(t) &= \frac{1}{48}(\beta-1)^3\left(2kt(6-k)\delta^3 + 9t^2(k-8)\delta^5 - 15t^3\delta^7\right), \\
A_{3,1}(t) &= \frac{1}{16}(\beta-1)^3\left(k\delta^3 - t(1+3k)\delta^5 + 6t^2\delta^7\right), \\
A_{3,2}(t) &= \frac{5}{192}(1-\beta)^3(k - t\delta^2)\delta^5.
\end{aligned}$$

Notice that for $\beta \neq 1$ (i.e. except for the BS model), $t > 0$ (i.e. for a strictly positive forward start date) and $k \neq 0$ (i.e. for out-of-the-money options), we have

$$\sigma_{t,\tau,n}(k) = \mathrm{O}\left(\tau^{-1}\right) \qquad \text{as } \tau \to 0^+,\ n = 2, 3.$$

To illustrate the accuracy of the expansion formulas, we compare our 3rd order forward implied volatility approximation with the high-precision numerical values obtained in [7] by Monte Carlo simulations. In the tests under consideration, the confidence interval widths are reduced to less than 1 basis point (i.e. $\pm 0.01\%$) for all the strikes and maturities.

The CEV parameters are set to $\delta = 0.2$ and $\beta = 0.5$. The forward start dates and maturities are the same as in [7]: specifically, we consider the forward start dates $t = 0$ (this corresponds to the spot implied volatility), $1M$ (one month), $3M$ (three months), $6M$ (six months) and $1Y$ (one year) and the forward maturities $\tau = 3M, 1Y, 5Y$ and $10Y$. The strikes $K = e^k$ are chosen with respect to the set maturities as in Table 1: in particular, the strikes approximately behave as $e^{q\delta\sqrt{\tau}}$ where q denotes the value of various quantiles of the standard normal law (from 1% to 99%) to cover around the money (i.e. $K \approx 1$) as well as far from the money options.

$\tau \setminus K$	1%	5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%
3M	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.25	1.30	1.35
1Y	0.55	0.65	0.75	0.80	0.90	0.95	1.00	1.05	1.15	1.25	1.40	1.50	1.80
5Y	0.25	0.40	0.50	0.60	0.75	0.85	1.00	1.15	1.35	1.60	2.05	2.50	3.60
10Y	0.15	0.25	0.35	0.50	0.65	0.80	1.00	1.20	1.50	1.95	2.75	3.65	6.30

Table 1: Set of maturities and strikes for the tests in the CEV model

From Table 2 we see that the 3rd order formula gives an excellent approximation of the forward implied volatility. We would like to emphasize again that our approximation involves only polynomials and can be computed without the need for numerical procedures or special functions.

4.2 Tests in the Heston model

In the Heston model [12], the dynamics of the asset price S are given by

$$\begin{aligned}
dS_t &= \sqrt{Z_t} S_t dW_t, \\
dZ_t &= \kappa(\theta - Z_t)dt + \delta\sqrt{Z_t} B_t, \\
d\langle W, B\rangle_t &= \rho dt,
\end{aligned}$$

with $\rho < 0$ to prevent a moment explosion (cf. [2]). To improve the approximation, we perform the change of variables

$$X_t = \log S_t, \qquad V_t = e^{\kappa t} Z_t,$$

so that we have

$$\begin{aligned}
dX_t &= -\frac{1}{2} e^{-\kappa t} V_t dt + \sqrt{e^{-\kappa t} V_t} dW_t, \\
dV_t &= \theta\kappa e^{\kappa t} dt + \delta\sqrt{e^{\kappa t} V_t} dB_t,
\end{aligned}$$

with $X_0 = \log S_0$ and $V_0 = Z_0$. The Kolmogorov operator of (X, V) is given by

$$\mathcal{A} = \frac{1}{2} e^{-\kappa t} v(\partial_{xx} - \partial_x) + \theta\kappa e^{\kappa t}\partial_v + \frac{1}{2}\delta^2 e^{\kappa t} v\partial_{vv} + \delta\rho v\partial_{xv}.$$

Using Corollary 4.1, we compute the expansion of the forward implied volatility $\sigma_{t,\tau}(k)$, evaluated at time zero, with $X_0 = 0$ and $V_0 = v > 0$, for the forward-start Call option with payoff $\left(e^{X_{t+\tau} - X_t} - e^k\right)^+$. By

