Displaced Historical Simulation is a Solution
for Negative-Valued Financial Risk Values:

Application to VaR in Times of Negative Government Bond Yields

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Abstract

In this paper we introduce the displaced relative change (DRC) model and the displaced filtered historical simulation (DFHS) model. These models make possible historical simulations based on potentially negative risk factors, such as interest rates or spreads. This is an issue of recent and major interest to the financial sector, both from a regulatory and financial institutions perspective, especially in light of observed negative values for government bond yields. Our empirical results show that compared to other models in the literature, models with our proposed displacement feature handle situations of close-to-zero or negative interest rates particularly well.

Keywords: Risk management, historical simulation, displacement model, negative risk variables, Value at Risk

JEL Classifications: C43, G17, G20, G30

1

1 Introduction

In this paper we propose an extended historical simulation approach that can handle situations in which a risk variable, such as interest rates, spreads, or security prices, can take on negative values. This is a situation that recently became of major interest when negative sovereign bond yields or negative interest rate spreads have been observed frequently.

In general, the historical simulation approach is used to construct a distribution of possible future risk variable realizations from the time series of past risk variable realizations. In the most standard form of historical simulation, first, a method for calculating changes of past realizations is chosen, for example, absolute or relative changes.[1] Next, independence of the past changes in the risk variable is assumed and the past changes are applied to the current data level to create a distribution of possible future values.

Historical simulations are widely employed in the financial industry to assess market risk. For example, under the Basel Accord, banks use Value-at-Risk (VaR) based on historical simulation to assess their market risk. Similarly, insurance companies and mutual funds frequently measure the market risk of their portfolio positions based on historical simulations. An advantage of historical simulation is that it is relatively easy to implement and has the potential to preserve structures hidden in the time-series data, e.g., correlation structures, that can become lost when fitting the data to a fully parametric model.[2] However, we show in an empirical application of historical simulation to Swiss government bond yield data and U.S. government bond spread data that standard historical simulation and currently available extensions have a difficult time coping with risk variables that can take on close-to-zero or even negative values. This is particularly the case for situations when relative changes, e.g., log returns, are assumed as the starting point of a historical simulation. This is important to notice since relative changes in the form of log returns constitute a standard choice in financial industry which is often applied uncritically when modeling changes to conduct historical simulations. The analysis in this paper is centered around extending historical simulation in such a way that it can handle close-to-zero or negative risk variables.

The contributions of this paper are the following. First, we propose a simple one-parameter "historical simulation model" that we call the *displaced relative change model* (DRC). The displaced relative change model encompasses the two most prominent change concepts–relative and absolute changes–as special cases and can be viewed as a continuous interpolation between them. In fact, employing the DRC model makes it unnecessary to explicitly choose between using either absolute or relative changes since the DRC model interpolates between the two concepts depending on the needs of the data. This feature of the model makes it particularly interesting for financial institutions from a regulatory perspective. Changing, e.g., from absolute to

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[1] For a more detailed discussion on choosing a method to calculate change when performing historical simulations, see Fries et al. (2013).

[2] Another approach to generating a distribution of possible future realizations of a risk variable is to use a parametric model dependent approach. That is, one assume some parametric data-generating model, estimates the model parameters based on a time-series of past realizations, and then generates a distribution of possible future values using the fitted model, e.g., analytically or through a Monte Carlo simulation. In such an approach, the resulting distribution is by necessity strongly influenced by the choice of the parametric model imposed on the structures in the data.

relative shifting when conducting a historical simulation poses an operational model change which, in general, has to be approved by the regulator. An advantage of the DRC model is that it makes an automatic and, even more important from the viewpoint of regulatory supervision, methodologically sound decision as to whether to apply absolute or relative changes to the data.

Second, the DRC model is designed to handle negative risk factors, such as negative interest rates or negative spreads, a property other historical simulation models lack. This is of particular interest since we currently face a market regime in which negative interest rates, e.g., for sovereign bonds, and negative spreads are frequently observed. Furthermore, we extend the *filtered historical simulation* (FHS) model introduced by Barone-Adesi et al. (1999) and Hull and White (1998) to overcome its inability to handle negative state variables. To this end, we combine our displaced relative change approach with FHS and propose the *displaced filters historical simulation* (DFHS), a model fully capable of handling negative state variables.

Finally, we conduct an empirical test of our proposed models. We apply models equipped with the displaced change feature to a VaR application. We set up a back-testing framework for VaR calculations using Swiss government bond yield (2Y) data for the period 1999–2012 and U.S. government bond spread data (10Y-3M) for the period 1992–2012. We compare the forecasting performance of the DFHS model and a combination of a standard GARCH model combined with our DRC model with that of a standard historical simulation model, a FHS, and a GARCH model. Note that our proposed models are not restricted to VaR applications but can, in fact, be applied to any other situation where distributions of state variables need to be considered.

The results of our empirical analysis reveal that the models with our proposed displacement feature outperform the models without the displacement extension. The outperformance is particularly acute for situations where risk factors are edging close to zero or even taking on negative values. The models with the displacement features also outperform in the case of shorter historical calibration periods such as one year. This implies a better practical applicability of these models since in practice calibration periods are usually relatively short. Furthermore, we obtain the very robust result that the standard historical simulation completely fails to handle non-positive risk factors.

There is a vast literature on applying historical simulation to financial issues. One strand that is particularly related to our paper attempts to overcome the major assumption of historical simulation that historical observations are identically independently distributed. This assumption is not supported by empirical data since we commonly observe the phenomenon of clustered volatility (see Bollerslev (1986)), which cannot be picked up by historical simulation.[3] To overcome this drawback Boudoukh et al. (1998) propose an exponential weighting of historical time series that puts more weight on more recent observations. Several other papers, e.g., see Hull and White (1998) and Barone-Adesi et al. (1999), suggest a filtered historical simulation approach, in which historical observations are passed through a filter before being used to generate forecasting distributions in order to account for time-series pattern in the variance of observations. The general idea is that GARCH-type models are fitted to historical data and estimated volatility is used to adjust the observed data. For example, Hull and White (1998) use estimated volatility to update the current level of

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[3] For a general discussion of the drawbacks of historical simulation, see, e.g., Christoffersen (2003) or Dowd (2002).

volatility when calculating a forecasting distribution and thereby are able to improve forecasts on stock indices. Barone-Adesi et al. (1999) use the errors resulting from fitting historical data to a GARCH process as shifts when generating a forecasting distribution. Multivariate extensions for filtered historical simulation models are proposed in Christoffersen (2009) and Audrino and Barone-Adesi (2005). Our suggested displaced filtered historical simulation model is in the spirit of the FHS models and is another extension of this model class.

Indeed, a particularly prominent application of historical simulation found in the literature has to do with VaR and management of market risk, which is why we chose VaR as an application in our empirical study also. There are a great many empirical and theoretical studies related to VaR that use historical simulation as a benchmark case. See, e.g., Kuester et al. (2006), Bao et al. (2006), Barone-Adesi et al. (2002), and van den Goorbergh and Vlaar (1999).

The remainder of the paper is structured as follows. Section 2 introduces the notational and methodological framework of historical simulation. Section 3 presents our proposed displaced relative change model that, in Section 4, is combined with a filtered historical simulation approach to create a new displaced filtered historical simulation model. Section 5 lays out the estimation and back-testing procedure for testing the models empirically. Data and empirical results are presented and discussed in Section 6 and Section 7 summarizes and concludes.

2   Methodological Framework

In this section we introduce our notational framework and offer a formal definition of a historical simulation. Let t_i, with $t_i < 0$ and $i = 1, \ldots, n+1$, denote a series of past discrete observation times and x_{t_i} denotes a time series of $n + 1$ past observations of some state variable (risk variable) at the respective times t_i. $\{x_{t_i} \mid i = 1, \ldots, n+1\}$ can be thought of as a particular sample path, denoted with ω_0, of some (unknown) stochastic process X, i.e.,

$$x_{t_i} \;=\; X(t_i, \omega_0) \qquad \text{for all } i = 1, \ldots, n+1.$$

A historical simulation consists of generating n possible future scenario samples $x^i_{T_1} := X(T_1, \omega_i)$, with $i = 1, \ldots, n$, as variations of a given state $x^0 = X(T_0, \omega_0)$ by applying data changes derived from n "historical" changes in a time series $\{x_{t_i}\}$ to the current state x^0.

That is, $x^i_{T_1}$ and x_{t_i} are understood to be samples of the same stochastic process X, namely, samples with respect to time (for a fixed path ω_0)

$$x^i_{T_1} \;=\; X(T_1, \omega_i) \tag{1}$$

4

and samples with respect to the state space (at a fixed time T_1)

$$x_{t_i} \; := \; X(t_i, \omega_0), \tag{2}$$

where $i = 1, 2, \ldots$, $t_i \leq T_0 < T_1$, and $\omega_i \in F_0 \in \mathcal{F}_{T_0}$ for a fixed F_0 (and X being adapted to the filtration $\mathcal{F}$).[4] Given that the process X is known, the two equations allow a time series from the past (x_{t_i} defined by varying the point in time, i.e., varying t_i), to be used to generate a distribution in the future ($x_{T_1}^i$ defined by varying the path, i.e., varying ω_i).

Note that we still left open how we derive the past changes from $\{x_{t_i}\}$ and how we apply them to the current state x^0 to generate possible future scenarios $x_{T_1}^i$. Two prominent options for generating a possible future distribution $X(T_1)$ are absolute and relative changes.

