Hedging mortality claims with longevity bonds

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Abstract

We study mean-variance hedging of a pure endowment, a term insurance, and general annuities by trading in a longevity bond with continuous rate payments proportional to the survival probability. In particular, we discuss the introduction of a gratification annuity as an interesting insurance product for the life insurance market. The optimal hedging strategies are determined via their Galtchouk-Kunita-Watanabe decompositions under specific, yet sufficiently general model assumptions. The results are then further illustrated by assuming a general affine structure of the mortality intensity process. The optimal hedging strategies as well as the residual hedging error of a gratification annuity and a simple life annuity are finally investigated with numerical simulations, which illustrate the nice features of the gratification annuity for the insurance industry.

1 Introduction

A life market for both traded mortality and longevity securities offers interesting risk transfer alternatives to more traditional actuarial schemes and has recently been gaining more and more attention, see e.g. Blake et al. [11]. The possibility of risk mitigation on the one hand, as well as diversification chances on the other creates a good potential for a liquid market. Yet, there are also some critical points of view, see e.g. Norberg [22].

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The respective securities typically involve a publicly accessible longevity index from which the mortality intensities for a wide range of age cohorts can be read.
In the present paper we consider two basic life insurance contracts, namely a pure endowment, i.e. a contract which pays out one unit if the insured person is alive at a pre-specified maturity, and a term insurance, i.e. a payment of one unit in case the insured person dies before the maturity of the contract. Moreover, we consider general annuities, paying out continuous rates as long as the insured person is alive. In this context, we specify a new type of (insurance) contract which we call a gratification annuity. This insurance contract would pay increasing annuity rates, proportional to the mortality probability of the insured person's own age cohort, inferred from the aforementioned longevity index. Broadly speaking, a policyholder gets gratified for being healthier or for belonging to a sicker age cohort than was originally expected. The concept of the gratification annuity may also be interesting because it allows for diversifying unsystematic insurance risk while transferring important parts of the systematic insurance risk to the policyholder, see also Norberg [21] and Wadsworth et al. [26] in this context. Therefore, the authors are convinced that such type of insurance contracts could be interesting for the life insurance market. Moreover, the description of new insurance products which are related to mortality risk can be fruitful with regard to the ongoing discussion on the introduction of government-issued longevity bonds as e.g. in Blake et al. [12].
Faced with the stochastic claims, the issuing institution is interested to hedge its risk exposure by purchasing a (coupon based) longevity bond on the financial market. This is an instrument which has continuous rate payments proportional to the survival probability, again inferable from a longevity index. Such kind of bonds have recently been discussed and recommended to be introduced to the markets, see e.g. Blake et al. [12], and have originally been proposed by Blake and Burrows [9] for hedging purposes. The combined position in one of the claims and the bond resembles various types of mortality swaps, see Dahl et al. [16] for a related concept, where the floating leg (realized mortality) is exchanged versus a fixed leg (related to some mortality projection). For a detailed overview of the securitization of mortality risk we refer to Barrieu and Albertini [3], as well as Blake et al. [10].
In the present paper we study mean-variance hedging of the respective claims by trading in the longevity bond. Regarding hedge effectiveness, this method provides solutions which are optimal by means of expected quadratic error, see e.g. Schweizer [25]. We study a general setting, where the filtration $\mathbb{G}$, describing the complete information of an insurance company, is generated by both the individual life history $\mathbb{H}$ of the insured person and a Brownian reference filtration $\mathbb{F}$, to which the hazard process is adapted to, and provide the Galtchouk-Kunita-Watanabe (GKW) decompositions of the claims and the longevity bond, and hence the optimal hedging strategies. The main mathematical problem within these model specifications is that in general the considered securities do not amount to so-called simple claims, and it is a priori not clear how to find martingale representations,

