Collateralization and Funding Valuation Adjustments (FVA) for Total Return Swaps

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Abstract

In this paper we consider the valuation of total return swaps (TRS). Since a total return swap is a collateralized derivative referencing the value process of an uncollateralized asset it is in general not possible that both counter parties agree on a unique value. Consequently it is not possible to have cash collateralization of the total return swap matching each counterparts valuation. The total return swap is a collateralized derivative with a natural funding valuation adjustment.

We develop a model for valuation and risk management of TRS where we assume that collateral is posted according to the mid average (or convex combination) of the valuations performed by both counterparts. This results in a coupled and recursive system of equations for the valuation of the TRS.

The main result of the paper is that we can provide explicit formulas for the collateral and the FVA, eliminating the recursiveness which is naturally encountered in such formulas, by assuming a natural *collateralization scheme*.

Although the paper focuses on total return swaps, the principles developed here are generally applicable in situations where collateralized assets reference uncollaterlized or partially collateralized underlings.

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1 Introduction

1.1 Total Return Swaps

A total return swap (TRS) exchanges the cash flows or total return of an uncollateralized underlying asset M against plain vanilla floating rate cash flows (LIBOR plus deal spread) with notional $\mathcal{N}$. At maturity T, typically five to ten years, the final underlying market value $\mathcal{M}(T)$ is exchanged against the TRS notional $\mathcal{N}$, see the top two legs in Figure 1.

Total return swap are usually subject to an ISDA master agreement with credit support annex (CSA) and daily cash margining, i.e. a collateral rate (typically EONIA) is paid on the cash collateral.

A total return swap is often "associated" with a set of other transactions, which - in the common setup - are as follows:

- Counterparty A is providing a loan with notational $\mathcal{N}$ and maturity T to counterparty B. Note that such a loan is typically associated with floating interest rate payments which are given by coupon of LIBOR plus an additional spread on the notional.
- Counterparty B is providing an asset $\mathcal{M}$ to counterparty A, whose initial value corresponds to that of the loan.

The asset $\mathcal{M}$ is provided to mitigate the counterparty risk in the load and in view of this setup total return swaps are used to mitigate *market risk* of the bond collateral in this transaction.

The above setup with a loan and a bond collateral is not part of the total return swap transaction and the TRS will be valued independently of it. On the other hand, though not specified in the TRS contract, in practice the TRS payer replicates the cash flows by purchasing and funding the underlying asset at trade inception time. At trade inception time typically the deal spread will be higher than the TRS payer's funding spread and lower than the TRS receiver funding spread, so that both counterparties calculate a positive present value. The two counter parties share a mutual funding benefit. Often the payer purchases the bond from the receiver: thus a TRS can be considered as a long dated repo and the TRS deal spread is a long dated repo rate.

Since the total return swap is a collateralized derivative referencing a funding transaction, it constitutes a collateralized derivative where an FVA naturally enters in to the valuation.

1.2 Lack of Perfect Collateralization

The role of collateralization in a TRS is - at least in some situation - different from a classical collateralized swap in that the collateral may come as an exogenously prescribed value process. In this situation there is no equilibrium between collateralization and valuation - as in perfect cash collateralization (which leads to so call *OIS discounting*). In this situation we find the TRS is a cash collateralized derivative for which both counterparties still calculate a different present value and full collateralization for a TRS is impossible. This makes collateral management and valuation nontrivial. Partial collateralization means that funding valuation adjustments need to be taken into account. The funding valuation adjustments will be calculated differently by both counterparties.

1.3 Collateralization Schemes

In contrast to classical swaps (referencing cash flows and not value processes), the type of collateralization is a degree of freedom for the TRS. Therefore we will discuss different collateralization schemes. One such scheme considered is that collateral is posted according to the mid average (or convex combination) of the valuations performed by both counterparts. We will show that this collateralization scheme can be considered as a fair partial collateralization scheme, where funding benefits are shared equally among both counterparts. Since the valuation is itself a function of the collateralization, we arrive at a system of recursive equations, for which we derive simple analytic valuation formulas.

While this type of collateralization is "fair" since mutual funding benefits are shared, it is required that both counter parties have knowledge of the other counterparts funding costs / FVA. To circumvent this requirement and avoid collateral management conflicts, in some TRS contracts a *repo style collateral agreement* is stated. According to these contracts the collateral is calculated as the difference of the TRS notional amount (plus possibly last period accrued LIBOR plus deal spread) and the underlying asset market value. Although this simple collateral calculation ignores the economic value of the deal, because future deal spread margin and funding costs are neglected, in many market situations it yields quite similar results and can sometimes even be preferable to the total return payer than a fair economic value collateral scheme.

Acknowledgment

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2 Basic Model Setup

2.1 Defintion of the Total Return Swap

A total return swap exchanges the total return of an underlying asset $\mathcal{M}$ with vanilla LIBOR plus spread payments $L_j + \mathrm{spr}$ on the TRS notional $\mathcal{N}$. At maturity T the final value of the underlying $\mathcal{M}(T)$ is exchanged with the TRS notional $\mathcal{N}$, which is usually equal to the markt value $\mathcal{M}(t_0)$ of the underlying asset at trade inception t_0.

The total return of the underlying asset means that the TRS exchanges the cash flows of this asset against vanilla flows. These cash flows are depicted in Figure 1 (top two legs)[1]. In other words the market value of the underlying is exchanged against vanilla flows.

Additional cash flows are exchanged due to the cash collateral agreement. Thus the TRS exchanges future changes in market value of the underlying asset in cash.

2.2 Valuation of the Total Return Swap

We perform the valuation of the total return swap in two steps. We will first value the TRS as an uncollateralized derivative. We will then value the collateral cash flows. All valuations are performed as (so called risk neutral) replication cost.

Since we first consider the TRS as an uncollateralized derivative, we value it with respects to each counterparts "funding numeraire" (i.e., using funded replication, [4]).

It might be tempting to directly consider an OIS discounted cash flow valuation, since the TRS is a OIS-collateralized instrument. However, this approach does not work out cleanly. Valuing the TRS as an uncollateralized derivative has the striking advantage that we immediately obtain the value of the total returns of the underlying asset $\mathcal{M}$ in terms of its value $\mathcal{M}(t)$ since that asset is uncollateralized too.[2]

[1] Although it is not part of the TRS contract, in the bottom to legs of Figure 1 the bond purchase and funding transactions are shown. For discussion of funding strategies and valuation of the total package (TRS plus funding of purchased bond) see section **??**.

[2] For the time being, we make the assumption that any intrinsic funding capabilities (like being itself repo-able) are included in the valuation of the underlying asset $\mathcal{M}$ and that the two counter parties agree on a unique market price $\mathcal{M}(t)$.

2.2.1 Valuation of the Plain LIBOR Flows

The funded replication cost for the plain vanilla leg is

$$\begin{aligned} \mathrm{Leg}^{r}_{\mathrm{Plain}}(t) \ := \ & \left(\sum_{i=i(t)}^{n} E_t \left[e^{-\int_t^{t_{i+1}} r(s)ds} \tau_i \left(L_i(t_i) + \mathrm{spr} \right) \right] \mathcal{M}(t_0) \right) \\ & + E_t \left[e^{-\int_t^{T} r(s)ds} \right] \mathcal{M}(t_0) \end{aligned}$$

where r denotes the (unsecured) funding curve[3], t_i and t_{i+1} denote LIBOR fixing and payment dates, spr is the deal spread,

$$i(t) := \min\{ j \mid t_{j+1} \geq t \}$$

denotes the index of the first LIBOR period which will have payment time after valuation time t.

