Extension of Normed Call Prices for Negative Strikes and Forwards

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Abstract

Financial crisis of recent times have opened up strange financial possibilities and it is not uncommon these days for popular interest rates derivatives such as swaptions to trade at negative strikes. In this note, irrespective of the asset classes, we provide an extension of Normed Call Prices (NCP) in Gope and Fries (2011) to negative strikes and forwards.

Introduction

Normed call prices (NCP) introduced in Gope and Fries (2011) are an useful tool for construction for arbitrage-free calibration of global volatility surfaces by way of smoothing techniques as in Fengler (2009). However, the NCP definition there is specified under log-normal like diffusion of the underlying which additionally requires that the forward price of the underlying be always positive. Besides, one may also be interested in negative strikes irrespective of the sign of the forward, which particularly makes sense for interest rate derivatives such as swaptions. In this note, we extend the NCP definition and properties to include non-positive forwards and negative strikes. Only Ito type diffusion of the underlying is assumed. However, we focus on the implied distributions with finite left tail.

Notation and Preliminaries

- $S(t)$ is the underlying process at time t, which follows Ito-type diffusion process.
- $F(t,T)$ the forward price of the underlying at time t for delivery at T, $t \leq T$.
- $D_{dom}(t,T)$, the domestic discount factor, is the time-t price of a zero-bond that pays 1 unit of domestic currency at T.
- $D_{for}(t,T)$ is the foreign time-t discount factor. $D_{dom}(t,T)$ and $D_{for}(t,T)$ depend on domestic and foreign rates, respectively. In particular,

$$F(t,T) = S(t)D_{for}(t,T)/D_{dom}(t,T) \tag{1}$$

- Q^T is the the T-terminal measure, with the numeraire process $N(t,T) = D_{dom}(t,T)$
- $N(.)$ is the normal cdf.

Forward Factor

Define $ForFac(T_1,T_2) > 0$, for maturities T_1 and $T_2, T_2 \geq T_1$, such that $F(0,T_2) = F(0,T_1)ForFac(T_1,T_2)$.

Implied Distributions with Finite Left Tail

As stated earlier, log-normal like distributions of the underlying $S(t)$ is not assumed, so that $S(t)$ can be zero or negative for any t including time-zero. When the time-zero spot is zero or negative, one can have zero or negative forwards, which is not a very desirable situation. This problem is circumvented by considering shifts in the strike dimension. For each maturity T, consider forward shift factors $x(T)$ and strike shift factors $s(T)$ such that the following recursive relationship holds:

$$\begin{aligned} x(0) &< s(0) = S(0) \\ x(T) &= x(t)ForFac(t,T) \text{ for } T > t \\ s(T) &= \sup_z \{z, S(T) \geq z \text{ a.s.}\} \end{aligned} \tag{2}$$

A particular choice would be: $s(0) - x(0) = 1$. In any case, (2) may be used to calibrate *a priori* the forward shift factors $x(T)$ and strike shift factors $s(T)$, and ensure the following:

$$\begin{aligned} S(T) - s(T) &\geq 0 \qquad \text{a.s.} \\ F(0,T) - x(T) &> 0 \end{aligned} \tag{3}$$

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Shifted Forward

Define T-*shifted forward* for maturity T

$$ShiftFor(0,T) = F(0,T) - x(T) \quad (4)$$

Clearly by (2), $ShiftFor(0,T) > 0$ for any T.

Shifted Forward Moneyness

Shifted forward moneyness, $\kappa(K,T)$, for strike K and maturity T, is defined as follows:

$$\kappa(K,T) = \frac{K - x(T)}{ShiftFor(0,T)} \quad (5)$$

Further define

$$\kappa_{\min}(T) = \frac{s(T) - x(T)}{ShiftFor(0,T)} \quad (6)$$

As usual one may use κ in place of $\kappa(K,T)$, whenever the meaning is clear from the context.