instrumental in the design of the GKW decompositions.
The mean-variance approach (coinciding in our case with local risk minimization) has already been applied to the hedging of financial insurance derivatives in several works such as Barbarin [2], Dahl and Møller [15], Dahl et al. [16], Møller [19] or Møller [20]. However there are some fundamental differences with respect to our paper that we would like to emphasize. First of all we use general techniques in order to determine the GKW-decomposition (the Föllmer-Schweizer decomposition respectively) for a given derivative as we follow the approach of Bielecki and Rutkowski [6] instead of computing the decompositions directly with respect to the underlying fundamental martingales as in Dahl and Møller [15], Dahl et al. [16], Møller [19] and Møller [20]. Moreover, our method allows to shorten the computations considerably and works in a general setting without assuming restrictive model assumptions. The results are also obtained without requiring the independence of the filtrations $\mathbb{F}$ and $\mathbb{H}$.
Furthermore, our computations do not require the existence of a mortality intensity, but hold also under the more general hypothesis that the hazard process Γ is continuous and increasing. This is due to the fact that we can apply Corollaries 5.1.1 and 5.1.3 as well as Proposition 5.1.3 of Bielecki and Rutkowski [6], since in our setting $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, τ is totally inaccessible and hypothesis (H) holds (Lemma 6.1.2 of Bielecki and Rutkowski [6]).
Regarding the hedging of life insurance contracts with longevity bonds, similar results can also be found in Barbarin [2] and Barbarin [1], where the zero-coupon longevity bond is modeled in a Heath-Jarrow-Morton framework. In contrast, we consider coupon paying longevity bonds in an intensity-based framework. This setting allows for more explicit results which can be investigated with the help of numerical simulations.
Finally another difference of our paper with respect to Dahl and Møller [15], Dahl et al. [16], Møller [19] and Møller [20] is that we do not restrict ourselves to the case of a specific affine model for the mortality rate such as the CIR model, but we compute the optimal strategy in a general affine framework for the mortality intensity dynamics.
The optimal hedging strategies are first calculated for a single life status and then generalized to hedging strategies for a whole portfolio of insured persons following the work of Biffis and Millossovich [8].
We note that our decompositions could also be derived from the results in e.g. Barbarin [2] or Blanchet-Scalliet and Jeanblanc [13] for pure endowments, in Barbarin [2] for term insurance and in Barbarin [2] or Biagini and Cretarola [4] for general annuities. In our setting, however, we work under specific but still very general model assumptions which allow to compute the GKW-decompositions explicitly. In particular, the setting allows to illustrate the results for an affine specification for the mortality intensity process. This assumption is very popular in the literature about modeling mortality intensities and has been suggested for example in Biffis [7], Biffis and Millossovich [8], Dahl and Møller [15], Dahl et al. [16] or Schrager [24]. Here, we can relate the optimal hedging strategies to the solutions of well known Riccati ODE's and analyse the results with numerical simulations.

These simulations are carried out for two specifications of the mortality intensity, following in the first case an Ornstein-Uhlenbeck process and in the second case a Feller process. Both processes are considered to be non mean-reverting, an assumption suggested by Luciano and Vigna [18] or Blake et al. [10]. In this context, we compare the optimal hedging strategies and their residual hedging-error for a gratification annuity and a simple life annuity. The results show that the gratification annuity possesses nice properties, which could make such a product interesting for the life insurance market.

The paper is organized as follows. Section 2 establishes the modeling framework which is used for obtaining the optimal hedging strategies for both a single life status in Section 3.1 and insurance portfolios in Section 3.2. The specifications to affine models of the stochastic mortality intensity are provided in Section 4. In Section 5 we show numerical illustrations of the optimal hedging strategies and their residual hedging errors at time $t = 0$ for a gratification annuity and a simple life annuity.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T > 0$ some fixed maturity. The time of decease $\tau > 0$ of a person is modeled as a totally inaccessible random time with $P(\tau > t) > 0$ for any $t \in [0, T]$. Let $H_t = \mathbb{I}_{\{\tau \leq t\}}$ be the counting process of decease and $\mathbb{H}$ the filtration generated by H. We suppose that our probability space supports also the augmented natural filtration $\mathbb{F}$ of some Brownian motion W. Let $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. We assume the following martingale invariance property, often referred to as Hypothesis (H): every $\mathbb{F}$-martingale remains a martingale in the larger filtration $\mathbb{G}$. In particular, W is a martingale in $\mathbb{G}$, and then by Lévy's characterization a Brownian motion. The survival probability process G associated to τ is supposed to fulfill

$$G_t := P\left(\tau > t \,|\, \mathcal{F}_t\right) =: e^{-\Gamma_t} \ .$$

The $\mathbb{F}$-adapted process Γ can hence be interpreted as the stochastic hazard process of the random time τ. Because τ is assumed to be totally inaccessible (and therefore avoids any $\mathbb{F}$-stopping time) the process Γ is continuous and due to Hypothesis (H) it is also increasing, see e.g. Coculescu et al. [14].