Absolute data changes are defined as follows: given a discrete time series $\{x_{t_i}\}$ and a state x^0, we define a sample of states $x_{T_1}^i$ by adjusting the state x^0 according to the "model" of absolute changes as

$$x_{T_1}^{\mathrm{abs},i} \; := \; x^0 + (x_{t_i} - x_{t_{i-1}}) \tag{3}$$

for $i = 1, \ldots, n$. That is, we generate a possible future distribution of X based on absolute changes of the process in the past.[5]

A second prominent option is to consider relative changes calculated according to

$$x_{T_1}^{\mathrm{rel},i} \; := \; x^0 \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}} \right), \tag{4}$$

for $i = 1, \ldots, n$.

In practice, the choice of methodology to calculate and apply changes is often arbitrary. Frequently, relative changes are preferred and it corresponds to modeling rate of changes that are seen as more fundamental than absolute values. For example, for equity prices, one usually considers returns, i.e., relative changes of these prices. However, for spreads, i.e., quantities that are by definition differences (e.g., differences of interest rates), it is eminent that a relative change does not make sense. Spreads may become negative, in which case absolute changes appear more appropriate. We will refer to this issue in more detail in Section 3.2.

3   Concept of Displaced Relative Changes

In the following section we introduce the concept of displaced relative changes. We show that this concept arises naturally when applying historical simulation to the "*displaced diffusion model*" (DD).

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[4] Note that this cumbersome mathematical notation just means that we know the past, i.e., $X(t_i, \omega) = X(t_i, \omega_0)$ for all $\omega \in F_0$.

[5] Note that the usual situation is such that $\{x_{t_i}\}$ is a sequence of the predecessors of x^0. However, t_i could also denote any other time period in the past (e.g., a period with stressed market data). We discuss this special case in more detail in Section 3.2.1.

3.1   The Displaced Diffusion Model for Stochastic Processes

In modeling asset price processes, displacing a log-normal diffusion process is a common way of introducing a mixed log-normal/normal model; see Brigo and Mercurio (2001). Rubinstein (1983) introduces the displaced diffusion model into the area of option pricing and recently the model has received extensive use in the LIBOR market literature; see, e.g., Beveridge and Joshi (2009), Errais and Mercurio (2005), Brigo and Mercurio (2003). The model captures a skewed distribution for the underlying stochastic process, which leads to the desirable feature of skewed implied volatility patterns. The model is also appreciated for its analytical tractability, resulting in closed-form pricing solutions. The model may allow for negative values. This was considered an undesired effect in interest rate modeling.[6] However, in the post-credit crises, modeling with negative rates became a desirable feature. For us, too, the ability to model negative risk factors is an advantage and is why we consider the model.

Assume X follows the mixed log-normal/normal process

$$\mathrm{d}X(t) \;=\; ((1 - |\alpha|)X(t) + \alpha X_0)\sigma^* \mathrm{d}W(t), \tag{5}$$

with $-1 \leq \alpha \leq 1$. Then X is a log-normal process for $\alpha = 0$ having log-volatility σ^*, and X is a normal process for $\alpha = 1$ and $\alpha = -1$ having normal volatility $X_0\sigma^*$. For $0 < |\alpha| < 1$, the process is a mixture of a log-normal and a normal process. For $\alpha < 0$, the two processes are anti-correlated. Note that the standard procedure in the literature is to consider the case $0 \leq \alpha \leq 1$, but it is straight-forward to generalize the model. Here $X_0 > 0$ is just a scaling factor to allow different volatilities in the two limiting cases. Without loss of generality, we may assume $X_0 > 1$.

For $|\alpha| \neq 1$, we can rewrite Equation (5) in order to obtain the standard definition of the displaced diffusion model:

$$\begin{aligned}
\mathrm{d}X(t) \;&=\; \left(X(t) + \frac{\alpha}{(1 - |\alpha|)} X_0 \right)(1 - |\alpha|)\sigma^* \mathrm{d}W(t) \\
&=\; (X(t) + a)\sigma^{(a)} \mathrm{d}W(t),
\end{aligned} \tag{6}$$

where the coordinate transformation is given by

$$a \;=\; \frac{\alpha}{(1 - |\alpha|)} X_0 \qquad\qquad \sigma^{(a)} \;=\; (1 - |\alpha|)\sigma^*$$

$$\alpha \;=\; \frac{\frac{a}{X_0}}{(1 + |\frac{a}{X_0}|)} \qquad\qquad \sigma^* \;=\; \left(1 + |\frac{a}{X_0}| \right)\sigma^{(a)},$$

---

[6] With respect to interest rate modeling, Rebonato (2002) shows that for reasonable values of the displacement parameter, the probability of obtaining negative values is small. Joshi and Rebonato (2003) show that the displaced diffusion model has characteristics that are very similar to those of the constant elasticity of variance (CEV) model, which is known for its appealing economic properties.

with $-\infty < a < \infty$ corresponding to $-1 < \alpha < 1$ and (6) being equivalent to

$$\mathrm{d}(X(t) + a) = (X(t) + a)\sigma^{(a)}\mathrm{d}W(t). \tag{7}$$

We consider the model in the form

$$\mathrm{d}(X(t) + a) = \frac{(X(t) + a)}{(1 + |\frac{a}{X_0}|)}\sigma^*\mathrm{d}W(t). \tag{8}$$

The latter form has the advantage of allowing us to consider a log-normal model for $X(t) + a$ with a fixed (bounded) volatility parameter σ^* (note that $\sigma^{(a)}$ is unbounded for fixed σ^*). This is advantageous when applying numerical algorithms, e.g., when estimating σ^* through a GARCH model.

3.1.1 Displace Diffusion Model, Historical Simulation, and Relation to Relative Changes

Let us assume that we observe past values of $X(t_i, \omega_0)$ with $i = 1, \ldots, n$ and we wish to generate "realistic" future samples of $X(T_1)$ from it by means of historical simulation. Further, let us assume that X follows the stochastic process in Equation (8) for some displacement parameter a. Using the concept of historical simulation, what we ultimately wish to do is to extract the stochastic driver that generates the historical data to forecast the possible future distribution of X. In the case of the displaced diffusion model, we rearrange Equation (8) to back out the stochastic driver $\sigma^{(a)}\mathrm{d}W$ of the displaced diffusion model. That is, we calculate the stochastic driver as

$$\sigma^{(a)}\mathrm{d}W(t) = \frac{\mathrm{d}(X(t) + a)}{X(t) + a}.$$

In other words, to extract the stochastic driver of the displaced diffusion model from historical data, the displaced diffusion model implies that we have to calculate relative changes of the displaced historical values of the process X. On the other hand, the SDE (8) leads to

$$
\begin{aligned}
X(t + \mathrm{d}t) + a &= (X(t) + a) + (X(t) + a)\left[\sigma^{(a)}\mathrm{d}W(t)\right] \\
&= (X(t) + a)\left\{1 + \left[\frac{\mathrm{d}(X(t) + a)}{X(t) + a}\right]\right\}.
\end{aligned}
$$

That is, using a historical driver to calculate a possible value of the displaced diffusion process at time $t + dt$ is consistent with generating a forecasting distribution by means of relative changes of the displaced value of X.

To summarize: the notion of "change", consistent with the model approach (8) (equivalently (5)), is that of a "relative change of the displaced value with displacement parameter a". For our time-discrete process defined in Equations (1) and (2), using the relative displaced changes approach translates to

$$z_{t_i} := \frac{(x_{t_i} + a) - (x_{t_{i-1}} + a)}{(x_{t_{i-1}} + a)}$$

and

$$
\begin{aligned}
x^i_{T_1} + a \quad &:= \quad (x^0 + a) + (x^0 + a)(z_{t_i}) \\
&= \quad (x^0 + a)(1 + z_{t_i}).
\end{aligned}
$$

3.2 Displaced Relative Changes with Constant Displacements

Let us apply changes in the variable $V^i := x^i_{T_1} + a$ using relative changes of past scenarios $W_{t_i} := x_{t_i} + a$, i.e., we define

$$
V^i_{T_1} \; := \; V_0 \left(1 + \frac{W_{t_i} - W_{t_{i-1}}}{W_{t_{i-1}}} \right).
$$

In other words, we apply relative changes on a displaced variable, $X + a$ with displacement parameter a. By definition, this is equivalent to

$$
X^i_{T_1} + a \; := \; (x^0 + a) \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a} \right).
$$

From this we have

$$
\begin{aligned}
X^i_{T_1} \; :=& \; (x^0 + a)\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a}\right) - a \\
=& \; x^0 + (x^0 + a)\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a} \\
=& \; x^0\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a}\right) + a\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a} \\
=& \; x^0\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a}\right) + \frac{a}{x_{t_{i-1}} + a}\left(x_{t_i} - x_{t_{i-1}}\right) \\
=& \; x^0\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right)\frac{x_{t_{i-1}}}{x_{t_{i-1}} + a} + x^0\frac{a}{x_{t_{i-1}} + a} + \frac{a}{x_{t_{i-1}} + a}\left(x_{t_i} - x_{t_{i-1}}\right) \\
=& \; x^0\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right)\frac{x_{t_{i-1}}}{x_{t_{i-1}} + a} + \left(x^0 + \left(x_{t_i} - x_{t_{i-1}}\right)\right)\frac{a}{x_{t_{i-1}} + a} \\
=& \; \alpha \underbrace{\left[x^0\left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right)\right]}_{\text{relative changes on } x^0} + (1 - \alpha)\underbrace{\left[x^0 + \left(x_{t_i} - x_{t_{i-1}}\right)\right]}_{\text{absolute changes on } x^0}
\end{aligned}
$$

with $\alpha = \frac{x_{t_{i-1}}}{x_{t_{i-1}} + a}$.