2.2.2 Valuation of the Total Return of the Underlying

The replication costs for the underlying asset in practice is simply the market value or price $\mathcal{M}(t)$ of the underlying asset at valuation time t.

Hence the value of a uncollateralized total return swap V_u is simply

$$V_u(t) = \mathrm{Leg}^{r}_{\mathrm{Plain}}(t) - \mathcal{M}(t).$$

We note that V_u does not depend on the future dynamics of the underlying price process $\mathcal{M}(s)$, $s \geq t$. In contrast the cash collateralized value will depend on the future dynamics $\mathcal{M}(s)$, $s \geq t$.

2.2.3 Valuation of Collateralization

Next, we consider the cash flows generated by the cash collateral agreement, i.e. funding costs or benefit created by the collateral for both counterparties. Let $C(s)$ denote the amount of cash collateral at time $s \geq t$.

The valuation of the funding benefits or costs depends on the individual funding accounts (funding rate processes) of the counterparties. We introduce the corresponding notation: We denote the two counterparties by A and B and assume that A is the total return payer, i.e. A pays all cashflows generated by the underlying asset (coupons, principal repayments, final market value $\mathcal{M}(T)$ or recovery at default) to B. In return A receives from B vanilla payments, LIBOR plus deal spread, and at maturity T the value $\mathcal{N}$.

[3] For simplicity we assume that borrowing and investing is done with respect to the same curve r.

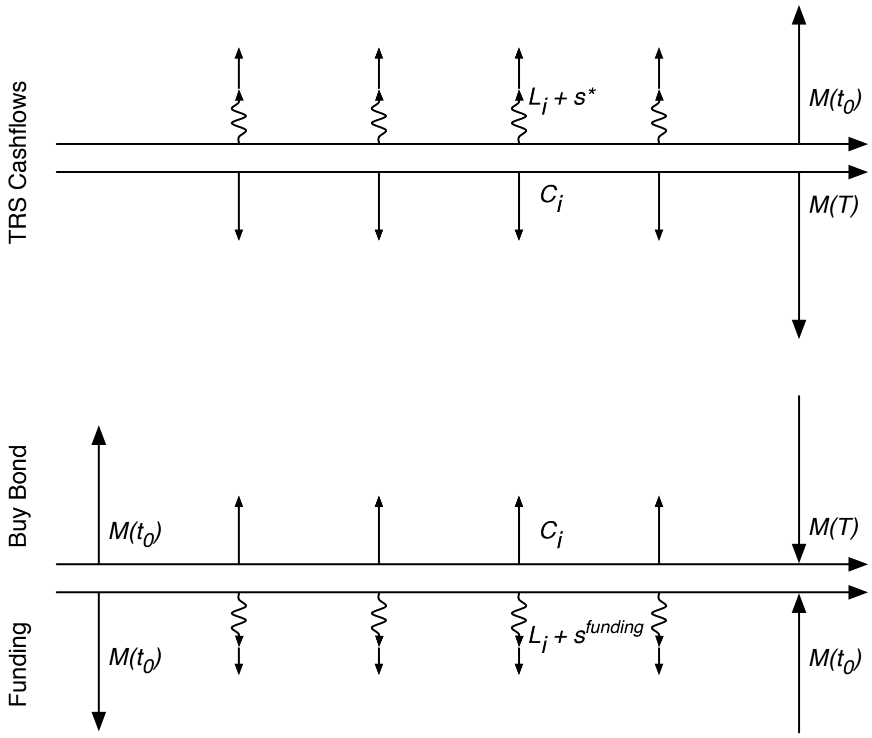

Figure 1: Cash flows of a total return swap (top two legs) together with the associated replication (bottom two leg) where the buying of the underlying asset is financed via a funding transaction. The funding spread depicted is smaller than the deal spread of the total return swap. The picture represents a more classical view on a total return swap, neglecting an important part of the product: The total return swap is collateralized and cash-flows generated from the collateral contract do represent an important aspect of the product.

Through this paper we will write down all valuations from the perspective of A. That is to say, a cash flow payed from A to B will be valued negative (even if we calculate its value for B).

Let r_{A} and r_{B} denote the funding rate processes for A and B, respectively.[4] For A the cash collateral $C(s) \geq 0$ ($C(s) < 0$) at time $s \geq t$ is invested (funded) at rate $r_{\mathsf{A}} ds$ and the collateral rate $r_{\mathsf{c}} ds$ is paid (earned).

Thus the value of the future cash collateral flows for A is:

$$E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A}}(x)dx} \left(r_{\mathsf{A}}(s) - r_c(s)\right) C(s) ds \mid \mathcal{F}_t\right].$$

At this point one clearly sees that, in contrast to an uncollateralized TRS, the valuation of a cash collateralized TRS will depend on the future dynamics of

[4] To be precise we assume that for counterparty A all funded cash flows are valued w.r.t. a numeraire N^{A}, where $dN^{\mathsf{A}} = r_{\mathsf{A}} N^{\mathsf{A}} \mathrm{d}t$, and respectively for B.

the underlying asset $\mathcal{M}(\cdot)$. As we will consider in more detail in section 4.1 and 4.2 There are different ways to define the collateral process $C(s)$. Either $C(s)$ may be exogeneously given by a simple formula depending directly on the market price $\mathcal{M}(\cdot)$ of the underlying, see section 4.2, or $C(s)$ may depend on a replicated future underlying price computed using a repo or counterparty specific funding rate r_{A} or r_{B}. Using a Risk neutral replication cost approach, this means that the rate of growth for the underlying $\mathcal{M}(s), s \geq t$ is the repo or funding rate for the underlying (minus an underlying coupon rate plus a default intensity rate), see [5] and the references there. For total return swaps, which have typical maturities from five to ten years, in general, there will be no unique repo rate applicable to both counterparties A and B. Often A and B need to apply their (unsecured) funding rate r_{A} and r_{B} to finance the underlying. An interesting example is that for certain underlyings A can perform cheap central bank funding short term and switch to unsecured funding long term as central bank changes its liquidity policy, but B can not do that because central bank does not accept the underlying from B (for example the underlying might be issued by B). For that example A could compute an attractive effective repo / funding rate for the underyling by combining the central bank curve and $\mathsf{A}'s$ unsecured funding curve[5]. One notes that an obvious funding benefit exists here.