τ	$t \setminus K$		1%	5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%
3M	0	MC	21.84	21.48	21.14	20.83	20.53	20.26	20.00	19.76	19.53	19.31	18.91	18.72	18.54
		approx	21.84	21.48	21.14	20.83	20.53	20.26	20.00	19.76	19.53	19.31	18.91	18.72	18.54
	1M	MC	21.96	21.56	21.2	20.87	20.57	20.29	20.03	19.78	19.56	19.35	18.97	18.80	18.64
		approx	21.96	21.57	21.2	20.87	20.57	20.29	20.03	19.79	19.56	19.35	18.97	18.80	18.64
	3M	MC	22.21	21.75	21.34	20.97	20.64	20.34	20.08	19.84	19.63	19.44	19.11	18.97	18.85
		approx	22.21	21.75	21.34	20.97	20.64	20.34	20.08	19.84	19.63	19.43	19.11	18.97	18.85
	6M	MC	22.59	22.03	21.54	21.12	20.75	20.43	20.16	19.92	19.73	19.56	19.32	19.23	19.17
		approx	22.59	22.03	21.54	21.11	20.74	20.43	20.15	19.92	19.72	19.56	19.32	19.23	19.17
	1Y	MC	23.35	22.59	21.95	21.41	20.96	20.60	20.31	20.09	19.93	19.82	19.74	19.75	19.80
		approx	23.36	22.59	21.94	21.41	20.96	20.60	20.31	20.09	19.93	19.82	19.73	19.76	19.81
1Y	0	MC	23.15	22.24	21.48	21.15	20.54	20.27	20.01	19.77	19.32	18.91	18.37	18.05	17.21
		approx	23.15	22.24	21.48	21.15	20.54	20.27	20.01	19.77	19.32	18.91	18.37	18.05	17.21
	1M	MC	23.24	22.30	21.52	21.18	20.57	20.29	20.03	19.79	19.34	18.95	18.42	18.11	17.31
		approx	23.24	22.30	21.53	21.18	20.57	20.29	20.03	19.79	19.35	18.95	18.42	18.11	17.31
	3M	MC	23.43	22.43	21.61	21.26	20.63	20.34	20.08	19.84	19.40	19.02	18.52	18.23	17.53
		approx	23.43	22.43	21.61	21.26	20.62	20.34	20.08	19.84	19.40	19.02	18.52	18.23	17.52
	6M	MC	23.72	22.63	21.75	21.37	20.71	20.43	20.16	19.92	19.49	19.12	18.67	18.42	17.85
		approx	23.71	22.62	21.74	21.37	20.71	20.42	20.16	19.92	19.49	19.12	18.67	18.41	17.84
	1Y	MC	24.29	23.01	22.01	21.59	20.88	20.58	20.32	20.08	19.67	19.34	18.97	18.79	18.49
		approx	24.28	23.01	22.01	21.59	20.88	20.58	20.32	20.08	19.67	19.34	18.97	18.79	18.48
5Y	0	MC	27.83	25.01	23.73	22.72	21.52	20.87	20.04	19.35	18.57	17.77	16.65	15.79	14.29
		approx	27.81	25.00	23.73	22.72	21.52	20.87	20.04	19.34	18.57	17.77	16.64	15.78	14.28
	1M	MC	27.92	25.06	23.78	22.76	21.55	20.89	20.06	19.37	18.60	17.80	16.69	15.85	14.40
		approx	27.90	25.06	23.77	22.75	21.55	20.89	20.06	19.37	18.60	17.80	16.69	15.84	14.40
	3M	MC	28.11	25.18	23.87	22.84	21.61	20.95	20.12	19.42	18.65	17.87	16.79	15.98	14.63
		approx	28.08	25.17	23.86	22.83	21.61	20.95	20.11	19.42	18.65	17.87	16.78	15.97	14.61
	6M	MC	28.39	25.37	24.02	22.95	21.71	21.03	20.19	19.50	18.74	17.97	16.93	16.18	14.97
		approx	28.33	25.35	24.00	22.94	21.70	21.03	20.19	19.50	18.73	17.96	16.92	16.16	14.94
	1Y	MC	28.96	25.73	24.30	23.19	21.89	21.20	20.35	19.65	18.90	18.17	17.22	16.57	15.66
		approx	28.87	25.70	24.29	23.18	21.89	21.20	20.34	19.65	18.90	18.17	17.21	16.55	15.60
10Y	0	MC	31.20	27.91	25.86	23.79	22.33	21.23	20.08	19.17	18.10	16.89	15.40	14.25	12.21
		approx	31.17	27.94	25.84	23.78	22.33	21.22	20.08	19.17	18.10	16.89	15.39	14.24	12.19
	1M	MC	31.28	27.97	25.91	23.82	22.36	21.25	20.10	19.19	18.12	16.92	15.45	14.31	12.33
		approx	31.23	27.94	25.89	23.81	22.36	21.25	20.10	19.19	18.12	16.92	15.44	14.30	12.31
	3M	MC	31.46	28.10	26.01	23.9	22.43	21.31	20.15	19.24	18.18	16.99	15.54	14.45	12.60
		approx	31.39	28.06	25.98	23.89	22.42	21.31	20.15	19.24	18.17	16.98	15.53	14.43	12.54
	6M	MC	31.72	28.29	26.16	24.02	22.52	21.39	20.23	19.32	18.26	17.09	15.69	14.66	12.99
		approx	31.62	28.24	26.13	24.00	22.52	21.38	20.22	19.31	18.25	17.08	15.69	14.62	12.90
	1Y	MC	32.23	28.67	26.46	24.25	22.71	21.56	20.38	19.47	18.42	17.29	15.98	15.07	13.77
		approx	32.10	28.61	26.42	24.23	22.70	21.55	20.38	19.46	18.41	17.28	15.95	15.01	13.63