We can see that the displaced relative changes model is just a linear interpolation of relative and absolute changes applied to the underlying stochastic process without displacement. The two limit cases can be recovered. For $a = 0$, we obtain relative changes; for $a = \infty$, we obtain absolute changes.

Two features of our model are worth special note at this point. First, we can use the DRC model to discover

whether it is more appropriate to apply relative or absolute changes to a data series by the following approach. Apply the DRC model to the considered data series and estimate the parameter a (with respect to some optimality criteria, like a maximum likelihood). Then, for $a = 0$, relative changes should be used, and for $a = \infty$, absolute changes should be used. Such a method provides an automatic and, even more important, methodologically sound (in the sense of a regulatory supervision) basis for deciding whether to apply absolute or relative changes to the data. Second, our DRC model has the ability to handle negative risk factors, an attribute that has become of major importance recently. Note that in general it is not sensible to apply the model of relative changes to negative risk factors. The DRC model handles this situation in the following intuitive way. In an empirical application, the displacement parameter is estimated in such a way that it shifts the risk factor into the positive realm of $\mathbb{R}$; see Section 5. Thereby, it is again possible to apply the relative changes to the risk factor and a historical simulation can be conducted.

3.2.1 Displaced Relative Shifting with Time Dependent Displacements

We can generalize the result from Section 3.2. Let us consider that the current regime of x^0 follows a model with displacement a, but the past regime of x_{t_i}, from which we calculate the data changes, follows a model with displacement b.

This situation can arise when there is a larger time lag between the observation time of x^0 and x_{t_i}. For example, this is the case if we want to calculate a Stress-VaR, taking the samples from a past stress period.

Let us consider the general case of a local, time-dependent displacement parameter where we calculate the $a(i)$-displaced relative change of x_{t_i} and apply it as an $a(0)$-displaced relative shift to x^0 to generate X_{t_i}. In other words, we apply relative changes to the variable $V^i := x^i_{T_1} + a(0)$ using relative changes of scenarios $W_{t_i} := x_{t_i} + a(i)$, i.e., we define

$$V^i_{T_1} := V_0 \left(1 + \frac{W_{t_i} - W_{t_{i-1}}}{W_{t_{i-1}}} \right).$$

From

$$X^i_{T_1} + a(0) := (x^0 + a(0)) \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a(i)} \right)$$

we then, analogously to the derivation in Section 3.2, obtain

$$
\begin{aligned}
X_{T_1}^i &:= x^0 \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right) \frac{x_{t_{i-1}}}{x_{t_{i-1}} + a(i)} + x^0 \frac{a(i)}{x_{t_{i-1}} + a(i)} \\
&\quad + \frac{a(0)}{x_{t_{i-1}} + a(i)} \left(x_{t_i} - x_{t_{i-1}}\right) \\
&= x^0 \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right) \frac{x_{t_{i-1}}}{x_{t_{i-1}} + a(i)} + \left(x^0 + \left(x_{t_i} - x_{t_{i-1}}\right)\right) \frac{a(i)}{x_{t_{i-1}} + a(i)} \\
&\quad + \left(x_{t_i} - x_{t_{i-1}}\right) \frac{a(0) - a(i)}{x_{t_{i-1}} + a(i)} \\
&= \alpha \underbrace{\left[x^0 \left(1 + \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}\right)\right]}_{\text{relative changes on } x^0} \\
&\quad + (1-\alpha) \underbrace{\left[x^0 + \left(x_{t_i} - x_{t_{i-1}}\right)\right]}_{\text{absolute changes on } x^0} + \underbrace{\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a(i)} \left(a(0) - a(i)\right)}_{\text{regime change}}
\end{aligned}
$$

with $\alpha = \frac{x_{t_{i-1}}}{x_{t_{i-1}} + a(i)}$.

Thus, by taking different displacements into account we obtain an additional term. This term can be reinterpreted in terms of the continuous model. The additional regime change term is the product of the relative changes of our displaced variable and the change of the displacement from the observed historical regime ($a(i)$) to the current regime ($a(0)$).

4 Displaced Relative Changes with GARCH Filtering

The displaced relative changes model discussed in Section 3 defines a method for calculating "changes" in a historical simulation by calculating relative changes on a displaced variable. Instead of considering "standard returns" of the form $\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}}$, we considered $\frac{(x_{t_i} + a) - (x_{t_{i-1}} + a)}{x_{t_{i-1}} + a}$.

We now propose an extension of the model class of filtered historical simulation models by combining the displaced relative changes model and the filtered historical simulation approach proposed in Barone-Adesi et al. (2002). We call this the *displaced filtered historical simulation* (DFHS) model. That is, we apply a GARCH-type filtering to displaced relative changes. To show how this model extension is constructed, we (1) reproduce the original model by Barone-Adesi et al. (2002), (2) rewrite the model slightly so as to (3) apply the displacement extension to the model.

In Barone-Adesi et al. (1999), an ARMA-GARCH(1,1) model is considered:

$$
y_{t_{i+1}} = \mu \cdot y_{t_i} + \theta \cdot \sigma_{t_i} \epsilon_{t_i} + \sigma_{t_{i+1}} \epsilon_{t_{i+1}} \tag{9}
$$

$$
\sigma_{t_{i+1}}^2 = \omega + \alpha \cdot (\sigma_{t_i} \epsilon_{t_i} + \gamma)^2 + \beta \cdot \sigma_{t_i}^2, \tag{10}
$$

where μ denotes the auto-regressive, θ denotes the moving-average factor, and σ_{t_i} denotes the standard GARCH volatility; $\sigma_{t_{i+1}}\epsilon_{t_{i+1}}$ is the error term with $\epsilon_{t_{i+1}} \sim \mathcal{N}(0,1)$ and ω, α, and γ are the usual constants controlling the behavior of $\sigma_{t_{i+1}}$.

To remove the path dependency introduced by the time-lagged ϵ_{t_i} in Equation (9) and (10), we apply a state space extension and rewrite equations (9) and (10) using $m_{t_i} := \sigma_{t_i}\epsilon_{t_i}$, as explained below.

Using Equation (9) shifted back in time by one period, that is,

$$y_{t_i} = \mu y_{t_{i-1}} + \theta \cdot m_{t_{i-1}} + m_{t_i}$$

we obtain

$$m_{t_i} = -\theta \cdot m_{t_{i-1}} + (y_{t_i} - \mu y_{t_{i-1}})$$

and thus (replacing all $\sigma_{t_i}\epsilon_{t_i}$ by m_{t_i})

$$
\begin{aligned}
y_{t_{i+1}} &= \mu y_{t_i} + \theta \cdot m_{t_i} + \sigma_{t_{i+1}}\epsilon_{t_{i+1}} \\
\sigma_{t_{i+1}}^2 &= \omega + \alpha \cdot (m_{t_i} + \gamma)^2 + \beta \cdot \sigma_{t_i}^2 \\
m_{t_i} &= -\theta m_{t_{i-1}} + (y_{t_i} - \mu y_{t_{i-1}}).
\end{aligned}
$$

After rewriting, the state space transition is driven by a single factor $\epsilon_{t_{i+1}}$ only, which is the component that is to be sampled by means of historical simulation, where σ and m are interpreted as model parameters know at time t_i.

The historical simulation producing equations based on relative changes can then be written as:

$$
\begin{aligned}
x_{T_1}^i &:= x_{t_i}\left(1 + \mu \cdot \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}} + \theta m_{t_i} + \sigma_{t_{i+1}}\sqrt{\Delta t_{i+1}}z_{t_{i+1}}\right) \\
\sigma_{t_{i+1}}^2 &= \omega + \alpha\,(m_{t_i} + \gamma)^2 + \beta\sigma_{t_i}^2 \\
m_{t_i} &= -\theta m_{t_{i-1}} + \left(\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}}} - \mu\frac{x_{t_{i-1}} - x_{t_{i-2}}}{x_{t_{i-2}}}\right).
\end{aligned}
$$

In the last step, we combine filtered historical simulation with our proposed displacement extensions, resulting in the following system of equations.

$$
\begin{aligned}
x_{T_1}^i &:= (x_{t_i} + a)\left(1 + \mu \cdot \frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a} + \theta m_{t_i} + \sigma_{t_{i+1}}\sqrt{\Delta t_{i+1}}z_{t_{i+1}}\right) \\
\sigma_{t_{i+1}}^2 &= \omega + \alpha\,(m_{t_i} + \gamma)^2 + \beta\sigma_{t_i}^2 \\
m_{t_i} &= -\theta m_{t_{i-1}} + \left(\frac{x_{t_i} - x_{t_{i-1}}}{x_{t_{i-1}} + a} - \mu\frac{x_{t_{i-1}} - x_{t_{i-2}}}{x_{t_{i-2}} + a}\right).
\end{aligned}
$$

This resulting model encompasses all the approaches that we consider in our empirical analysis. For $\alpha = \beta =$

$\theta = 0$ and $\mu = 1$, we recover the DRC model, which itself comprises the modeling of relative changes and absolute changes through the parameter a. For $a = \theta = 0$ and $\mu = 1$, we recover a FHS model based on a GARCH-type filtering. For $a = 0$, we recover a FHS model based on an ARMA-GARCH-type filtering.

Moreover, every parameter set can be associated, to some extent, with an interpretation. The parameter a decides which quantity is assumed to be i.i.d.-returns or absolute changes. The parameter μ and θ decide whether, on that quantity, a drift (or trend) is considered. The parameters α, β decide whether filtering based on a GARCH volatility estimation will be performed.