Normed Call Prices

Normed Call Price (NCP), $\widetilde{C}(\kappa,T)$, for moneyness κ and maturity T, can now be extended from the definition given in Gope and Fries (2011) as follows:

$$\widetilde{C}(\kappa,T) = \frac{C(\kappa ShiftFor(0,T) + x(T), T)}{ShiftFor(0,T) D_{dom}(0,T)} \quad (7)$$

$C(\kappa ShiftFor(0,T) + x(T), T)$ above denotes the price of a vanilla European call with strike $K = \kappa ShiftFor(0,T) + x(T)$ and maturity T.

Monotonicity and Convexity Constraints

Following no-arbitrage considerations,

$$\begin{aligned}
\widetilde{C}(\kappa,T) &= \mathrm{E}^{\mathrm{Q}^T}\left[\left(\frac{S(T) - \kappa ShiftFor(0,T) - x(T)}{ShiftFor(0,T)}\right)^+\right] \\
&= \mathrm{E}^{\mathrm{Q}^T}\left[\left(\frac{S(T) - x(T)}{ShiftFor(0,T)} - \kappa\right)^+\right] \\
&= \mathrm{E}^{\mathrm{Q}^T}\left[(Y(T) - \kappa)^+\right],\ Y(t) = \frac{S(t) - x(t)}{ShiftFor(0,T)} \\
&= \int_\kappa^\infty (y - \kappa) q^T(T,y) dy \quad (8)
\end{aligned}$$

where $q^T(T,y)$ is the transition probability density of the random variable $Y(T)$ under Q^T. Differentiating (8), one obtains

$$\begin{aligned}
\frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} &= -\int_\kappa^\infty q^T(T,y) dy \\
&= -\mathrm{Q}^T(Y(T) \geq \kappa) \\
&= -\mathrm{Q}^T(S(T) \geq \kappa ShiftFor(0,T) + x(T)) \\
&= -\mathrm{Q}^T(S(T) \geq K) \quad (9)
\end{aligned}$$

Thus the negative of the first derivative of the normed call price function with respect to moneyness is the Q^T-proability of the option ending up in -the-money at maturity, with no scaling due to the rates being involved and no underlying assumption of deterministic rates. Note that when rates are deterministic, the T-terminal measure is identical to the risk neutral measure $\mathbb{Q}$, and then the negative of the first derivative of normed call price function is the terminal in-the-money risk neutral probability, which is the traditional interpretation (with some scaling due to the rates) of the first derivative of the call price function with respect to strike. From (9) it follows that

$$-1 \leq \frac{\partial \widetilde{C}(\kappa,T)}{\partial \kappa} \leq 0 \quad (10)$$

Now further differentiate (9) to obtain

$$\frac{\partial^2 \widetilde{C}(\kappa,T)}{\partial \kappa^2} = q^T(T,\kappa) \geq 0. \quad (11)$$

Note that strict convexity is not assumed here.

Properties of Normed Call Price Function due to monotonicity and convexity constraints

Note that the monotonicity and convexity constraints imply the following properties of the normed call price function $\widetilde{C}(\kappa,T)$, which are much simpler (though equivalent) compared to similar properties that apply to the (unnormed) call price function $C(K,T)$.

Proposition 0.1 *Let* $\kappa_{\min} = \kappa_{\min}(T)$. $\widetilde{C}(\kappa_{\min},T) = 1 - \kappa_{\min}$, $\widetilde{C}(\kappa,T) = 1 - \kappa$ *for* $\kappa < \kappa_{\min}$, *and* $\widetilde{C}(\kappa,T) \to 0$ *as* $\kappa \to \infty$.