Hence in general we need to take into account two dynamic underlying processes $\mathcal{M}_{\mathsf{A}}(\cdot)$ and $\mathcal{M}_{\mathsf{B}}(\cdot)$[6]. Thus we obtain the following system of valuation equations

$$
\begin{aligned}
V_{\mathsf{A}}(t) &= \mathrm{Leg}^{r_{\mathsf{A}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{A}}(t) \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A}}(x)dx}\left(r_{\mathsf{A}}(s) - r_c(s)\right)C(s)ds \mid \mathcal{F}_t\right] \qquad (1)\\
V_{\mathsf{B}}(t) &= \mathrm{Leg}^{r_{\mathsf{B}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{B}}(t) \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{B}}(x)dx}\left(r_{\mathsf{B}}(s) - r_c(s)\right)C(s)ds \mid \mathcal{F}_t\right]. \qquad (2)
\end{aligned}
$$

[5] We assume that haircuts are included in the effective rate of growth of the underlying. For example a 10% haircut means that 90% can be funded using cheap central bank funding where the remaining 10% have to be funded unsecured

[6] There might be other reasons to consider different underlying processe $\mathcal{M}_{\mathsf{A}}(\cdot)$ and $\mathcal{M}_{\mathsf{B}}(\cdot)$. For example the underlying might be a default free fix coupon bond which has maturity equal to the TRS maturity. Then A could replicate the cash flows of the underlying bond by buying back zero bonds (issued by A or an entity which has the same risk as A). The process would be given as $\mathcal{M}_{\mathsf{A}}(s) := e^{-\int_s^{t_i} r_{\mathsf{A}}(x)dx}c_i$, where t_i are coupon (including final notional) payment dates. This process completely ignores the present and future market price $\mathcal{M}(s)$ of the underlying. This replication can be much cheaper than purchasing the underlying, which might be obervalued (for example German government bond recently were sold at negative yield).

Both counterparties discount all occurring future cash flows with their own funding rate and subtract the replicated bond value $\mathcal{M}_{\mathsf{A}}(t)$ and $\mathcal{M}_{\mathsf{B}}(t)$, respectively. Note that if $C(s)$ is a function of $V_{\mathsf{A}}(s)$ and $V_{\mathsf{B}}(s)$, then we have a system of recursive equations for V_{A} and V_{B}, where the coupling of the equations is due to the collateral $C(s)$.[7]

Remark 1 (Relation to perfectly collateralized derivatives): If $\mathcal{M}(t)$ would denote the value of an uncollateralized future cash flow, the value of that cashflow would depend on the funding, hence collateralization itself. This is the classical situation and in this case it is possible to find an equilibrium collateral such that both counter parties agree on the same value. This is not possible here, since $\mathcal{M}(t)$ is considered as an exogenously prescribed process.

[7] Note again that all valuation are perform from A's perspective, hence there is no change in sign.

3 Valuation of Derivatives with Exogenous Collateral Process

The valuation of collateralized and partially collateralized derivatives is well known, see Lemma 12 in the Appendix and references there. Here we give a generalization of this result, which will be used in the valuation of total return swaps in Section 4

Lemma 2: Let X denote a Levy process (jump-diffusion process)[8] with $dX(t) = 0$ for $t > T$. Let $g_r(t)$ denote the valuation of the cash flows $dX(t)$ in t over the time $[t, T]$, i.e.,[9]

$$g_r(t) \ := \ E\left[\int_t^\infty e^{-\int_t^s r(x)dx} dX(t) \mid \mathcal{F}_t\right].$$

Let f denote another Levy process (jump-diffusion process) (i.e., we assume $dt\, df = 0$) and let $f^T := f\mathbf{1}_{t<T}$. Then we have:

Suppose C satisfies for all $t \leq T$ the following equation

$$C(t) \ = \ f(t) \ + \ g_r(t) \ + \ E\left[\int_t^T e^{-\int_t^s r(x)dx} \left(r(s) - r_c(s)\right) C(s) ds \mid \mathcal{F}_t\right],$$

where r,r_c are Ito processes and X_i are fixed cash flows in t_i. Then C has the equivalent representations

$$C(t) \ = f(t) \ + \ g_{r_c}(t) \ + \ E\left[\int_t^\infty e^{-\int_t^s r_c(x)dx} \left(r(s) - r_c(s)\right) f^T(s) ds \mid \mathcal{F}_t\right] \tag{3}$$

$$= g_{r_c}(t) \ + \ E\left[\int_t^\infty e^{-\int_t^s r_c(x)dx} \left(r(s) f^T(s) ds - df^T(s)\right) \mid \mathcal{F}_t\right]. \tag{4}$$

Remark 3: The form (3)-(4) is just a notationally more elegant version of

[8] This lemma holds under more general assumptions. In fact, we only need that the stochastic processes are semi-martingales such that an integration by parts formula holds, see, e.g., [7, Theorem 21, page 278] and that $s \mapsto e^{-\int_t^s r(x)dx}$, $s \mapsto e^{-\int_t^s r_c(x)dx}$ are continuous semi-martingales.

[9] The process X is just a more general form of defining cash flows. For example, if X is the piecewise constant function $X(t) = \sum X_i \mathbf{1}_{t>T_i}$, then $\int_t^\infty e^{-\int_t^s r(x)dx} dX(t) = \sum e^{-\int_t^{T_i} r(x)dx} X_i$, i.e., the integral denotes a sum of discounted cash flows.

(using f in place of f^T)

$$
\begin{aligned}
C(t) &= f(t) \ + \ g_{r_c}(t) \ + \ E\left[\int_t^T e^{-\int_t^s r_c(x)dx} \left(r(s) - r_c(s)\right) f(s)ds \mid \mathcal{F}_t\right] \\
&= g_{r_c}(t) \ + \ E\left[e^{-\int_t^T r_c(x)dx} f(T) \mid \mathcal{F}_t\right] \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_c(x)dx} \left(r(s)f(s)ds - df(s)\right) \mid \mathcal{F}_t\right].
\end{aligned}
$$

Proof: Define $\tilde{C}(t) := C(t) - f(t)$. Then we have

$$
\tilde{C}(t) \ = \ E\left[\int_t^T e^{-\int_t^s r(x)dx} \left(r(s) - r_c(s)\right) \left(\tilde{C}(s) + f(s)\right) ds \mid \mathcal{F}_t\right].
$$

Applying Lemma 12 with $V = C = \tilde{C}$ we get

$$
\begin{aligned}
\tilde{C}(t) &= E\left[\int_t^T e^{-\int_t^s r_c(x)dx} \left(r(s) - r_c(s)\right) f(s)ds \mid \mathcal{F}_t\right] \\
&= E\left[\int_t^T e^{-\int_t^s r(x)dx} f(s) \, de^{-\int_t^s (r_c(x) - r(x))dx} \mid \mathcal{F}_t\right]
\end{aligned}
$$

and with integration by parts we find

$$
\begin{aligned}
\tilde{C}(t) &= E\left[e^{-\int_t^T r_c(x)dx} f(T) \mid \mathcal{F}_t\right] - f(t) \\
&\qquad + E\left[\int_t^T e^{-\int_t^s r_c(x)dx} \left(r(s)f(s)ds - df(s)\right) \mid \mathcal{F}_t\right].
\end{aligned}
$$

□|

4 Collateralization Schemes

Given equations (1)-(2) we are now in the situation that the collateral process C is a free parameter.

In some cases the amount of the collateral is explicitly specified as part of the contract, see Section 4.2 for an example. But often the collateral is not explicitly prescribed and C is the result of an (undefined) day-to-day consensus between the two counterparts. In any case, the valuations will come with a residual *funding valuation adjustment*.