5   Estimation and Testing

In a practical application of a "historical simulation," one would choose a data change model (say, relative or absolute changes) and then test its predictive power, e.g., via model back testing. Endowed with our one-parameter family of models, we are able to replace the choice of the data change model with an estimation of the displacement parameter a.

In this section we describe the techniques related to estimating the displacement parameter and explain how to test its properties. For parameter estimation, we use maximum likelihood estimation. A more formal derivation of the maximum likelihood estimator of our displaced change model is given in Section A of the Appendix. For testing, we consider back-testing Value-at-Risk calculation based on possible future scenarios generated by the different historical simulation models.

We use a likelihood function to calibrate parameters of the change function F and the function G of our historical simulation model. This likelihood function comes from a distributional assumption, namely, on the argument ΔZ of the function F. We choose this procedure because it is a practically feasible way of obtaining some estimates for model parameters. The likelihood function is just one way to define a target function for the optimization problem. Another way is to derive such a target function from a back-test procedure, a method that works without any distributional assumption on ΔZ. However, due to space constraints, improvements to the estimation process are left for future research.

5.1   Maximum Likelihood Estimation for the Displaced Lognormal Model

In this section we derive the likelihood function for the displaced lognormal model given in Equation (6). We do not consider the GARCH filtering here, but this likelihood function can be used to estimate the GARCH parameters as well, and for the DFHS model we use this likelihood function to jointly estimate all parameters.

Following Section 3.1, we consider a time series x_i and assume that it follows the (time-discrete) model

$$y_i \; := \; \log(x_{i+1} + a) - \log(x_i + a) \; = \; \mu \Delta t + \sigma \sqrt{\Delta t} Z_i,$$

where Z_i are samples from a standard normal distribution. We wish to estimate a, μ and σ in such a way as to maximize the likelihood of observing y_i. We find for fixed a that μ and σ are determined by

$$\mu(a)\Delta t := \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\sigma(a)^2\Delta t := \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu(a))^2.$$

These formulas provide the maximum likelihood estimators for samples of a normally distributed random variable with mean $\mu(a)\Delta t$ and variance $\sigma^2(a)\Delta t$.

Given $\mu(a)$ and $\sigma(a)$, we may determine a as

$$a := \arg\max_{a} \mathcal{L}(a) = \arg\max_{a}\left(\sum_{i=1}^{n}\log\left(f(x_{i+1}; x_i, a)\right)\right), \tag{11}$$

with $\mathcal{L}(a)$ denoting the log-likelihood function and where

$$f(x_{i+1}; x_i, a) := \phi(x, \mu(a), \sigma(a))\frac{\mathrm{d}x}{\mathrm{d}x_{i+1}}|_{x=\log(x_{i+1}+a)-\log(x_i+a)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma(a)}\exp\left(-\frac{1}{2}\frac{(x-\mu(a))^2}{2\sigma(a)^2}\right)\frac{\mathrm{d}x}{\mathrm{d}x_{i+1}}|_{x=\log(x_{i+1}+a)-\log(x_i+a)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma(a)(x_{i+1}+a)}\exp\left(-\frac{1}{2}\frac{(\log(x_{i+1}+a)-\log(x_i+a)-\mu(a))^2}{2\sigma(a)^2}\right).$$

Note that f is just the probability density of the log-normal distribution. There is a noteworthy subtlety to the likelihood function for the displaced model: when using log returns, the usual procedure is to consider the likelihood function of the normal distribution applied to the log-returns. The difference between the two approaches is the factor $\frac{1}{(x_{i+1})}$, which is a constant with respect to a parameter optimization and hence not relevant in the maximum likelihood optimization. Here, the factor $\frac{1}{(x_{i+1}+a)}$ is not a constant with respect to the parameter optimization.

5.2 Backtesting the Historical Simulation

In the following subsection, we describe the back-testing used to test the models.

Average Backtest Hit Statistic

We take a fixed date t_{k-1} based on which scenarios for the future date t_k are generated.[7] To generate future scenarios we use the historical simulation approach on the previous N dates t_i (calibration period), with

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[7] For illustrative purpose, one can think of t_{k-1} being the current data point.

$k - N \leq i \leq k - 1$, and where N determines the size of the calibration window. A back-test hit is defined as the value at date t_k falling below the p-quantile predicted from the scenarios derived from the simulation. That is, the quantile hit indicator process q is defined such that $q(t_k) = 1$ if the value at t_k falls below the p-quantile and $q(t_k) = 0$ otherwise.

In theory, the probability of $q(t_k)$ being 1 is p. In our empirical analysis we generate $q(t_k)$ predictions for M consecutive dates up to t_n, that is, $n - M < k \leq n$. For the date t_n, we then calculate the average of the back-test hit indicator values by summing up the series of M back-test hit indicator values and dividing them by M. We call this number the *average back-test hit statistic* (ABHS) (a moving average over a sequence of 0s and 1s), which, ideally, should remain close to the analytically value of the quantile p.[8] Then the model is tested by comparing the ABHS to the theoretical value p. That is, we check whether

$$\text{ABHS} = \frac{1}{M} \sum_{j=n-M+1}^{n} q(t_j) \approx p.$$

Note that this test does not require any distributional assumption about the time series.

Back-Test Stability for Different Calibration Window Sizes

In the first step of our analysis we choose $N = M = 250$ working days, that is, the size of calibration window equals the size of the averaging window. Our choice of the averaging period $M = 250$ is motivated by practice: back-test deviations are usually monitored and reported on an annual basis. However, the choice of the calibration window size $N = 250$ is arbitrary and thus we also analyze the overall stability of the model as a function of the calibration window size. We vary N from 100 days to 1,000 days and compare model performance to the theoretical value, as well as comparing the performance of the different models.

6   Numerical Results

6.1   Data

To empirically analyze our modelâĂŹs performance for different security classes, we use Swiss government bond yield (2Y and 3Y) data for the period 1999–2012 obtained from the Swiss National Bank, U.S. government bond spread data (10Y-3M) for the period 1992–2012, and U.S. dollar-Euro exchange ratio data for the period 1999–2012. We choose this data series since one particular strength of our model is its ability to handle non-positive risk factors and these data have had non-positive values frequently in the past, as illustrated by Figure 1.

---

[8] Note that if the data set to be analyzed is of length L, the first $N + M$ observations are needed to calculate a first ABHS value and in total we can calculate $L - (N + M)$ ABHS values.
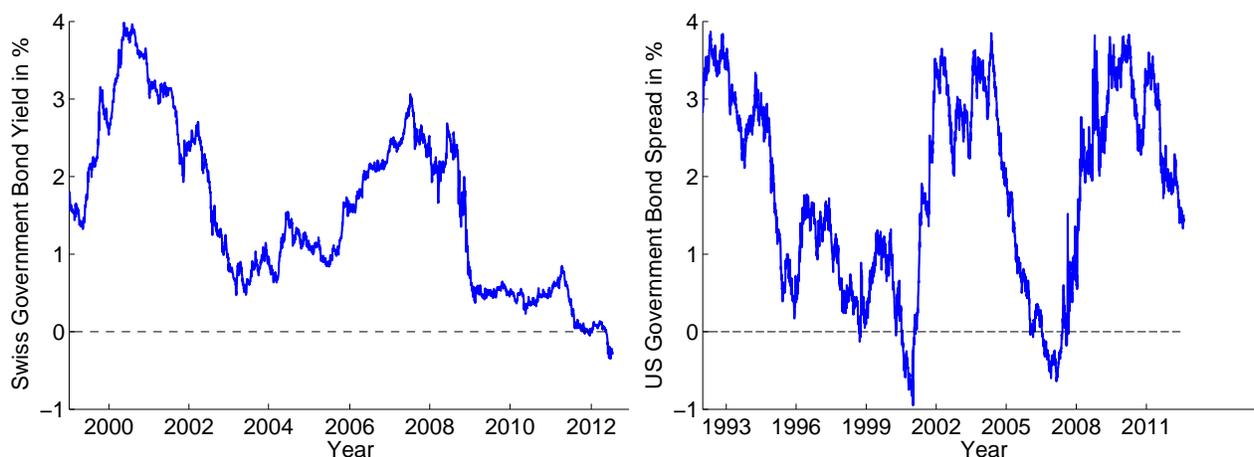
Figure 1: Swiss Government 2-Year Bond Yield Data for 1999– 2012 (left) and U.S. Government 10-Year 3-Month Bond Yield Spread Data for 1992–2012 (right).

6.2 Average Back-Test Hits Statistic Results

For our empirical analysis we implement following models: the FHS model as proposed by Barone-Adesi et al. (1999), the DFHS model, a standard GARCH(1,1) model (G), a displaced version of the GARCH model (DG), the displaced relative change model (DRC), and a standard historical simulation model (HS) with relative changes. In Figure 2, the 1% quantile back-test results for the Swiss government 2-year bond yield data are presented. In the five graphs of Figure 2, from left to right and from top to bottom, we test the models against each other in different combinations: the DFHS model against the FHS model, the displaced GARCH model against the GARCH model, the DFHS model against the displaced GARCH model, the DFHS model against the HS model, the displaced GARCH model against the HS model, and, finally, DRC model against HS model.