Table 2: Forward implied volatilities (Monte Carlo and 3rd order approximation) in the CEV model with parameters $\beta = 0.5$, $\delta = 0.2$, with forward start date t, maturity $t + \tau$ and the range of strikes in Table 1

expanding around the points $\bar{x} = \bar{\bar{x}} = X_0 = 0$ and

$$\bar{v}(t) = \bar{\bar{v}}(t) = E\left[V_t\right] = v + \theta\kappa \int_0^t e^{\kappa s} ds,$$

we find the following approximation of the forward implied volatility $\sigma_{t,\tau}(k)$ at orders zero and one:

$$\begin{aligned}
\sigma_{t,\tau,0}(k) =& \sqrt{\frac{(-e^{-\kappa(t+\tau)} + e^{-\kappa t})(v-\theta)}{\kappa\tau} + \theta}, \\
\sigma_{t,\tau,1}(k) =& \frac{e^{-\frac{1}{2}\kappa(t+\tau)}\rho\delta\big(-v + ve^{\kappa\tau} + \theta - e^{\kappa\tau}\theta + e^{\kappa(t+\tau)}\kappa(2k - 2r\tau + \theta\tau)\big)}{4\sqrt{\tau}\big(\kappa(-v + ve^{\kappa\tau} + \theta - e^{\kappa\tau}\theta + e^{\kappa(t+\tau)}\theta\kappa\tau)\big)^{3/2}} \cdot \\
& \cdot \big(ve^{\kappa\tau} + \theta + e^{\kappa t}\theta - e^{\kappa\tau}\theta + \theta\kappa\tau + e^{\kappa(t+\tau)}\theta(-1+\kappa\tau) - v(1+\kappa\tau)\big).
\end{aligned}$$

The expression for $\sigma_{t,\tau,2}$ is too long to be reported, however its explicit formula is provided in the Mathematica notebook on the authors' website.

As in the CEV model the approximation explodes as the forward maturity τ tends to zero, in case $k \neq 0$ and $t > 0$. In fact, we have the following limit values of the smile expansion:

$$\begin{aligned}
\lim_{\tau \to 0^+} \sigma_{t,\tau,0}(k) &= \sqrt{e^{-t\kappa}(v-\theta) + \theta}, \\
\lim_{\tau \to 0^+} \sigma_{t,\tau,1}(k) &= \frac{k e^{\frac{t\kappa}{2}}\delta\rho}{4\sqrt{v - \theta + e^{t\kappa}\theta}}, \\
\lim_{\tau \to 0^+} \tau\sigma_{t,\tau,2}(k) &= \frac{\delta^2 k^2 \left(e^{\kappa t} - 1\right)\left(\theta\left(e^{\kappa t} - 1\right) + 2v\right)}{16\kappa\left(\theta\left(e^{\kappa t} - 1\right) + v\right)^2 \sqrt{e^{-\kappa t}\left(\theta\left(e^{\kappa t} - 1\right) + v\right)}}, \\
\lim_{\tau \to 0^+} \tau\sigma_{t,\tau,3}(k) &= \frac{5\delta^3 k^3 \rho e^{\kappa t}\left(1 - e^{\kappa t}\right)\left(\theta e^{\kappa t} - \theta + 2v\right)}{64\kappa\sqrt{\theta - \theta e^{-\kappa t} + v e^{-\kappa t}}\left(\theta e^{\kappa t} - \theta + v\right)^3}.
\end{aligned}$$