The counting process martingale M associated with the one-jump process H is given as

$$M_t = H_t - \int_0^t (1 - H_u)\, d\Gamma_u \ .$$

Note that the focus of our study is on mortality and we therefore work for simplicity with a fixed constant short rate r. More generally, one could assume that the short rate is a stochastic process, independent of the mortality. For an adept study of this, see Dahl et al. [16].

We assume that an insurance company can sell the following products at time $t = 0$:

- A *pure endowment* with present value

$$C^{pe} = e^{-rT}\mathbb{I}_{\{\tau>T\}} = e^{-rT}(1 - H_T).$$

  Here one unit of cash will be paid to the policyholder, given that the insured person is still alive at maturity T.

- A *term insurance* with present value

$$C^{ti} = e^{-r\tau}\mathbb{I}_{\{\tau\leq T\}} = e^{-r\tau}H_T,$$

  where one unit is paid at the event of decease, given that this happens before maturity.

- A *general annuity* with present value

$$C^Y = \int_0^T e^{-ru}\,(1 - H_u)\,Y_u\,du\,, \tag{1}$$

  where Y is a positive, bounded, $\mathbb{F}$-adapted stochastic process. Here, the insured person receives general annuity payments as long as she is alive with at a rate given by Y.
  In particular, we introduce a new insurance product, which we call *gratification annuity*, where $Y_t = 1 - G_t$, $t \in [0, T]$. As G_t can be inferred from the longevity index which itself bases on realized mortality of some representative group, such an instrument rewards an insured person's higher longevity (e.g. due to healthier life style) as was originally expected.
  In order to compare this product's characteristics numerically to an existing insurance product, we furthermore consider a *simple life annuity*, where $Y_t = 1$ for all $t \in [0, T]$.

We now assume that it is possible to trade on the financial market in an instrument called a longevity bond which has present value

$$B_t = \int_0^t e^{-ru}G_u\,du.$$

The payment, generated by this (coupon based) bond, has also the form of an annuity where the declining rate is given by the survival probability for the age cohort of the insured person. It does not depend on her individual life history, in contrast to the payouts of the considered claims. The (discounted) value process associated with the longevity bond is thus given by the conditional expectation

$$V_t = E\left[\int_0^T e^{-ru}G_u\,du \,\middle|\, \mathcal{G}_t\right].$$

Here we have implicitly assumed that P is some pricing measure, reflecting the market price of risk. Our goal is now to hedge the risk exposure from having sold either a pure endowment, a term insurance or a general annuity by trading dynamically in the longevity bond with value process V.
Let us first collect some technical notations and assumptions: we assume

$$e^{\Gamma_T} \in L^2(P). \tag{2}$$

The spaces $L^2(W)$, $L^2(M)$ consist of all predictable θ, ψ such that

$$E\left[\int_0^T \theta_s^2 \, ds\right] < \infty, \quad E\left[\int_0^T \psi_s^2 \, d\Gamma_s\right] < \infty.$$

The space Θ of admissible strategies consists of all predictable ϑ such that

$$E\left[\int_0^T \vartheta_s^2 \, d\langle V\rangle_s\right] < \infty.$$

If $\vartheta \in \Theta$, then $\int_0^\cdot \vartheta_s \, dV_s$ is a square-integrable martingale.

3 Dynamic hedging with longevity bonds

3.1 Single life status

We start the analysis of optimal hedging strategies by considering a single life status. We shall use the mean-variance hedging approach, see Schweizer [25] for an overview: for a given discounted claim $C \in L^2(P)$, we want to solve

$$\min_{c,\vartheta} E\left[\left(C - c - \int_0^T \vartheta_s \, dV_s\right)^2\right],$$

where we minimize over all constants c and $\vartheta \in \Theta$. It results that the fair price in this framework is given by $c = E[C]$, and the optimal hedging strategy $\vartheta^* \in \Theta$ can be found via the Galtchouk-Kunita-Watanabe (GKW) decomposition

$$E[C \mid \mathcal{G}_t] = c + \int_0^t \vartheta_s^* \, dV_s + V_t^\perp, \tag{3}$$

where $V^\perp$ is a martingale strongly orthogonal to V (i.e. $VV^\perp$ is a local martingale). It follows by the uniqueness of the GKW decomposition and strong orthogonality that once we have found a decomposition as in (3) in the sense that the terms on the r.h.s. are local martingales, then they are automatically square-integrable martingales.