In each graph, the horizontal and vertical axis show the time period of the data set analyzed (in this case, 1999–2012) and the value of the average back-test hit statistics (ABHS), respectively. In the upper-left graph we analyze the ABHS for the DFHS (solid black line) and the FHS (solid gray line) model. The horizontal black dashed line shows the theoretical quantile based on which we test the models. If the ABHS value is below the dashed line, it means the model predicts quantile values that overestimate the actually occurring realizations of the state variable. That is, if we compare the predicted value of the state variable, then on average the occurring realization will be less negative than the predicted quantile value. In this sense, the model can be interpreted as giving a conservative prediction for the respective quantile realizations of the risk factor. Vice versa, if the ABHS value is above the dashed line, the model underestimates actually occurring quantile realizations of the risk factor: the occurring realizations violate the predicted quantile values more frequently than implied by the theoretical quantile.

Although these graphs present only a superficial visual comparison, they are indicative of some interesting results. When looking at the upper-left graph again, we observe that for the time period mid 2011 to mid 2012 the FHS model severely fails to predict the future quantile realizations. Even more noticeable, however, is
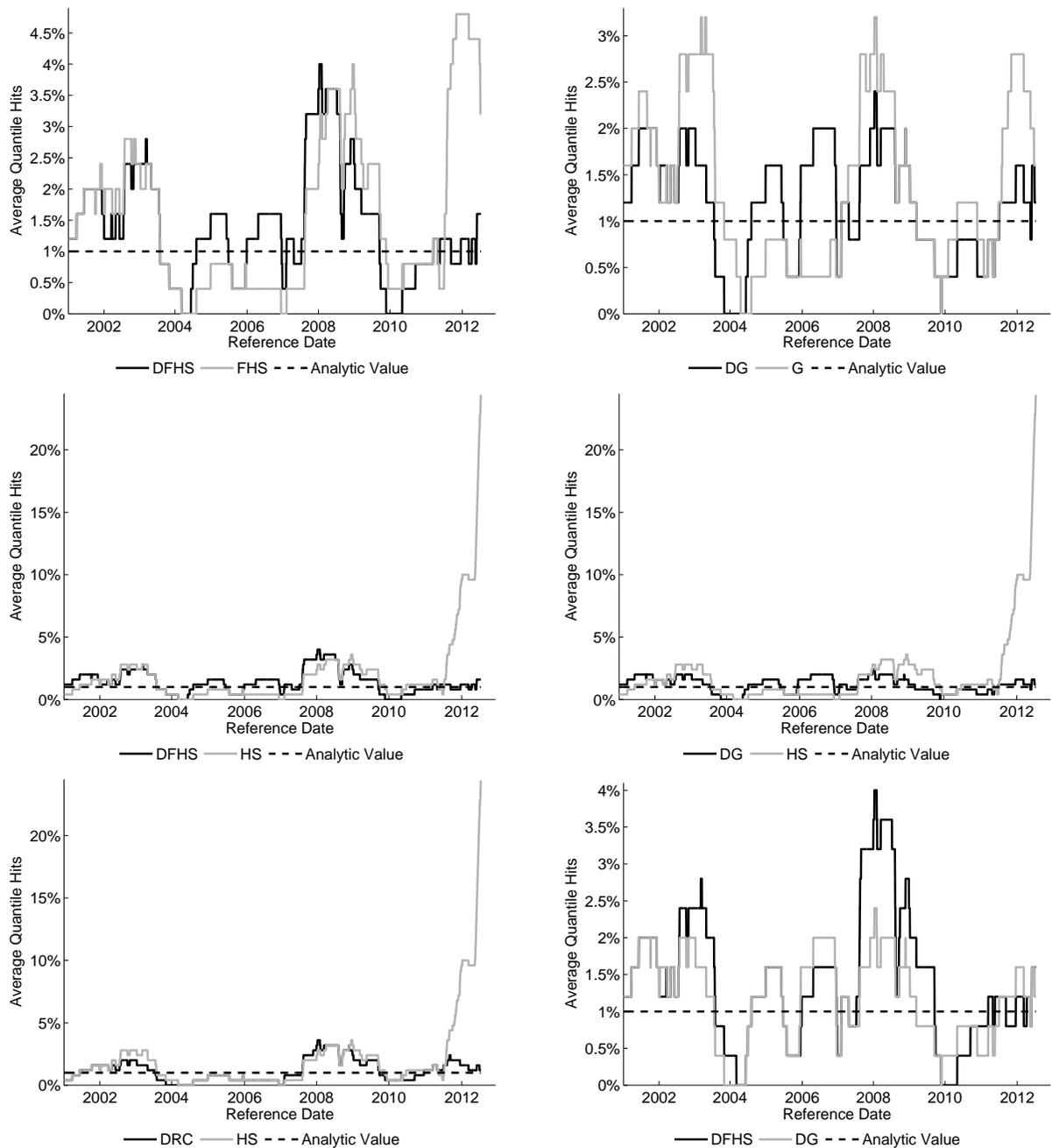
15

Figure 2: 1% Quantile Back-Test Results for Swiss Government 2-Year Bond Yield Data for 1999–2012
The figure presents the 1% quantile back-test results for the Swiss government 2-year bond yield data. The six graphs from left to right and from top to bottom show pair-wise model comparison for the DFHS model against the FHS model, the displaced GARCH model against the standard GARCH model, the DFHS against the HS model, the displaced GARCH against the HS model, the DRC against the HS model, and the DFHS model against the displaced GARCH model. In each graph, the calendar day of the forecast (Reference Date) on the x-axis is plotted against the value of the average back-test hit statistics (ABHS) on the y-axis. The ABHS is calculated based on a calibration window size of 250 days and an averaging time period of the same length.

Figure 3: 5% Quantile Back-Test Results for Swiss Government 2-Year Bond Yield Data for 1999–2012
The figure presents the 5% quantile back-test results for the Swiss government 2-year bond yield data. The six graphs from left to right and from top to bottom show pair-wise model comparison for the DFHS model against the FHS model, the displaced GARCH model against the standard GARCH model, the DFHS against the HS model, the displaced GARCH against the HS model, the DRC against the HS model, and the DFHS model against the displaced GARCH model. In each graph, the calendar day of the forecast (Reference Date) on the x-axis is plotted against the value of the average back-test hit statistics (ABHS) on the y-axis. The ABHS is calculated based on a calibration window size of 250 days and an averaging time period of the same length.
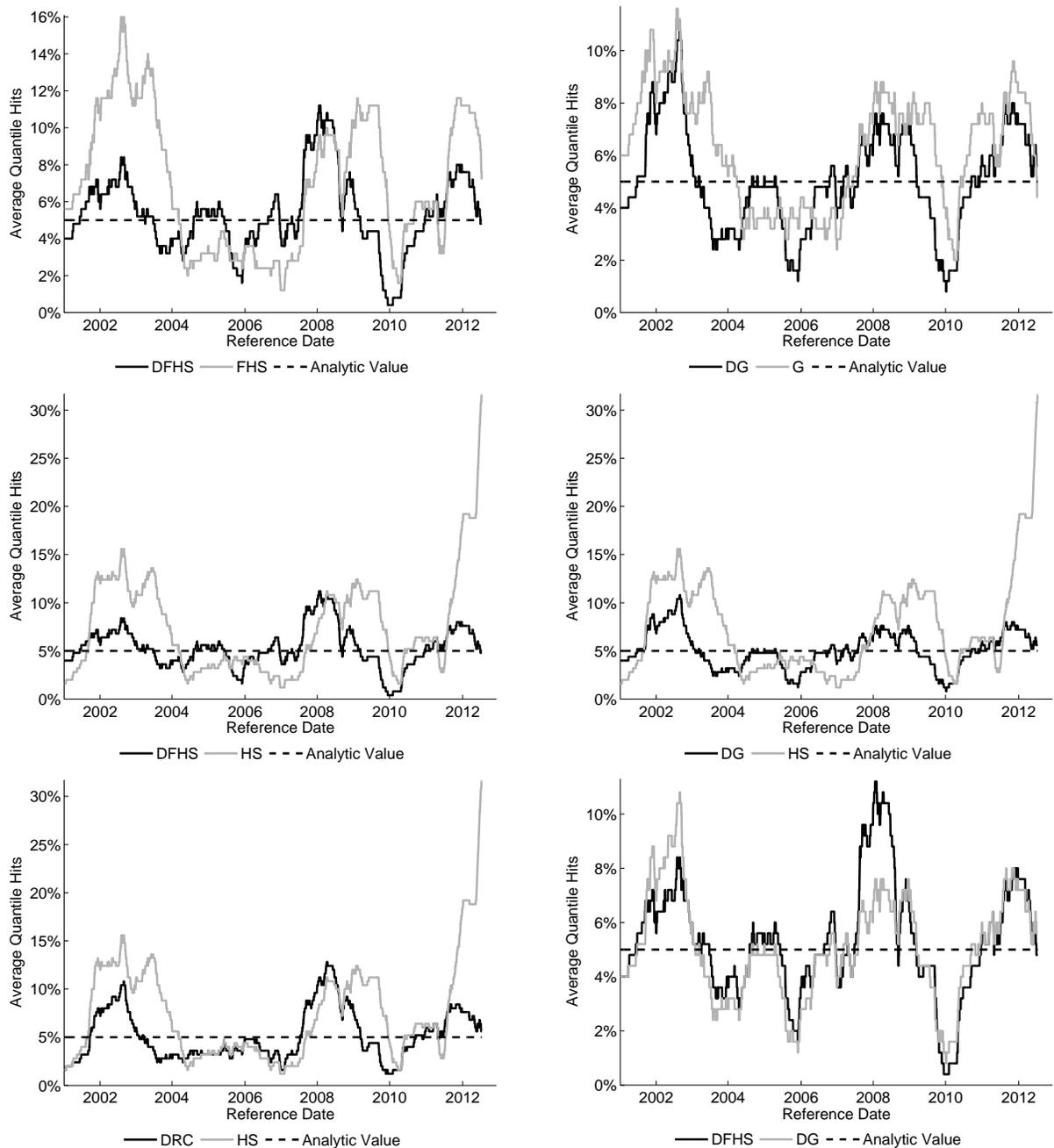
that the FHS model consistently underestimates the quantile values, which produces a bias in the prediction that will expose anyone using that model for any kind of risk management to much larger risks than expected. Since during this period, negative Swiss government bond yields are observed, it is a particularly important period for testing our suggested model extension. And, in fact, the DFHS model does appear to do a much better job in dealing with these negative yields since the ABHS moves closely around the theoretic quantile value. The same result can be seen in the upper-right graph where the displaced GARCH model is compared with the standard GARCH model. The result is even more obvious in the two middle graphs and the lower-left graph where the DFHS, the displaced GARCH, and the DRC model are compared to the standard historical simulation. It is clear that the standard historical simulation model totally fails at handling close-to-zero and negative bond yields.