In this section we will thus consider different collateralization schemes and discuss their properties.

4.1 Fair partial collateralization

Let us consider the model of future collateral given as a convex combination (for example mid average) of the two counterparty valuations

$$C(s) := pV_{\mathsf{A}}(s) + (1-p)V_{\mathsf{B}}(s). \tag{5}$$

Our following result gives the solution to the model equations (1)-(2) with the collateral model (5) explicitly.

Before we state the result we need to introduce a combined funding rate r_{AB}. Let us define the stochastic discount factor for the combined rate as the weighted average of the stochastic discount factors for r_{A} and r_{B}:

$$e^{-\int_t^T r_{\mathsf{AB}}(s)ds} = p\, e^{-\int_t^T r_{\mathsf{A}}(s)ds} + (1-p)\, e^{-\int_t^T r_{\mathsf{B}}(s)ds},$$

i.e. the rate r_{AB} is a convex combination of the rates r_{A} and r_{B} with time dependent weights:

$$r_{\mathsf{AB}}(t) := \frac{p\, e^{-\int_t^T r_{\mathsf{A}}(s)ds}\, r_{\mathsf{A}}(t) + (1-p)\, e^{-\int_t^T r_{\mathsf{B}}(s)ds}\, r_{\mathsf{B}}(t)}{p\, e^{-\int_t^T r_{\mathsf{A}}(s)ds} + (1-p)\, e^{-\int_t^T r_{\mathsf{B}}(s)ds}}.$$

Moreover let us define the convex combined underlying process

$$\mathcal{M}_{\mathsf{AB}} := p\,\mathcal{M}_{\mathsf{A}} + (1-p)\,\mathcal{M}_{\mathsf{B}}.$$

In the following we will use the notational short form A/B for A or B, respectively.

Proposition 4 (Valuation under Fair Partial Collateralization): The value of a total return swap on $\mathcal{M}$ with with the equilibrium collateral model (5) is given as

$$V_{\mathsf{A/B}}(t) = C(t) + \mathsf{FVA}_{\mathsf{A/B}}, \tag{6}$$

where the collateral process is given by

$$\begin{aligned}C(t) &= \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{AB}}(t) \\ &\quad - E\left[\int_t^T e^{-\int_t^s r_{\mathsf{C}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_{\mathsf{C}}(s)\right)\mathcal{M}_{\mathsf{AB}}(s)ds \mid \mathcal{F}_t\right] \\ &= \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) - E\left[e^{-\int_t^T r_c(x)dx}\mathcal{M}_{\mathsf{AB}}(T) | \, \mathcal{F}_t\right] \\ &\quad - E\left[\int_t^T e^{-\int_t^s r_{\mathsf{C}}(x)dx}\left(r_{\mathsf{AB}}(s)\mathcal{M}_{\mathsf{AB}}(s)ds - \mathrm{d}\mathcal{M}_{\mathsf{AB}}(s)\right) \mid \mathcal{F}_t\right],\end{aligned} \tag{7}$$

and the funding valuation adjustment is

$$\begin{aligned}\mathsf{FVA}_{\mathsf{A/B}}(t) &:= \mathcal{M}_{\mathsf{AB}}(t) - \mathcal{M}_{\mathsf{A/B}}(t) \\ &\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A/B}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_{\mathsf{A/B}}(s)\right)\mathcal{M}_{\mathsf{AB}}(s)ds \mid \mathcal{F}_t\right].\end{aligned} \tag{8}$$

Remark 5 (Formulas at initial valuation time): The formulas above hold for any $t_v \leq t \leq T$, where t_v denotes inital valuation time. At initial valuation time the underlying processes $\mathcal{M}_{\mathsf{AB}}$, $\mathcal{M}_{\mathsf{A}}$ and $\mathcal{M}_{\mathsf{B}}$ are all equal to the underlying market value $\mathcal{M}$. I.e. if $t = t_v$ is initial valuation time, then

$$\mathcal{M}_{\mathsf{AB}}(t) = \mathcal{M}_{\mathsf{A}}(t) = \mathcal{M}_{\mathsf{B}}(t) = \mathcal{M}(t)$$

and the FVA and collateral formulas are

$$\mathsf{FVA}_{\mathsf{A/B}}(t) = E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A/B}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_{\mathsf{A/B}}(s)\right)\mathcal{M}_{\mathsf{AB}}(s)ds \mid \mathcal{F}_t\right],$$

$$\begin{aligned}C(t) &= \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) - \mathcal{M}(t) \\ &\quad - E\left[\int_t^T e^{-\int_t^s r_{\mathsf{C}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_{\mathsf{C}}(s)\right)\mathcal{M}_{\mathsf{AB}}(s)ds \mid \mathcal{F}_t\right].\end{aligned}$$

Remark 6 (Explicity of Collateral and FVA): The remarkable feature for Proposition 4 is that we give the collateral C and the FVAs $\mathrm{FVA}_{\mathsf{A}}$ and $\mathrm{FVA}_{\mathsf{B}}$ explicitly in terms of $\mathcal{M}$. Note that the standard expression of partially collateralized products involve implicit equations for the collateral (as in Lemma 12).

Remark 7 (Mutual Funding Benefit): Before we turn to the proof, note that Proposition 4 immediately shows that for $r_{\mathsf{B}} > r_{\mathsf{AB}} > r_{\mathsf{A}}$ the two counter parties enter into a win-win situation. Due to $r_{\mathsf{AB}} > r_{\mathsf{A}}$ we have that $\mathsf{FVA}_{\mathsf{A}} > 0$[10], hence counterpart A positively values a funding benefit.

[10] We assumed that $\mathcal{M}_{\mathsf{A/B}} > 0$.

Due to $r_{\mathsf{AB}} < r_{\mathsf{B}}$ we have that $\mathsf{FVA}_{\mathsf{B}} < 0$, and since all valuation are performed from A's perspective, the negative value implies that counterpart B positively values a funding benefit too.

In other words: the two counterparts share a mutual funding benefit by agreeing on an average funding rate r_{AB} and this agreement is done via the collateral contract!

The TRS provides a mutual funding benefit as long as $r_{\mathsf{B}} > r_{\mathsf{A}}$, see Figure 2.