The decomposition (3) of the martingales associated to the various claims can be found by simple algebra once we have established representations of the martingales $E\left[C \mid \mathcal{G}_{\cdot}\right]$ and V in terms of stochastic integrals with respect to the Brownian motion W and the counting process martingale M.
To establish these representation formulas, we define

$$L_t = (1 - H_t) e^{\Gamma_t}$$

and note that $H_0 = 0$ and $L_0 = 1$, moreover $L_t = 0$ for $t \geq \tau$. By Proposition 5.1.3, equation (5.28), of Bielecki and Rutkowski [6], L is the stochastic exponential of $(-M)$ and satisfies the equation

$$dL_t = -L_{t-}\, dM_t.$$

Definition 1 *A* simple claim *is a random variable of the form* $(1 - H_T)\, Z$ *for some integrable* $\mathcal{F}_T$*-measurable random variable* Z *and* $T \geq 0$*. We define the* $(\mathcal{F}_t)$*-martingale* U *and the predictable process* ψ *by*

$$U_t := E\left[e^{-\Gamma_T} Z \,\middle|\, \mathcal{F}_t\right] = E\left[e^{-\Gamma_T} Z\right] + \int_0^t \psi_s \, dW_s, \tag{4}$$

where the second equality follows from the martingale representation theorem with respect to a Brownian filtration.

For simple claims, a martingale representation can then be found by integration by parts as in the proof of Proposition 5.2.2 in Bielecki and Rutkowski [6]:

Proposition 2 *Let* $X = (1 - H_T)\, Z$*,* $T \geq 0$*, be a simple claim. Then*

$$E\left[X \mid \mathcal{G}_t\right] = E\left[e^{-\Gamma_T} Z\right] + \int_0^t \zeta_s^W \, dW_s + \int_{0+}^t \zeta_s^M \, dM_s$$

where $\zeta_s^W = \psi_s L_{s-}$ *and* $\zeta_s^M = -L_{s-} U_s$*.*

Obviously, the pure endowment C^{pe} is a simple claim. However, both term insurance as well as general annuities have to be dealt with differently.
A general representation result, see Bielecki and Rutkowski [6] Proposition 5.2.2, can be obtained by approximating $\mathcal{G}_T$-measurable random variables in $L^2(P)$ by simple claims and by using that the spaces of stochastic integrals of admissible integrands in $L^2(W)$, or $L^2(M)$ respectively, are closed in $L^2(P)$. The result then states that each square-integrable $(\mathcal{G}_t)$-martingale N can be written as

$$N_t = N_0 + \int_0^{\tau \wedge t} \zeta_s^W \, dW_s + \int_{0+}^{\tau \wedge t} \zeta_s^M \, dM_s \,,$$

for admissible integrands ζ^W and ζ^M. It does not, however, give any information how to calculate the integrands for claims which are not simple.

Proposition 3 *The GKW-decomposition for a* ***pure endowment*** C^{pe} *is*

$$E\left[e^{-rT}(1-H_T)\,\middle|\,\mathcal{G}_t\right] = c^{pe} + \int_0^t \alpha_s^W\,dW_s + \int_{0+}^t \alpha_s^M\,dM_s\,, \tag{5}$$

where the predictable integrands α^W *and* α^M *are given as*

$$\begin{aligned} \alpha_s^W &= \psi_s L_{s-}\,, \\ \alpha_s^M &= -U_s L_{s-}\,. \end{aligned} \tag{6}$$

and $c^{pe} = E\left[e^{-rT}e^{-\Gamma_T}\right]$*.*
Here ψ *corresponds to the integrand in* (4) *for the choice* $Z = e^{-rT}$*.*

Proof. C^{pe} is a simple claim with $Z = e^{-rT}$. Hence, Proposition 2 yields the result. ■

We turn now our attention to term insurance. For completeness, we provide its Galtchouk-Kunita-Watanabe decomposition in our setting. The results could also be derived by applying some results of Barbarin [2] to our setting.
Let us first observe that by martingale representation, there exists a constant c^{ti} and $\chi \in L^2(W)$ such that

$$E\left[\int_0^T e^{-ru}e^{-\Gamma_u}\,d\Gamma_u\,\middle|\,\mathcal{F}_t\right] = c^{ti} + \int_0^t \chi_u\,dW_u. \tag{7}$$