Looking at the upper two graphs for the rest of the overall sample time period reveals, at most, that the displaced model alternative moves, on average, more closely around the theoretical quantile than the model without displacement and that it is more conservative in its predictions. Of course, drawing any conclusions from this rough visual test must be done with caution; however, the result is confirmed in a later analysis (see Section 6.3). If we directly compare the two displaced model alternatives in the lower-right graph we again are tempted to say that the displaced GARCH model moves on average more closely around the theoretical quantile than does the DFHS model. This result, too, hinted at by the smaller amplitude of the ABHS for the GARCH-type models compared to the FHS-type models, is confirmed by results presented in Section 6.3.

All results discussed for the 1% quantile backtest hold in similar fashion for the 5% quantile back-test results, which are presented in Figure 3. If we look at the two graphs the superior performance of the displaced models over the models without the displacement features becomes even more pronounced. Now, we can identify several time periods, e.g., years 2000–2004 in the left-upper graph, in which the bond yield is not negative and yet the DFHS model still outperforms the FHS model. The same result can be seen in the two middle graphs and the lower-left one comparing the performance of the DFHS, displaced GARCH, and the DRC model with that of the standard HS model. However, directly comparing DFHS against DG models (lower-right graph) does not result in the same obvious interpretation, as was also the case for the 1% quantile.

Looking at graphs (see 4 and 5) for the U.S. government spread data reveals results similar to, and even more consistent than, those derived for the Swiss government bond yield data. In general, the models with the displacement feature consistently outperform the models without this feature, both for the 1% and 5% quantile case. The displacement modelsâĂŹ stronger performance is especially pronounced for periods when the spread is close to zero or negative, for example, 1996, 1999, 2001, and 2007. Also the inability of the standard HS model to cope with situations where the spreads are close to zero or negative is clearly visible in the middle graphs and the lower-left graph of figures 4 and 5. Again, however, there is no major difference in performance between the DFHS and the DG models (see 4 and 5).

To summarize, our results indicate that, on average, a model with the displacement feature performs better than a model without this feature, most especially when risk factors are close to zero or even become negative. Among the two displaced model alternatives, the displaced GARCH model occasionally seems to do slightly
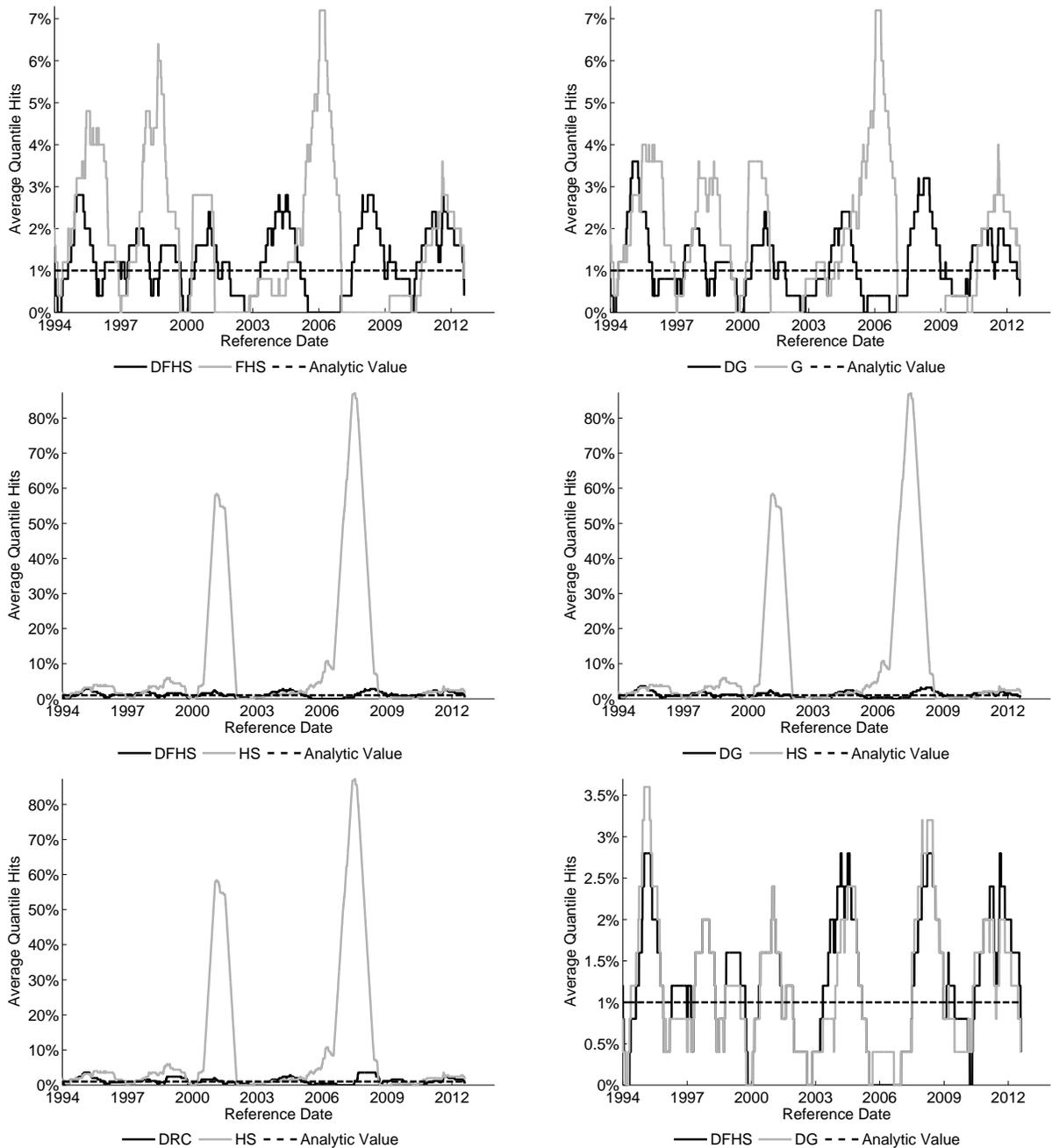
Figure 4: 1% Quantile Back-Test Results for U.S. Government Spreads (10Y-3M) for 1992–2012
The figure presents the 5% quantile back-test results for the U.S. government spreads (10Y-3M) data. The six graphs from left to right and from top to bottom show pair-wise model comparison for the DFHS model against the FHS model, the displaced GARCH model against the standard GARCH model, the DFHS against the HS model, the displaced GARCH against the HS model, the DRC against the HS model, and the DFHS model against the displaced GARCH model. In each graph, the calendar day of the forecast (Reference Date) on the x-axis is plotted against the value of the average back-test hit statistics (ABHS) on the y-axis. The ABHS is calculated based on a calibration window size of 250 days and an averaging time period of the same length.