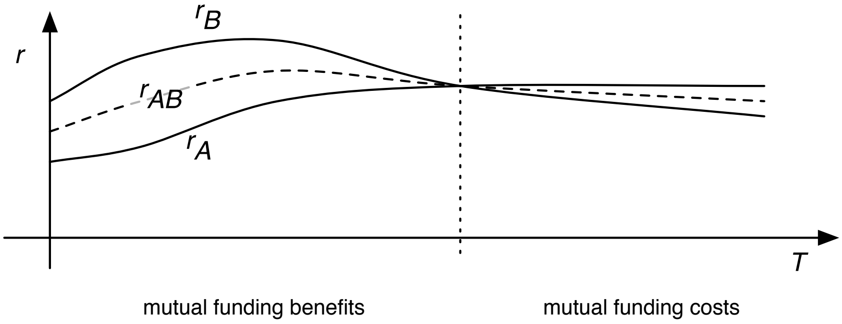

Figure 2: Funding curves of the two counterparts and the average funding curve r_{AB}. The TRS provides a mutual funding benefit as long as $r_{\mathsf{B}} > r_{\mathsf{A}}$

Remark 8 (Equal sharing of funding benefit ($p = \frac{1}{2}$)): By construction we have

$$\begin{aligned} C(t) &= pV_{\mathsf{A}}(t) + (1-p)V_{\mathsf{B}}(t) \\ &= C(t) + p\mathsf{FVA}_{\mathsf{A}} + (1-p)\mathsf{FVA}_{\mathsf{B}}. \end{aligned}$$

Hence

$$p\mathsf{FVA}_{\mathsf{A}} = -(1-p)\mathsf{FVA}_{\mathsf{B}}.$$

Suppose A has cheaper funding than B, i.e. $r_{\mathsf{A}}(s) < r_{\mathsf{B}}(s)$ for $t \leq s \leq T$. Consider $p = 2$, i.e. both counterparties agree that collateral is posted according to the average of the valuations calculated by A and B. Then funding benefits are shared equally among both counterparties, i.e. both counterparties calculate the same funding benefit and FVA adjustment, respectively, to the fair value collateral $C(t)$:

$$\mathsf{FVA}_{\mathsf{A}} = -\mathsf{FVA}_{\mathsf{B}}.$$

Proof: Multiplying (1) with p and (2) with $1-p$ and adding we get

$$\begin{aligned} C(s) &= p\,\mathrm{Leg}^{r_{\mathsf{A}}}_{\mathrm{Plain}}(t) - p\mathcal{M}_{\mathsf{A}}(s) \\ &\quad + pE\left[\int_s^T e^{-\int_s^u r_{\mathsf{A}}(x)dx}\left(r_{\mathsf{A}}(u) - r_c(u)\right)C(u)du \mid \mathcal{F}_t\right] \\ &\quad + (1-p)\mathrm{Leg}^{r_{\mathsf{B}}}_{\mathrm{Plain}}(t) - (1-p)\mathcal{M}_{\mathsf{B}}(s) \\ &\quad + (1-p)E\left[\int_s^T e^{-\int_s^u r_{\mathsf{B}}(x)dx}\left(r_{\mathsf{B}}(u) - r_c(u)\right)C(u)du \mid \mathcal{F}_t\right]. \end{aligned}$$

From the definition of r_{AB} we find

$$p\,\mathrm{Leg}^{r_{\mathsf{A}}}_{\mathrm{Plain}}(t) + (1-p)\,\mathrm{Leg}^{r_{\mathsf{B}}}_{\mathrm{Plain}}(t) \;=\; \mathrm{Leg}^{r_{\mathsf{AB}}}_{\mathrm{Plain}}(t)$$

and hence

$$\begin{aligned} C(t) &= \mathrm{Leg}^{r_{\mathsf{AB}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{AB}}(t) \\ &\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{AB}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_c(s)\right)C(s)ds \mid \mathcal{F}_t\right] \end{aligned}$$

Applying Lemma 2 with $g_r = g_{r_{\mathsf{AB}}} = \mathrm{Leg}^{r_{\mathsf{AB}}}_{\mathrm{Plain}}(t)$ and $f = -\mathcal{M}_{\mathsf{AB}}(t)$ gives

$$\begin{aligned} C(t) &= \mathrm{Leg}^{r_{\mathsf{c}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{AB}}(t) \\ &\quad - E\left[\int_t^T e^{-\int_t^s r_{\mathsf{c}}(x)dx}\left(r_{\mathsf{AB}}(s) - r_{\mathsf{c}}(s)\right)\mathcal{M}_{\mathsf{AB}}(s)ds \mid \mathcal{F}_t\right] \\ &= \mathrm{Leg}^{r_{\mathsf{c}}}_{\mathrm{Plain}}(t) - E\left[e^{-\int_t^T r_c(x)dx}\mathcal{M}_{\mathsf{AB}}(T) \mid \mathcal{F}_t\right] \\ &\quad - E\left[\int_t^T e^{-\int_t^s r_{\mathsf{c}}(x)dx}\left(r_{\mathsf{AB}}(s)\mathcal{M}_{\mathsf{AB}}(s)ds - \mathrm{d}\mathcal{M}_{\mathsf{AB}}(s)\right) \mid \mathcal{F}_t\right], \end{aligned} \tag{9}$$

i.e. (7).

We now define the FVA for counterpart A as $\mathsf{FVA}_{\mathsf{A}}$ and for counterparty B as $\mathsf{FVA}_{\mathsf{B}}$ via

$$\mathsf{FVA}_{\mathsf{A/B}}(t) \;:=\; V_{\mathsf{A/B}}(t) - C(t).$$

From the definition of $V_{\mathsf{A}}(t)$ we have

$$\begin{aligned} \mathsf{FVA}_{\mathsf{A}}(t) \;&=\; \mathrm{Leg}^{r_{\mathsf{A}}}_{\mathrm{Plain}}(t) - \mathcal{M}_{\mathsf{A}}(t) \\ &\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A}}(x)dx}\left(r_{\mathsf{A}}(s) - r_c(s)\right)C(s)ds \mid \mathcal{F}_t\right] \\ &\quad - C(t) \end{aligned} \tag{10}$$

To eliminate the integral term which includes $C(s)$, observe that we can apply

Lemma 2 to (9) again in reverse, but now with

$$
\begin{aligned}
g_r \; = \; g_{r_\mathsf{c}} \; &= \mathrm{Leg}^{r_c}_{\mathrm{Plain}}(t) - E\left[e^{-\int_t^T r_c(x)dx} \mathcal{M}_\mathsf{AB}(T) |\, \mathcal{F}_t \right] \\
&\quad - E\left[\int_t^T e^{-\int_t^s r_\mathsf{c}(x)dx} \left(r_\mathsf{AB}(s) \mathcal{M}_\mathsf{AB}(s) ds - \mathrm{d}\mathcal{M}_\mathsf{AB}(s) \right) \,|\, \mathcal{F}_t \right],
\end{aligned}
$$

and $f = 0$ to get

$$
\begin{aligned}
C(t) \; &= \mathrm{Leg}^{r_\mathsf{A}}_{\mathrm{Plain}}(t) - E\left[e^{-\int_t^T r_\mathsf{A}(x)dx} \mathcal{M}_\mathsf{AB}(T) |\, \mathcal{F}_t \right] \\
&\quad - E\left[\int_t^T e^{-\int_t^s r_\mathsf{A}(x)dx} \left(r_\mathsf{AB}(s) \mathcal{M}_\mathsf{AB}(s) ds - \mathrm{d}\mathcal{M}_\mathsf{AB}(s) \right) \,|\, \mathcal{F}_t \right] \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_\mathsf{A}(x)dx} \left(r_\mathsf{A}(s) - r_c(s) \right) C(s) ds \,|\, \mathcal{F}_t \right] \\
&= \mathrm{Leg}^{r_\mathsf{A}}_{\mathrm{Plain}}(t) - \mathcal{M}_\mathsf{AB}(t) \\
&\quad - E\left[\int_t^T e^{-\int_t^s r_\mathsf{A}(x)dx} \left(r_\mathsf{AB}(s) - r_\mathsf{A}(s) \right) \mathcal{M}_\mathsf{AB}(s) ds \,|\, \mathcal{F}_t \right] \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_\mathsf{A}(x)dx} \left(r_\mathsf{A}(s) - r_c(s) \right) C(s) ds \,|\, \mathcal{F}_t \right]
\end{aligned}
\tag{11}
$$