Proof

$$\widetilde{C}(\kappa_{\min},T) = \mathrm{E}^{Q^T}\left[(Y(T)-\kappa_{\min})^+\right] = \mathrm{E}^{Q^T}\left[\left(\frac{S(T)-s(T)}{ShiftFor(0,T)}\right)^+\right]$$
$$= \mathrm{E}^{Q^T}\left[\frac{S(T)-s(T)}{ShiftFor(0,T)}\right] = \frac{F(0,T)-s(T)}{ShiftFor(0,T)}$$
$$= 1 - \frac{s(T)-x(T)}{F(0,T)-x(T)} = 1-\kappa_{\min} \qquad (12)$$

as $S(T) \geq s(T)$ a.s. and $ShiftFor(0,T) = F(0,T) - x(T) > 0$. For $\kappa < \kappa_{\min}$, $\widetilde{C}(\kappa,T) = \mathrm{E}^{Q^T}\left[(Y(T)-\kappa_{\min}+(\kappa_{\min}-\kappa))^+\right] = \mathrm{E}^{Q^T}\left[(Y(T)-\kappa_{\min}+(\kappa_{\min}-\kappa))\right] = 1-\kappa$, as $Y(T) \geq \kappa_{\min}$ a.s. and $\mathrm{E}^{Q^T}\left[(Y(T)\right] = 1$. That as $\kappa \to \infty$, $\widetilde{C}(\kappa,T) \to 0$ follows similarly, and also immediately from arbitrage point of view, as with $\kappa \to \infty$, the option becomes worthless.

Proposition 0.2 $\max(1-\kappa,0) \leq \widetilde{C}(\kappa,T)$

Proof By Jensen's inequality, $\widetilde{C}(\kappa,T) \geq (\mathrm{E}^{Q^T}\left[Y(T)-\kappa\right])^+ = \max(1-\kappa,0)$, as the max function is convex.

It is important to note that $\widetilde{C}(\kappa,T) \geq \max(1-\kappa,0)$ also follows directly: Let $g^{(T)}(\kappa) = \widetilde{C}(\kappa,T) - (1-\kappa)$. It follows that $\frac{d}{d\kappa}g^{(T)}(\kappa) = \frac{\partial\widetilde{C}(\kappa,T)}{\partial\kappa} + 1 \geq 0$, due to the monotonicity constraint. So, $g^{(T)}(\kappa) \geq g^{(T)}(\kappa_{\min}) = 0$ for $\kappa \geq \kappa_{\min}$, due to Proposition 0.1, thus proving the assertion.

Excursion: Behavior of the NCP Function with no Forward and Strike Adjustment

A simplified definition (7) by taking away the shift adjustments would be as follows, assuming $F(0,T) \neq 0$:

$$\hat{C}(\kappa,T) = \frac{C(\kappa F(0,T),T)}{F(0,T)D_{dom}(0,T)} \qquad (13)$$

No assumption is made for log normal dynamics of the underlying $S(t)$, so $S(t)$ can be negative. By imposing the condition $F(0,T) \neq 0$, it is just assumed that $S(0) \neq 0$.

Positive Forward Let $\kappa_{\min}$ be an attachment point for the implied distribution of the underlying for maturity T. What this means is that $S(T) \geq F(0,T)\kappa_{\min}$ a.s. Note that $\kappa_{\min}$ can be negative. It follows that $\hat{C}(\kappa,T) = 1-\kappa$ for $\kappa \leq \kappa_{\min}$, and $\hat{C}(\kappa,T) \to 0$ as $\kappa \to \infty$. Further, $\frac{\partial\widetilde{C}(\kappa,T)}{\partial\kappa} = -\mathrm{Q}^T(S(T) \geq K = F(0,T)\kappa)$. $\frac{\partial^2\hat{C}(\kappa,T)}{\partial\kappa^2} \geq 0$, as the second derivative is a probability density. The entire dynamics is captured in Figure 1.