better at predicting the VaR quintiles, but there is no indication yet that this is true for longer time periods.
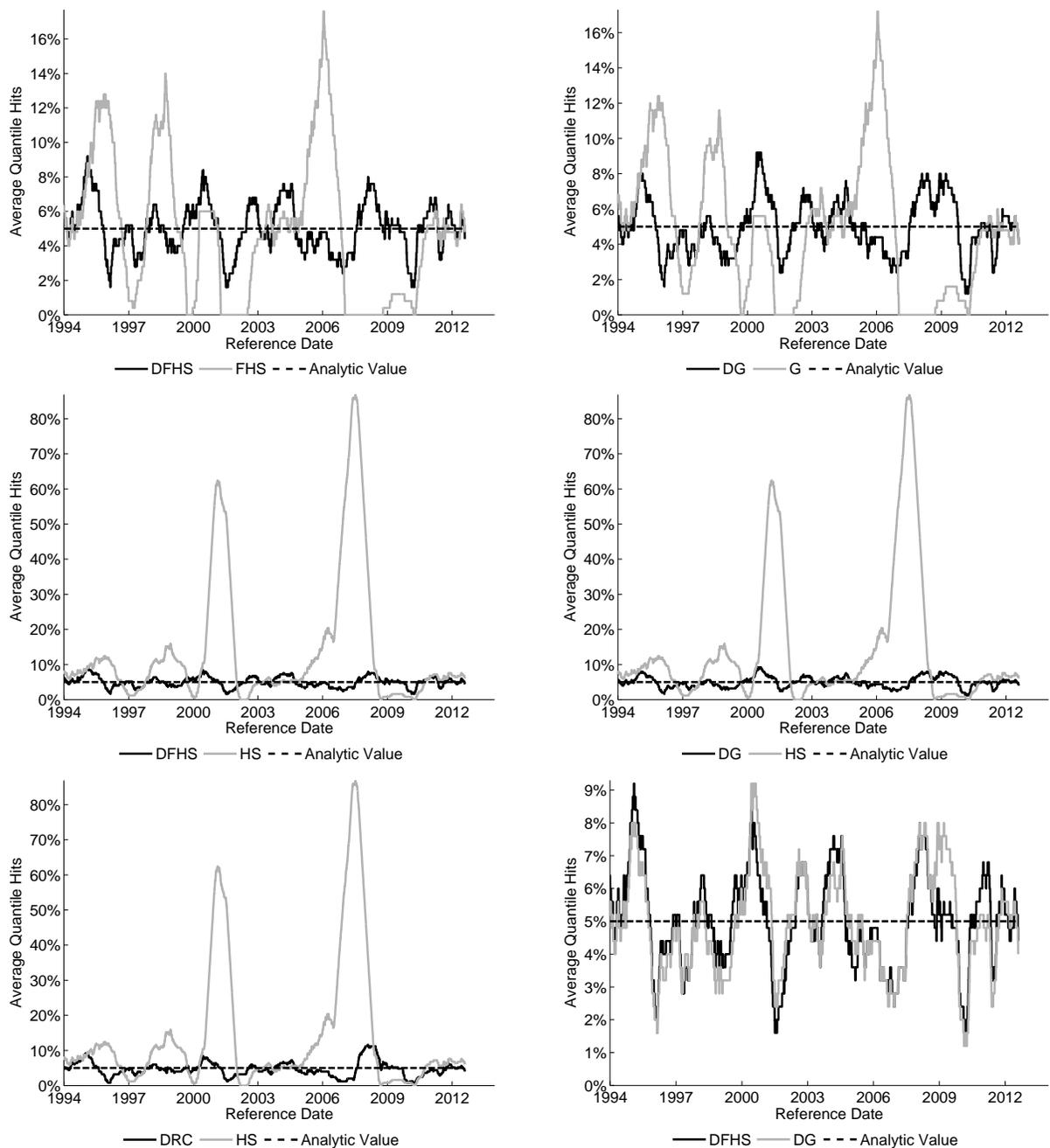
Figure 5: 5% Quantile Back-Test Results for U.S. Government Spreads (10Y-3M) for 1992–2012
The figure presents the 5% quantile back-test results for the U.S. government spreads (10Y-3M) data. The six graphs from left to right and from top to bottom show pair-wise model comparisons for the DFHS model against the FHS model, the displaced GARCH model against the standard GARCH model, the DFHS against the HS model, the displaced GARCH against the HS model, the DRC against the HS model, and the DFHS model against the displaced GARCH model. In each graph, the calendar day of the forecast (Reference Date) on the x-axis is plotted against the value of the average back-test hit statistics (ABHS) on the y-axis. The ABHS is calculated based on a calibration window size of 250 days and an averaging time period of the same length.

6.3   Back-Test Stability for Different Calibration Window Sizes

The back-test results presented in Section 6.2 are based on a averaging window size of $M = 250$, which, as mentioned in Section 5.2, is motivated by the use of historical simulation methods in practice. However, the choice of the calibration window $N = 250$ is to some extent arbitrary. Therefore, in this section we test the robustness of our results with respect to calibration window size, N, by varying it from 100 to 1,000. To make results comparable across data sets with varying time periods, we proceed as follows. First, we calculate all daily ABHS values for the whole data set based on a specific calibration window size. Then we sum up all daily ABHS values and divide them by the number of daily ABHS values, resulting in what we call the "summarized ABHS."< A summarized ABHS value of, e.g., 0.5 means that, on average, the daily ABHS value deviates from the theoretical quantile by 0.5%.

Table 1 reports results of the summarized ABHS for the 1% quantile back test of the Swiss 2-year government bond yields. For varying calibration periods, Columns 2–7 give the results for the DFHS, FHS, displaced GARCH, standard GARCH, DRC, and a standard historical simulation model. To compare the models to each other, we calculate several combinations of ratios of summarized ABHS values in Columns 8–13. These ratios can be interpreted as percentage relations between the performances of the models. If the ratio is less than 1, the model in the numerator is best; if it is larger than 1, the model in the denominator is best. A ratio of, e.g., 0.8 means that the model in the numerator has on average a 20 % lower daily deviation from the theoretical quantile than the model in the denominator.

A comparison of the results for the DFHS and FHS models in Columns 2 and 3, and the displaced GARCH and the GARCH model in Columns 4 and 5, reveals an interesting result. For calibration window sizes up to 500 days, the model with the displaced feature always shows a smaller average daily summarized ABHS than the model without the displaced feature. This means that the model with the displaced feature performs better than the alternative model. This result can also be seen in Columns 8 and 9, which present the calculated ratios of summarized ABHS values. When comparing the DFHS model with the FHS model in Column 8, the DFHS model shows, on average, between 13 to 57% less daily deviation from the theoretical quantile for calibration window sizes up to 500 days. Comparing the displaced GARCH model with the standard GARCH model brings to light that the displaced GARCH model exhibits, on average, between 3 and 56% less daily deviation from the theoretical quantile for calibration window sizes up to 500 days. However, for longer calibration window sizes, the values of displacement and alternative models converge and the relationship even reverses slightly. From a practical perspective, this result increases the attractiveness of the models with the displacement feature. Needing less data to calibrate the models makes them easier to handle. Furthermore, the models are able to produce predictions for the quantiles that are better than or at least equal to the predictions from models without the displacement feature, but they need less historical data to do so. This is helpful in situations when there is no long historical time series for a risk factor. Also, from a practical point of view, it is desirable to use only the latest historical observations to calibrate the model rather than outdated market events.

Table 1
Swiss Government 2-Year Bond Yield Back Test:
Rolling Windows, 1% Quantile

Wind	DFHS	FHS	DG	G	DRC	HS	DFHS/FHS	DG/G	DFHS/DG	DFHS/HS	DG/HS	DRC/HS
100	0.533	0.616	0.463	0.599	0.460	0.787	0.866 ***	0.773 ***	1.153 ***	0.677 ***	0.588 ***	0.548 ***
150	0.493	0.782	0.461	0.472	0.429	0.995	0.630 ***	0.976	1.069 ***	0.495 ***	0.463 ***	0.431 ***
200	0.531	0.895	0.424	0.695	0.549	1.181	0.593 ***	0.610 ***	1.251 ***	0.449 ***	0.359 ***	0.464 ***
250	0.677	0.985	0.505	0.696	0.648	1.297	0.687 ***	0.725 ***	1.341 ***	0.522 ***	0.389 ***	0.500 ***
300	0.870	1.031	0.485	0.966	0.784	1.610	0.844 ***	0.502 ***	1.794 ***	0.540 ***	0.301 ***	0.487 ***
350	0.452	1.060	0.493	0.644	0.799	1.582	0.426 ***	0.765 ***	0.916 ***	0.285 ***	0.311 ***	0.504 ***
400	0.523	0.740	0.536	1.227	0.818	1.823	0.706 ***	0.437 ***	0.974	0.287 ***	0.294 ***	0.449 ***
450	0.527	1.226	0.579	0.642	0.864	1.863	0.430 ***	0.902 ***	0.910 ***	0.283 ***	0.311 ***	0.464 ***
500	0.680	1.300	0.491	0.779	0.894	1.945	0.521 ***	0.630 ***	1.385 ***	0.349 ***	0.252 ***	0.459 ***
550	0.551	0.553	0.490	0.678	0.973	2.020	0.996	0.723 ***	1.122 ***	0.272 ***	0.242 ***	0.481 ***
600	0.652	0.560	0.634	0.521	0.910	2.118	1.165 ***	1.215 ***	1.029	0.308 ***	0.299 ***	0.430 ***
650	0.613	0.618	0.525	0.554	0.913	2.274	0.992	0.946 ***	1.168 ***	0.269 ***	0.231 ***	0.401 ***
700	0.629	0.582	0.558	0.533	0.910	2.262	1.080 ***	1.047 ***	1.127 ***	0.278 ***	0.246 ***	0.402 ***
750	0.576	0.569	0.572	0.550	0.835	2.291	1.011	1.039 **	1.006	0.251 ***	0.249 ***	0.364 ***
800	0.598	0.547	0.567	0.479	0.861	2.356	1.093 ***	1.182 ***	1.055 ***	0.254 ***	0.240 ***	0.365 ***
850	0.524	0.481	0.640	0.521	0.843	2.407	1.090 ***	1.227 ***	0.818 ***	0.217 ***	0.266 ***	0.350 ***
900	0.511	0.493	0.554	0.500	0.865	2.420	1.037 *	1.107 ***	0.922 ***	0.211 ***	0.229 ***	0.357 ***
950	0.517	0.515	0.542	0.512	0.878	2.447	1.005	1.057 **	0.954 ***	0.211 ***	0.221 ***	0.359 ***
1000	0.535	0.446	0.550	0.570	0.955	2.405	1.197 ***	0.964 **	0.972	0.222 ***	0.228 ***	0.396 ***

This table presents the 1% quantile back-test results for the Swiss government 2-year bond yield data for 1990–2012. The back-test statistic described in Section 5.2 is calculated for calibration window sizes varying from 100 to 1,000 days. To make the results comparable across several data sets with different time spans, the test statistic is summed over all daily ABHS values and normalized by the number of ABHS values. Results for the displaced filtered historical simulation (DFHS) model, filtered historical simulation (FHS) model, displaced GARCH (DG) model, standard GARCH (G) model, the displaced relative change model (DRC), and a standard historical simulation (HS) model are presented in Columns 2–7. Columns 8–13 present percentage relationships between the test statistics of the different models. We also test whether the average daily ABHS of the model in the numerator and that of the model in the denominator are significantly different. The stars (*,**,***) denote significantly different values at the 10