Plugging (11) into (10)

$$
\begin{aligned}
\mathsf{FVA}_\mathsf{A}(t) \; &:= \mathcal{M}_\mathsf{AB}(t) - \mathcal{M}_\mathsf{A}(t) \\
&\quad + E\left[\int_t^T e^{-\int_t^s r_\mathsf{A}(x)dx} \left(r_\mathsf{AB}(s) - r_\mathsf{A}(s) \right) \mathcal{M}_\mathsf{AB}(s) ds \,|\, \mathcal{F}_t \right].
\end{aligned}
$$

The corresponding expression for $\mathsf{FVA}_\mathsf{B}(t)$ follows likewise. □|

4.1.1 Fair partial collateralization under a simple jump-diffusion model for the bond dynamics

We will now derive a corollary to proposition 4 making an explicit model assumption on the underlying dynamics $\mathcal{M}$. For the underlying bond price process $\mathcal{M}$ we assume that

$$
\mathcal{M}(s) \; = \; M(s) \mathbf{1}_{\tau_\mathcal{M} > s},
$$

where $\tau_{\mathcal{M}}$ is the first jump of a Cox process with stochastic intensity $\lambda_{\mathcal{M}}$[11] and $M(s)$ is the pre-default bond value process with dynamics

$$dM(s) = r_M(s)M(s)ds + \sigma_M(s, M(s))dW_M,$$

i.e. the full dynamics is

$$d\mathcal{M}(s) = r_M(s)M(s)\mathbf{1}_{\tau_{\mathcal{M}}>s}ds + \sigma_M(s, M(s))\mathbf{1}_{\tau_{\mathcal{M}}>s}dW_M + M(s)d\mathbf{1}_{\tau_{\mathcal{M}}>s}$$

in risk neutral measure.

The drift r_M is used to model the pull-to-par of an underlying bond: $r_M(s)$, $s \geq t$ is negative (positive) if the bond value $M(s)$ is over (under) par at time t. Clearly, the drift will be an important model parameter driving the collateral dynamics of the TRS. For a zero coupon bond r_M is the zero bond yield, which is obtained from the associated zero coupon bond curve. For a coupon paying bond the continuously compounding bond coupon rate has to be subtracted from the zero bond yield, i.e., coupons are modeled via a continuously compounding dividend rate.

Suppose A and B associate the specific funding (repo) rates $r_{M,\mathsf{A}}$, $r_{M,\mathsf{B}}$ for funding the underlying. For example $r_{M,\mathsf{A}}$ ($r_{M,\mathsf{B}}$) is A's (B's) unsecured funding rate. Another example is that $r_{M,\mathsf{A}}$ is equal to a central bank rate, assuming A can obtain cheap central bank funding for the underlying[12].

Let us consider a funded replication risk neutral approch: then in the economy of counterparty A (B) the bond needs to have the effective growth rate (drift) $r_{M,\mathsf{A}}$ ($r_{M,\mathsf{B}}$) [5].

Hence we define the associated counterparty bond dynamics by applying the corresponding drift adjustment:[13]

$$\mathcal{M}_{\mathsf{A}}(s) := \mathcal{M}(s)\frac{e^{\int_t^s r_{M,\mathsf{A}}(x)}}{e^{\int_t^s r_0(x)dx}}, \quad \mathcal{M}_{\mathsf{B}}(s) := \mathcal{M}(s)\frac{e^{\int_t^s r_{M,\mathsf{B}}(x)}}{e^{\int_t^s r_0(x)dx}}. \tag{12}$$

The drift adjustments in (12) can be interpreted as fx rates using a cross currency analogy [4, 2]. The fx rates are

$$fx_{\mathsf{A}}(s) := \frac{e^{\int_t^s r_{M,\mathsf{A}}(x)dx}}{e^{\int_t^s r_0(x)dx}}, \quad fx_{\mathsf{B}}(s) := \frac{e^{\int_t^s r_{M,\mathsf{B}}(x)dx}}{e^{\int_t^s r_0(x)dx}}.$$

Let us define the convex combined fx rate

$$fx_{\mathsf{AB}}(s) := p\, fx_{\mathsf{A}}(s) + (1-p)\, fx_{\mathsf{B}}(s).$$

[11] If the reader prefers he can for simplicity assume that $\lambda_{\mathcal{M}}$ is deterministic, so that $\tau_{\mathcal{M}}$ is the first jump of an inhomogeneous Poisson process.

[12] If A can obtain central bank funding, but B can not due to regulatory reasons, this is an example of regulatory arbitrage. The arbitrage or funding benefit can be shared by adjusting the deal spread in the TRS.

[13] here we assume that t is initial valuation time, i.e. $t = t_v$

Corollary 9: The value of a total return swap on $\mathcal{M}$ with the equilibrium collateral model (5) and where $\mathcal{M}$ follows the above model is given as

$$V_{\mathsf{A/B}}(t) \ = \ C(t) + \mathsf{FVA}_{\mathsf{A/B}}, \tag{13}$$

where

$$\begin{aligned} C(t) \ &= \mathrm{Leg}^{r_{\mathsf{c}}}_{\mathrm{Plain}}(t) - E\left[e^{-\int_t^T (r_{\mathsf{c}}(x)+\lambda_{\mathcal{M}}(x))dx} f x_{\mathsf{AB}}(T) M(T) \mid \mathcal{F}_t\right] \mathbf{1}_{\tau_{\mathcal{M}}>t} \\ &\quad - E\left[\int_t^T e^{-\int_t^s (r_{\mathsf{c}}(x)+\lambda_{\mathcal{M}}(x))dx} (r_{\mathsf{AB}}(s) - (r_M(s) - r_0(s) - \lambda_{\mathcal{M}} \right. \\ &\quad \left. + p\, r_{M,\mathsf{A}} + (1-p)\, r_{M,\mathsf{B}})) f x_{\mathsf{AB}}(s) M(s) ds \mid \mathcal{F}_t \right] \mathbf{1}_{\tau_{\mathcal{M}}>t} \\ &= \mathrm{Leg}^{r_{\mathsf{c}}}_{\mathrm{Plain}}(t) - \mathcal{M}(t) - E\left[\int_t^T e^{-\int_t^s (r_{\mathsf{c}}(x)+\lambda_{\mathcal{M}}(x))dx} \right. \\ &\quad \left. (r_{\mathsf{AB}}(s) - r_{\mathsf{c}}(s))\, f x_{\mathsf{AB}}(s) M(s) ds \mid \mathcal{F}_t \right] \mathbf{1}_{\tau_{\mathcal{M}}>t}, \end{aligned} \tag{14}$$

and the funding valuation adjustment is

$$\begin{aligned} \mathsf{FVA}_{\mathsf{A/B}} :=& E\left[\int_t^T e^{-\int_t^s (r_{\mathsf{A/B}}(x)+\lambda_{\mathcal{M}}(x))dx} (r_{\mathsf{AB}}(s) - r_{\mathsf{A/B}}(s)) \right. \\ & \left. f x_{\mathsf{AB}}(s) M(s) ds \mid \mathcal{F}_t \right] \mathbf{1}_{\tau_{\mathcal{M}}>t}. \end{aligned} \tag{15}$$