This is consistent with the result proved by [13] but with a different asymptotic: in [13] it is proved that the forward implied volatility behaves asymptotically as $\mathrm{O}\big(\tau^{-1/4}\big)$ as the maturity approaches zero. This discrepancy is not unexpected since by the results in [20], also in the spot case, the smile expansion is asymptotically convergent only inside the at-the-money region $\{|x-k| \leq \lambda\sqrt{\tau}\}$, for any fixed $\lambda > 0$. Even so, the approximation formulas seem to be generally quite accurate as the following numerical tests indicate.

Next we compare our 3rd order approximation with the forward implied volatilities computed using standard Fourier methods and then inverting numerically the Black-Scholes formula. To this end, we consider the set of parameters calibrated to market data, recently proposed by [10]:

$$\kappa = 1, \quad \theta = 0.08, \quad \delta = 0.39, \quad \rho = -0.93, \quad v = 0.245^2. \tag{4.8}$$

This set is obtained by calibrating the Heston model to daily observations of implied volatility surfaces for the S&P500-index over the period 2005-2009, synchronized with the index itself and with estimated complete term structures of interest rates and dividend yields. We consider the forward start dates $t = 0, 1M, 3M, 6M$ and $1Y$ and maturities from one week ($1W$) to ten years ($10Y$). The set of strikes is given with respect to the set of maturities in Table 3.

Globally the results in Table 4 are very good and show that the approximation is accurate also for long maturities. In this set of tests we included a short forward maturity ($1W$) that is particularly significant

$\tau \setminus K$	1%	5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%
1W	0.95	0.958	0.967	0.975	0.983	0.992	1.00	1.007	1.015	1.022	1.029	1.036	1.044
3M	0.85	0.875	0.90	0.925	0.95	0.975	1.00	1.017	1.033	1.05	1.067	1.083	1.10
1Y	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.033	1.067	1.10	1.133	1.167	1.20
10Y	0.20	0.333	0.467	0.60	0.733	0.867	1.00	1.30	1.60	1.90	2.20	2.50	2.80

Table 3: Set of maturities and strikes for the tests in the Heston model

because of the exploding behaviour of the forward implied volatility: despite of this, the tests show that the approximation is accurate in this case as well.

To conclude, we would like to emphasize again that the strength of the approach is that we can incorporate in a generic stochastic volatility model, also variable coefficients (local volatility) and multi-dimensional risk factors without any further complication: indeed we still get approximation formulas that are fully explicit and involve only polynomials and elementary functions.