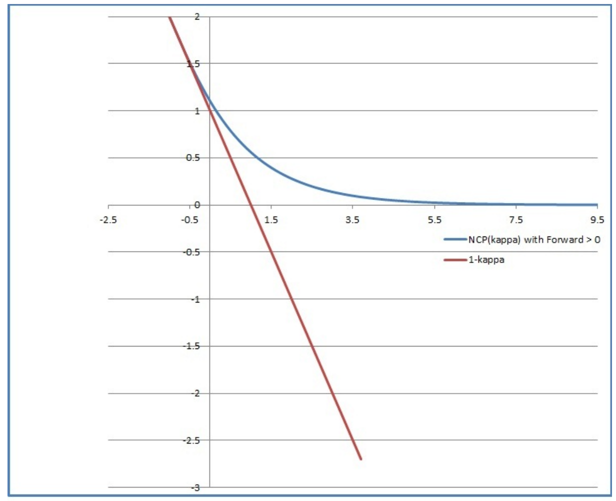

Figure 1: $\hat{C}(\kappa,T)$ as a function of κ, where $\kappa_{\min} = -0.9$ and $F(0,T) > 0$

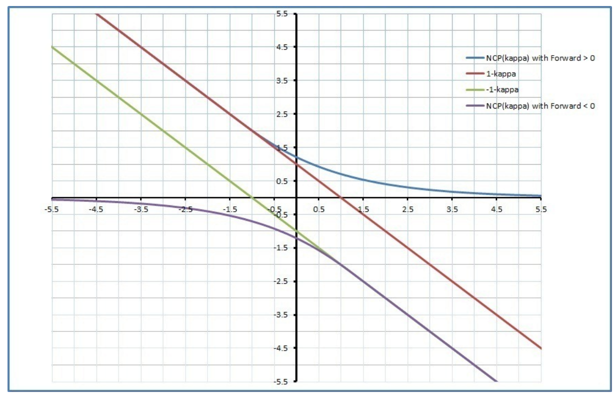

Figure 2: $\hat{C}(\kappa,T)$ as a function of κ, where $\kappa_{\max} = 1.4$ and $F(0,T) < 0$

Negative Forward Let $\kappa_{\max}$ be an attachment point for the implied distribution of the underlying for maturity T. What this means is that $S(T) \geq F(0,T)\kappa_{\max}$ a.s. Note that here $F(0,T) < 0$ and $\kappa_{\max}$ would be usually positive. Obviously here $\hat{C}(\kappa,T) \leq 0$, as $F(0,T) < 0$. It follows that $\hat{C}(\kappa,T) = 1-\kappa$ for $\kappa \geq \kappa_{\max}$, and $\hat{C}(\kappa,T) \to 0$ as $\kappa \to -\infty$. Further, $\frac{\partial\widetilde{C}(\kappa,T)}{\partial\kappa} = -\mathrm{Q}^T(S(T) \geq K = F(0,T)\kappa)$. However, here $\frac{\partial^2\hat{C}(\kappa,T)}{\partial\kappa^2} \leq 0$. This dynamics is captured in Figure 2.

In any case, however, the simplification described here is not very helpful. For one thing, the normed call prices can get negative for negative forwards, and for the second, the simplification fails to apply when the forward is traded at zero.

Calendar Arbitrage

It is not enough for an arbitrage-free global normed call price surface to be constrained only in terms of moneyness gradients. Since one considers a continuum of maturities, there are implications for the maturity gradient of the normed call price function.

These implications are actually an immediate consequence of Kolmogorov's Forward Equation and Feynman-Kac Theorem (See Feller (1949) and Kac (1949) which leads to constraints on the T-derivative of the transition probability density function. Fengler (2009) has coined the term *calendar arbitrage* for implication of this phenomenon on call prices. Under the assumptions

$$\begin{aligned} D_{for}(T_1, T_2) &= \frac{D_{for}(0, T_2)}{D_{for}(0, T_1)} \\ D_{dom}(T_1, T_2) &= \frac{D_{dom}(0, T_2)}{D_{dom}(0, T_1)} \end{aligned} \tag{14}$$