Proof: The result follows from Proposition 4 and the definition of $\mathcal{M}$ and the relation (see [8, page 122])

$$E\left[d\mathbf{1}_{\tau_{\mathcal{M}}>s} \mid \mathcal{F}_s\right] = -1_{\tau_{\mathcal{M}}>s} \lambda_{\mathcal{M}}(s) ds.$$

□|

Remark 10 (Zero bond): If the underlying is a zero bond we have $r_M = r_0 + \lambda_{\mathcal{M}}$. Thus we get (assuming that the spread $r_{M,\mathsf{A/B}} - r_c$ and the remaining processes are independent, i.e. neglecting a convexity adjustment)

$$\begin{aligned}
C(t) &= \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) \\
&\quad - \mathcal{M}(t) E\left[e^{-\int_t^T r_{\mathsf{C}}(x)dx} \left(p\, e^{\int_t^T r_{M,\mathsf{A}}(x)dx} + (1-p)\, e^{\int_t^T r_{M,\mathsf{B}}(x)dx} \right) \mid \mathcal{F}_t \right] \\
&\quad - \mathcal{M}(t) E\Bigg[\int_t^T e^{-\int_t^s r_{\mathsf{C}}(x)dx} \left(r_{\mathsf{AB}}(s) - (p\, r_{M,\mathsf{A}} + (1-p)\, r_{M,\mathsf{B}}) \right) \\
&\quad \left(p\, e^{\int_t^s r_{M,\mathsf{A}}(x)dx} + (1-p)\, e^{\int_t^s r_{M,\mathsf{B}}(x)dx} \right) ds \mid \mathcal{F}_t \Bigg] \\
&= \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) - \mathcal{M}(t) \\
&\quad - \mathcal{M}(t) E\Bigg[\int_t^T e^{-\int_t^s r_{\mathsf{C}}(x)dx} \left(r_{\mathsf{AB}}(s) - r_{\mathsf{C}}(s) \right) \\
&\quad \left(p\, e^{\int_t^s r_{M,\mathsf{A}}(x)dx} + (1-p)\, e^{\int_t^s r_{M,\mathsf{B}}(x)dx} \right) ds \mid \mathcal{F}_t \Bigg],
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{FVA}_{\mathsf{A/B}} :=& \mathcal{M}(t) E\Bigg[\int_t^T e^{-\int_t^s r_{\mathsf{A/B}}(x)dx} \left(r_{\mathsf{AB}}(s) - r_{\mathsf{A/B}}(s) \right) \\
& \left(p\, e^{\int_t^s r_{M,\mathsf{A}}(x)dx} + (1-p)\, e^{\int_t^s r_{M,\mathsf{B}}(x)dx} \right) ds \mid \mathcal{F}_t \Bigg].
\end{aligned}$$

In particular for $p = 1$ (i.e. the deal is fully collateralized for A) and $r_{M,\mathsf{A}} = r_{\mathsf{A}}$ we get the simple formula

$$V_{\mathsf{A}}(t) = C(t) = \mathrm{Leg}^{r_{\mathsf{C}}}_{\mathrm{Plain}}(t) - \mathcal{M}(t) E\left[e^{-\int_t^T r_{\mathsf{C}}(x)dx} e^{\int_t^T r_{\mathsf{A}}(x)dx} \mid \mathcal{F}_t \right].$$

Remark 11 (Wrong Way Funding Risk in the Collateral): The more explicit form with the bond dynamics allows for a qualitative discussion now: The expected local growth of $\mathcal{M}$ is

$$E\left[d\mathcal{M}(s) \mid \mathcal{F}_s \right] = \left(r_M(s) - \lambda_{\mathcal{M}}(s) \right) M(s) \mathbf{1}_{\tau_{\mathcal{M}} > s} ds.$$

Suppose $\lambda_{\mathcal{M}}(s)$ is negatively correlated to both r_{A} and r_{B} and that $r_{\mathsf{A}} << r_{\mathsf{B}}$. Then on paths where $\lambda_{\mathcal{M}}(s)$ becomes small with high probability r_{A}, r_{B} and r_{AB} become large and $M(s)$ increases. In that case the collateral $C(t)$ becomes significantly negative while the magnitude of both $\mathsf{FVA}_{\mathsf{A}}$ and $\mathsf{FVA}_{\mathsf{B}}$ become large as well. This means that A has to post large amounts of collateral to B while at the same time funding becomes expensive. This is an example of *wrong way collateral and funding valuation adjustments* for counterparty A, while the situation is in favor for counterparty B.

Likewise, if $\lambda_{\mathcal{M}}(s)$ is positively correlated to both r_{A} and r_{B} and $r_{\mathsf{A}} << r_{\mathsf{B}}$, the situation is an example of *wrong way collateral and funding valuation adjustments* for counterparty B, while the situation is in favor for counterparty A. Note that such a situation can naturally arise if the underlying asset and the TRS counterparty belong to the same economic sectors (e.g., financials).

One possibility to avoid such wrong way funding risk is a *resetting* of the total return swap (like it is market practice for cross-currency swaps), where the notional of $\mathcal{M}$ is adjusted according to its change in market value. We will discuss this case in Section 4.1.2.

4.1.2 Continuously Resetting Bond Notional

As became clear in Remark 11, a resetting of the notional of the underlying asset such that it's market value matches the notional or market value $M(t_0)$ at inception time t_0 can be a desirable feature: As soon as the market value changes, the volume of the asset is reduced or increased, respectively, to match the notional amount. In other words we have $\mathcal{M}(t) \equiv M(t_0)$ and hence

$$\begin{aligned} V_{\mathsf{A/B}}(t) &= \mathrm{Leg}^{r_{\mathsf{A/B}}}_{\mathrm{Plain}}(t) - M(t_0) \\ &\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A}}(x)dx} \left(r_{\mathsf{A/B}}(s) - r_c(s)\right) C(s) ds \mid \mathcal{F}_t\right] \end{aligned}$$

This is a special case ($r_M = \sigma_M = \lambda_{\mathcal{M}} = 0$, $r_{M,\mathsf{A}} = r_{M,\mathsf{B}} = r_0$) of the more general dynamic bond model setup of Section 4.1. For this simple special case we obtain as a Corollary of Corollary 9:

$$\begin{aligned} C(t) =& M(t_0) \sum_{i=i(t)}^{n} E\left[e^{-\int_t^{t_{i+1}} r_{\mathsf{c}}(s)ds} \tau_i \left(L_i(t_i) + \mathrm{spr}\right) \mid \mathcal{F}_t\right] \\ & - M(t_0) E\left[\int_t^T e^{-\int_t^s r_{\mathsf{c}}(x)dx} r_{\mathsf{AB}}(s) ds \mid \mathcal{F}_t\right]. \end{aligned}$$