5 Declaration of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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τ	$t \setminus K$		1%	5%	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%
1 W	0	Fourier	26.25	25.96	25.67	25.38	25.08	24.78	24.48	24.21	23.94	23.66	23.38	23.09	22.80
		approx	26.20	25.95	25.70	25.42	25.12	24.81	24.48	24.19	23.90	23.62	23.35	23.08	22.82
	1M	Fourier	26.97	26.45	25.95	25.46	25.00	24.58	24.21	23.96	23.78	23.67	23.63	23.64	23.71
		approx	26.88	26.40	25.92	25.44	25.00	24.59	24.23	23.97	23.76	23.61	23.54	23.55	23.63
	3M	Fourier	27.95	27.19	26.44	25.71	25.01	24.38	23.90	23.69	23.71	23.89	24.17	24.51	24.88
		approx	27.92	27.08	26.27	25.53	24.88	24.35	23.96	23.74	23.65	23.69	23.88	24.24	24.77
	6M	Fourier	28.82	27.91	26.99	26.08	25.21	24.43	23.86	23.70	23.89	24.27	24.75	25.29	25.85
		approx	28.86	27.78	26.74	25.79	24.98	24.32	23.86	23.65	23.62	23.81	24.22	24.89	25.82
	1Y	Fourier	29.73	28.71	27.69	26.68	25.71	24.84	24.22	24.10	24.37	24.87	25.47	26.13	26.81
		approx	29.88	28.59	27.37	26.27	25.32	24.57	24.06	23.85	23.87	24.16	24.74	25.62	26.83
3M	0	Fourier	28.97	28.22	27.46	26.69	25.90	25.09	24.25	23.68	23.09	22.49	21.87	21.22	20.56
		approx	29.10	28.29	27.50	26.71	25.91	25.10	24.26	23.69	23.11	22.51	21.90	21.27	20.63
	1M	Fourier	29.24	28.40	27.55	26.70	25.83	24.96	24.08	23.50	22.93	22.37	21.84	21.35	20.91
		approx	29.12	28.38	27.57	26.73	25.87	24.99	24.12	23.54	22.97	22.41	21.86	21.32	20.80
	3M	Fourier	29.68	28.73	27.77	26.80	25.84	24.88	23.94	23.35	22.81	22.35	22.00	21.74	21.57
		approx	29.27	28.58	27.75	26.83	25.87	24.90	23.97	23.39	22.83	22.32	21.86	21.44	21.09
	6M	Fourier	30.17	29.14	28.09	27.05	26.01	24.99	24.02	23.43	22.92	22.54	22.31	22.22	22.21
		approx	29.57	28.89	28.02	27.02	25.97	24.94	23.97	23.37	22.84	22.37	21.97	21.66	21.45
	1Y	Fourier	30.80	29.71	28.61	27.52	26.44	25.39	24.40	23.82	23.33	23.01	22.85	22.85	22.94
		approx	30.13	29.42	28.48	27.41	26.29	25.20	24.20	23.61	23.09	22.66	22.32	22.10	21.99
1Y	0	Fourier	31.37	30.11	28.87	27.63	26.39	25.14	23.88	23.02	22.15	21.27	20.38	19.48	18.57
		approx	31.80	30.36	29.01	27.72	26.46	25.19	23.91	23.05	22.17	21.27	20.36	19.43	18.49
	1M	Fourier	31.51	30.22	28.95	27.69	26.42	25.16	23.89	23.04	22.19	21.34	20.50	19.69	18.90
		approx	31.66	30.35	29.05	27.77	26.49	25.20	23.91	23.05	22.18	21.31	20.43	19.55	18.66
	3M	Fourier	31.76	30.43	29.12	27.82	26.53	25.25	23.97	23.13	22.30	21.50	20.73	20.02	19.40
		approx	31.53	30.38	29.15	27.87	26.57	25.26	23.96	23.09	22.24	21.40	20.57	19.75	18.95
	6M	Fourier	32.07	30.70	29.36	28.03	26.72	25.43	24.15	23.32	22.51	21.74	21.02	20.40	19.88
		approx	31.52	30.51	29.33	28.05	26.72	25.39	24.07	23.21	22.37	21.56	20.78	20.02	19.30
	1Y	Fourier	32.49	31.10	29.74	28.39	27.07	25.77	24.49	23.67	22.87	22.13	21.46	20.89	20.45
		approx	31.70	30.80	29.65	28.36	27.02	25.66	24.33	23.48	22.66	21.87	21.13	20.43	19.78
10Y	0	Fourier	32.63	30.42	28.89	27.70	26.72	25.87	25.13	23.73	22.57	21.58	20.71	19.94	19.23
		approx	32.48	30.30	28.80	27.62	26.66	25.82	25.09	23.70	22.55	21.56	20.69	19.91	19.20
	1M	Fourier	32.65	30.44	28.91	27.72	26.74	25.90	25.16	23.75	22.60	21.61	20.75	19.98	19.28
		approx	32.50	30.32	28.82	27.65	26.68	25.85	25.12	23.73	22.58	21.59	20.72	19.95	19.24
	3M	Fourier	32.69	30.48	28.95	27.76	26.78	25.93	25.20	23.80	22.64	21.66	20.80	20.04	19.36
		approx	32.52	30.36	28.86	27.69	26.72	25.89	25.16	23.77	22.62	21.64	20.78	20.01	19.31
	6M	Fourier	32.74	30.53	29.00	27.81	26.83	25.98	25.25	23.85	22.70	21.72	20.87	20.12	19.45
		approx	32.56	30.41	28.91	27.73	26.77	25.94	25.21	23.82	22.68	21.70	20.84	20.08	19.38
	1Y	Fourier	32.81	30.60	29.07	27.88	26.89	26.05	25.32	23.92	22.78	21.81	20.97	20.22	19.56
		approx	32.61	30.47	28.97	27.80	26.84	26.01	25.28	23.90	22.76	21.79	20.94	20.18	19.49

Table 4: Forward implied volatility (Fourier and 3rd order approximation) in the Heston model with parameters as in (4.8), with forward start date t, forward maturity τ and the range of strikes in Table 3

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