Comparing the DFHS and displaced GARCH models (see Column 10) does not clearly point to one model outperforming the other. For very short calibration window sizes, the displaced GARCH model does better than the DFHS model, but the situation reverses for $N = 350 - 450$ and reverses again for $N = 500 - 800$ and then reverses again for $N = 850 - 1000$. However, we note that the displaced GARCH model exhibits better

Table 2
U.S. Government Spread (10Y-3M) Back Test:
Rolling Windows, 1% Quantile

Wind	DFHS	FHS	DG	G	DRC	HS	$\frac{DFHS}{FHS}$	$\frac{DG}{G}$	$\frac{DFHS}{DG}$	$\frac{DFHS}{HS}$	$\frac{DG}{HS}$	$\frac{DRC}{HS}$
100	0.489	3.372	0.425	3.072	0.452	7.936	0.145 ***	0.138 ***	1.149 ***	0.062 ***	0.054 ***	0.057 ***
150	0.581	2.016	0.459	1.985	0.484	8.261	0.288 ***	0.231 ***	1.267 ***	0.070 ***	0.056 ***	0.058 ***
200	0.693	1.318	0.519	1.125	0.627	8.620	0.526 ***	0.461 ***	1.334 ***	0.080 ***	0.060 ***	0.072 ***
250	0.648	1.352	0.635	1.244	0.722	8.938	0.479 ***	0.510 ***	1.021 ***	0.073 ***	0.071 ***	0.08 ***
300	0.729	1.420	0.587	0.902	0.745	9.127	0.513 ***	0.651 ***	1.241 ***	0.080 ***	0.064 ***	0.081 ***
350	0.631	0.787	0.641	1.116	0.832	9.352	0.801 ***	0.574 ***	0.984 ***	0.067 ***	0.069 ***	0.089 ***
400	0.646	0.854	0.773	0.938	0.833	9.589	0.757 ***	0.824 ***	0.836 ***	0.067 ***	0.081 ***	0.086 ***
450	0.711	1.279	0.852	0.993	0.923	9.802	0.555 ***	0.859 ***	0.834 ***	0.072 ***	0.087 ***	0.094 ***
500	0.677	0.972	0.646	0.968	0.898	9.854	0.697 ***	0.668 ***	1.047 **	0.069 ***	0.066 ***	0.091 ***
550	0.673	0.911	0.595	0.941	0.909	10.040	0.739 ***	0.632 ***	1.131 ***	0.067 ***	0.059 ***	0.090 ***
600	0.624	0.925	0.627	0.935	0.918	10.146	0.674 ***	0.671 ***	0.994 ***	0.061 ***	0.062 ***	0.090 ***
650	0.634	0.932	0.616	0.920	0.922	10.258	0.680 ***	0.669 ***	1.030 *	0.062 ***	0.060 ***	0.089 ***
700	0.637	0.927	0.606	0.938	0.935	10.328	0.687 ***	0.646 ***	1.051 ***	0.062 ***	0.059 ***	0.090 ***
750	0.617	0.909	0.589	0.922	0.951	10.395	0.680 ***	0.639 ***	1.048 ***	0.059 ***	0.057 ***	0.091 ***
800	0.574	0.946	0.568	0.941	0.961	10.547	0.607 ***	0.603 ***	1.011 ***	0.054 ***	0.054 ***	0.091 ***
850	0.585	0.872	0.557	0.845	0.976	10.674	0.671 ***	0.659 ***	1.050 ***	0.055 ***	0.052 ***	0.091 ***
900	0.569	0.865	0.489	0.869	1.012	10.815	0.658 ***	0.563 ***	1.165 ***	0.053 ***	0.045 ***	0.093 ***
950	0.525	0.847	0.488	0.858	0.980	10.948	0.620 ***	0.569 ***	1.075 ***	0.048 ***	0.045 ***	0.089 ***
1000	0.519	0.923	0.480	0.875	0.993	11.081	0.563 ***	0.548 ***	1.083 ***	0.047 ***	0.043 ***	0.089 ***

This table presents the 1% quantile back-test results for the U.S. government bond spreads (10Y-3M) for 1992–2012. The back-test statistic described in Section 5.2 is calculated for calibration window sizes varying from 100 to 1,000 days. To make the results comparable across several data sets with different time spans, the test statistic is summed over all daily ABHS values and normalized by the number of ABHS values. Results for the displaced filtered historical simulation (DFHS) model, filtered historical simulation (FHS) model, displaced GARCH (DG) model, standard GARCH (G) model, the displaced relative change model (DRC), and a standard historical simulation (HS) model are presented in Columns 2–7. Columns 8–13 present percentage relationships between the test statistics of the different models. We also test whether the average daily ABHS of the model in the numerator and that of the model in the denominator are significantly different. The stars (*,**,***) denote significantly different values at the 10

performance for a greater number of calibration window sizes than does the DFHS model. In Columns 11, 12, and 13, the performance of the DFHS, displaced GARCH, and DRC models is compared to the standard HS model. As already established by the graphical results in Section 6.2, on average the HS model is strongly outperformed by models having the displacement feature. However, this result is mainly driven by the fact that

the HS model fails in handling the close-to- or below-zero observations for the Swiss government bond yields around the years 2011 and 2012.

Furthermore, we test whether the results in Table 1 are statistically significantly different from each other. We perform a two-sample t-test on the difference of average daily ABHS values for two models. The results are indicated by the stars in Columns 8-13, where *,**,*** denote significantly different means at the 10

The results for the U.S. government spread data in Table 2 are in general very similar to those for the Swiss government bond yield data; the only noticeable difference is that variation in model performance is even more robust.

For example, looking at the DFHS and FHS models in Columns 2, 3, and 7 shows that the DFHS model consistently outperforms the FHS model for all calibration window sizes by having, on average, between 20 to 85% less daily deviation from the theoretical quantile. As for the displaced GARCH and GARCH models, the displaced GARCH model shows, on average, between 14 to 86% less daily deviation from the theoretical quantile. As we might by now expect, the comparison between the DFHS model and the displaced GARCH model is one of mixed results, with the displaced GARCH model showing now less daily deviation from the theoretical quantile for 16 out of 20 calibration size windows. And just as equally expected, the DFHS, the DRC, and the displaced GARCH model strongly and consistently outperform the standard HS model.

7 Conclusion

In this paper we propose the displaced relative change model in conjunction with a historical simulation approach as a solution for generating historical simulations for possibly non-positive-valued risk variables, such as interest rate spreads and government bond yields. Additionally, the displaced relative change model has desirable features from a regulatory perspective. When conducting historical simulations changing from the concept of relative changes to absolute changes due to a regime switch in the data poses a operational change between from one model to another which in general has to be approved by a regulator. The displaced relative change model automatically interpolates between relative and absolute changes driven by the needs of the data it and based on a sound methodological approach. We also extend the filtered historical simulation model originally proposed by Barone-Adesi et al. (1999) by adding our displacement feature, naming the result the "displaced filter historical simulation model." In an empirical study based on a VaR analysis we investigate the performance of the models that include this new displacement feature by applying them to Swiss government bond yield data and U.S. government spread (10Y-3M) data. We find that models with our displacement feature strongly outperform models without the displacement feature, particularly in situations when the risk factors are close to zero or even negative. Furthermore, the displaced models perform better for shorter calibration windows, a strong practical advantage. The standard historical simulation model totally fails to handle close-to-zero and non-positive risk factors in our empirical analysis.

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A   Maximum Likelihood Estimation for Itô Processes

Below is a short derivation of the maximum likelihood method used to estimate the displacement parameter.

Consider a numerical scheme

$$X(t_{i+1}) \;=\; F(\Delta W(t_i); X(t_i); \alpha_1, \ldots, \alpha_l),$$

where $\Delta W(t_i)$ is a Brownian increment over the period $[t_i, t_{i+1}]$, $F : \mathbb{R}^{1 \times 1 \times l} \to \mathbb{R}$ constitutes the model, and $\alpha_1, \ldots, \alpha_l$ are model parameters. Given observed realizations $X(t_{i+1}) = x_{i+1}$, we want to determine the model parameters $\alpha_1, \ldots, \alpha_l$ such that the conditional likelihood of the observations is maximized. That is, for each i we maximize the likelihood of $X(t_{i+1}) = x_{i+1}$ given $X(t_i) = x_i$. In other words, we maximize

$$\sum_i \log(\psi(x_{i+1}|x_i)) \tag{12}$$

where $\psi(x_{i+1}|x_i)$ is the probability density of $F(\Delta W(t_i); X(t_i); \alpha_1, \ldots, \alpha_l)$.

In general, if Y has a probability density $y \to \phi(y)$ and $Z = f(Y)$ we find from the substitution

$$\phi(y)\mathrm{d}y \;=\; \phi(f^{-1}(z)) \frac{\mathrm{d}f^{-1}(y)}{\mathrm{d}z} \mathrm{d}z.$$

that Z has a probability density

$$\psi(z) \;:=\; \phi(f^{-1}(z)) \frac{\mathrm{d}f^{-1}(z)}{\mathrm{d}z}.$$

This allows us to write Equation (12) in terms of F and the density of $\Delta W(t_i, t_{i+1})$, which is known to be $\phi(y) = \frac{1}{\sqrt{2\pi\Delta t_i}} \exp(-\frac{x^2}{2\Delta t_i})$.