This formula means simply that LIBOR plus spread is earned and funding is paid, and that all flows are discounted with the collateral rate. The valuation formulas are:

$$V_{\mathsf{A/B}}(t) = C(t) + \mathsf{FVA}_{\mathsf{A/B}},$$

where the funding valuation adjustment is

$$\mathsf{FVA}_{\mathsf{A/B}} := M(t_0) E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A/B}}(x)dx} \left(r_{\mathsf{AB}}(s) - r_{\mathsf{A/B}}(s)\right) ds \mid \mathcal{F}_t\right].$$

4.2 Simplified Explicit Collateral (Repo Style)

To avoid complications in collateral management due to asymmetric valuations collateral can be explicitly specified. A common example is to use (see [1])

$$C(s) \ := \ \mathcal{M}(t_0) - \mathcal{M}(s). \tag{16}$$

That is, the collateral at each time $s \geq t_0$ is defined to be the difference of the TRS notional $\mathcal{M}(t_0)$ (market value at inception time) and the time s market value $\mathcal{M}(s)$ [14]. Thus market changes of the underlying asset are trivially collateralized, neglecting TRS margin and funding levels. Note that $\mathcal{M}(s)$ is an exogenous given market rate process, not a funded and replicated process $\mathcal{M}_{\mathsf{A/B}}(s)$. This means that the underlying process $\mathcal{M}(s)$ has to be considered like a quanto index for each counterparty A/B.

To value this contract one just solves (1)-(2):

$$\begin{aligned} V_{\mathsf{A/B}}(t) \ &= \mathrm{Leg}^{r_{\mathsf{A/B}}}_{\mathrm{Plain}}(t) - \mathcal{M}(t) \\ &\quad + E\left[\int_t^T e^{-\int_t^s r_{\mathsf{A/B}}(x)dx} \left(r_{\mathsf{A/B}}(s) - r_c(s)\right) C(s)ds \mid \mathcal{F}_t\right]. \end{aligned}$$

This can be easily done after bond dynamics has been calibrated. A very simple approach is to use a simple deterministic forward bond dynamics. In general one should use correlated stochastic spread models for the bond [4].

[14] In some contracts notional plus LIBOR and dealspread accrued minus underlying market value are defined as collateral. For simplicity we have neglected accrued.

5 Appendix

The following lemma is a well known result on the valuation (i.e., replication costs) of a collateralized transaction.

Lemma 12: Let X denote a cash-flow at maturity time T and let V,r,r_c be Ito stochastic processes with $V(T) = X$ [15] so that the following integral equation is satisfied for $t \leq T$

$$V(t) \;=\; E\left[e^{-\int_t^T r(s)ds} X + \int_t^T e^{-\int_t^s r(x)dx}\left(r(s) - r_c(s)\right) C(s)ds \mid \mathcal{F}_t\right]. \tag{17}$$

Then for $t \leq T$

$$V(t) \;=\; E\left[e^{-\int_t^T r_c(s)ds} X - \int_t^T e^{-\int_t^s r_c}\left(r(s) - r_c(s)\right)\left(V(s) - C(s)\right)ds \mid \mathcal{F}_t\right]. \tag{18}$$

Conversly, if $t \mapsto V(t)$ is given by (18), then $t \mapsto V(t)$ satisfies the integral equation (17).

A proof of Lemma 12 for the fully collateralized case $V = C$ can be found in [3, Appendix A]. Piterbarg [5, 6] gave a different proof, based on a self financing replication approach. We now give a simple alternative proof, which uses a foreign market cross currency viewpoint [4, 2]:

Proof: Consider a market which is governed by the rate dynamics r_c, and a (foreign) market which is governed by the rate dynamics r. Define the bank account numeraire $N_c(s) := e^{\int_t^s r_c(x)dx}$ and $N(s) := e^{\int_t^s r(x)dx}$, and the fx rate $FX(s) := e^{\int_t^s (r(x)-r_c(x))}dx$. Assets in the r_c market are transfered to the r market by multiplying with the fx rate. Thus assets seen in r have a drift which is increased by $(r(t) - r_c(t))dt$. Because $N(s) = FX(s)N_c(s)$ the risk neutral measures associated with N and N_c are equal. Write

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) := E\left[e^{-\int_t^T r_c(s)ds} X \mid \mathcal{F}_t\right]$$

and

$$V_2(t) := -E\left[\int_t^T e^{-\int_t^s r_c(x)dx}\left(r(s) - r_c(s)\right)\left(V(s) - C(s)\right)ds \mid \mathcal{F}_t\right]$$

According to the universal pricing theorem the N_c relative value $\frac{V_1(s)}{N_c(s)}$ is a

[15] We assume standard regularity conditions, e.g. r, r_c have bounded L^2 norm on $[0, T]$, which guarantee unique existence of solutions to the integral equation (17)

martingale. On the other hand

$$
\begin{aligned}
E\left(\frac{V(T)}{N(T)} - V(t) \mid \mathcal{F}_t\right) &= E\left(\int_t^T d\left(\frac{V}{N}\right) \mid \mathcal{F}_t\right) \\
&= E\left(\int_t^T d\left(FX(s)FX^{-1}(s)\frac{V(s)}{N(s)}\right) \mid \mathcal{F}_t\right) \\
&= E\left(\int_t^T d\left(FX^{-1}(s)\frac{V(s)}{N_c(s)}\right) \mid \mathcal{F}_t\right) \\
&= E\left(\int_t^T \frac{V(s)}{N_c(s)} d\left(FX^{-1}(s)\right) \mid \mathcal{F}_t\right) \\
&\quad + E\left(\int_t^T d\left(\frac{V_2(s)}{N_c(s)}\right) FX^{-1}(s) \mid \mathcal{F}_t\right)
\end{aligned}
$$

where the last equation holds because from the martingale property

$$
E\left(\int_t^T FX^{-1}(s) d\left(\frac{V_1(s)}{N_c(s)}\right) \mid \mathcal{F}_t\right) = 0.
$$

and $d\left(FX^{-1}(s)\right)\ d\left(\frac{V(s)}{N_c(s)}\right) = 0$ due to the absence of any diffusion in FX.[16] Because

$$
\begin{aligned}
d\left(\frac{V(s)}{N_c(s)}\right) &= - r_c(s)\frac{V(s)}{N_c(s)} ds + r_c(s)\frac{V(s)}{N_c(s)} ds \\
&\quad - \left(r(s) - r_c(s)\right)\left(V(s) - C(s)\right) \frac{1}{N_c(s)} ds
\end{aligned}
$$

we have (17). □|

Equation (17) has an intuitive meaning in terms of the universal pricing theorem: the collateral can be viewed as an additional contract which generates the instantanous cash flows $C(s)(r(s) - r_c(s))$. Equation (17) is a reformulation in terms of discounting with the collateral rate. Here the gap between value and collateral $V - C$ generates the instantaneous cashflows $-(r(s) - r_c(s))(V(s) - C(s))ds$.

If the cash-flow X is fully collateralized, i.e. $C(s) = V(s)$, then the value $V(t)$ is obtained by discounting the cash-flow X with the collateral rate r_c according to (18).

[16] Note that the numeraires N and N_c are locally risk free, hence is FX.

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