the following constraint has been proven in Gope and Fries (2011):

$$\frac{\partial \widetilde{C}(\kappa, T)}{\partial T} \geq 0 \tag{15}$$

The assumptions (14) are somewhat weaker than (though not substantially different from) assuming that the rates are deterministic: They mean that the rates have zero volatility on $[T_1, T_2]$, and indeed this is a reasonable approximation to make when the maturities T_1 and T_2 are not far apart. In the fully general case, when rates can be stochastic, the constraint (15) may not hold good, and the sign of $\frac{\partial \widetilde{C}(\kappa,T)}{\partial T}$ may depend on the correlation between the processes for underlying $S(t)$, the domestic interest rate process $r_{dom}(t)$ and the foreign interest rate process $r_{for}(t)$.

We do not address the problem that arises out of this correlations here, but instead prove (15) for the extension of normed call prices due to strike shifts. First note that

$$\begin{aligned} \widetilde{C}(\kappa, T_1) &= \mathrm{E}^{\mathrm{Q}^{T_1}} \left[\left(\frac{S(T_1) - x(T_1)}{ShiftFor(0, T_1)} - \kappa \right)^+ \right] \\ &= \frac{D_{dom}(0, T_2)}{D_{dom}(0, T_1)} \mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_1) - x(T_1)}{ShiftFor(0, T_1)} - \kappa \right)^+ \frac{D_{dom}(T_1, T_1)}{D_{dom}(T_1, T_2)} \right] \\ &= \mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_1) - x(T_1)}{ShiftFor(0, T_1)} - \kappa \right)^+ \right] \end{aligned}$$

Now by Jensen's inequality

$$\begin{aligned} &\mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_2) - x(T_2)}{ShiftFor(0, T_2)} - \kappa \right)^+ \Big| \mathcal{F}(T_1) \right] \\ &\geq \left(\mathrm{E}^{\mathrm{Q}^{T_2}} \left[\frac{S(T_2) - x(T_2)}{ShiftFor(0, T_2)} \Big| \mathcal{F}(T_1) \right] - \kappa \right)^+ \\ &= \left(\frac{F(T_1, T_2) - x(T_2)}{F(0, T_2) - x(T_2)} - \kappa \right)^+ \\ &= \left(\frac{S(T_1) - x(T_1)}{F(0, T_1) - x(T_1)} - \kappa \right)^+ \end{aligned}$$

as $F(T_1, T_2) = S(T_1) \frac{D_{for}(T_1, T_2)}{D_{dom}(T_1, T_2)} = S(T_1) ForFac(T_1, T_2)$ due to (14), and $x(T_2) = x(T_1) ForFac(T_2, T_2)$ due to (2). It follows that

$$\begin{aligned} \widetilde{C}(\kappa, T_2) &= \mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_2) - x(T_2)}{ShiftFor(0, T_2)} - \kappa \right)^+ \right] \\ &= \mathrm{E}^{\mathrm{Q}^{T_2}} \left[\mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_2) - x(T_2)}{ShiftFor(0, T_2)} - \kappa \right)^+ \Big| \mathcal{F}(T_1) \right] \right] \\ &\geq \mathrm{E}^{\mathrm{Q}^{T_2}} \left[\left(\frac{S(T_1) - x(T_1)}{F(0, T_1) - x(T_1)} - \kappa \right)^+ \right] = \widetilde{C}(\kappa, T_1) \end{aligned}$$

Concluding Remarks

It is clear that the normed call price function remains well-behaved even under the extension introduced here. It is to be noted that the particular utility of the NCPs is the no-arbitrage properties which are simple, direct, and nicely behaved in terms of bounds and asymptotic behavior. Extension for general Ito-type diffusion only changes the behavior for negative strikes: instead of being bounded by 1 from above, for strikes below the attachment point for the implied distribution, the upper bound is actually the forward contract. Independence on rates remains unchanged.

References

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