Proposition 4 *The GKW-decomposition for a* ***term insurance*** C^{ti} *is*

$$E\left[e^{-r\tau}H_T\,\middle|\,\mathcal{G}_t\right] = c^{ti} + \int_0^t \beta_s^W\,dW_s + \int_{0+}^t \beta_s^M\,dM_s\,, \tag{8}$$

where

$$\begin{aligned} \beta_s^W &= L_{s-}\chi_s, \\ \beta_u^M &= e^{-r(s\wedge\tau)} - L_{s-}\left(c^{ti} + \int_0^s \chi_u\,dW_u - \int_0^s e^{-ru}e^{-\Gamma_u}\,d\Gamma_u\right)\,. \end{aligned} \tag{9}$$

Proof. We write

$$E\left[e^{-r\tau}H_T\,\middle|\,\mathcal{G}_t\right] = H_t E\left[e^{-r\tau}H_T\,\middle|\,\mathcal{G}_t\right] + (1-H_t)\,E\left[e^{-r\tau}H_T\,\middle|\,\mathcal{G}_t\right] \tag{10}$$

and find the canonical decompositions of the two terms on the r.h.s. into a local martingale and a finite variation part separately.

Since $H_t H_T = H_t$, and $H_t e^{-r\tau}$ is $\mathcal{G}_t$-measurable, we get for the first term by integration by parts

$$
\begin{aligned}
H_t E\left[e^{-r\tau} H_T \,\middle|\, \mathcal{G}_t\right] &= H_t e^{-r\tau} = H_t e^{-r(t\wedge\tau)} \\
&= 0 + \int_0^t H_{s-}\, de^{-r(s\wedge\tau)} + \int_0^t e^{-r(s\wedge\tau)}\, dH_s \\
&= 0 + \int_0^t e^{-r(s\wedge\tau)}\, dM_s + X_t^1\ ,
\end{aligned} \tag{11}
$$

where

$$
X_t^1 = \int_0^t e^{-r(s\wedge\tau)}(1-H_s)d\Gamma_s = \int_0^t e^{-rs}(1-H_s)d\Gamma_s\ .
$$

Note that we used that $H_s = 0$ on $[0,\tau)$, and that $H_s = M_s + \Gamma_{s\wedge\tau}$.
For the second term of the r.h.s in (10), we get by Corollary 5.1.3 of Bielecki and Rutkowski [6] that

$$
\begin{aligned}
(1-H_t)\, E\left[e^{-r\tau} H_T \,\middle|\, \mathcal{G}_t\right] &= (1-H_t)\, E\left[\int_t^T e^{-rs} e^{\Gamma_t - \Gamma_s}\, d\Gamma_s \,\middle|\, \mathcal{F}_t\right] \\
&= L_t E\left[\int_t^T e^{-rs} e^{-\Gamma_s}\, d\Gamma_s \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

Again by integration by parts as well as the martingale representation (7),

$$
\begin{aligned}
L_t E\left[\int_t^T e^{-rs} e^{-\Gamma_s}\, d\Gamma_s \,\middle|\, \mathcal{F}_t\right] &= L_t \left(E\left[\int_0^T e^{-rs} e^{-\Gamma_s}\, d\Gamma_s \,\middle|\, \mathcal{F}_t\right] - \int_0^t e^{-rs} e^{-\Gamma_s}\, d\Gamma_s \right) \\
&= c^{ti} + \int_0^t \phi_s\, dW_s + \int_{0+}^t \nu_s\, dM_s + X_t^2\,,
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
\phi_s &= L_{s-}\chi_s, \\
\nu_s &= -L_{s-}\left(c^{ti} + \int_0^s \chi_v\, dW_v - \int_0^s e^{-rv} e^{-\Gamma_v}\, d\Gamma_v\right)\ ,
\end{aligned}
$$

and

$$
X_t^2 = -\int_0^t L_s e^{-rs} e^{-\Gamma_s} d\Gamma_s = -X_t^1\ .
$$

The result now follows by combining (11) and (12). ■

Now we turn to a general annuity. As stated in the introduction, the following decompositions could also be derived by applying results in Barbarin [2] or Biagini and Cretarola [4] to our setting. Here they are computed explicitly under our specific model assumptions. By martingale representation, for each $u \in [0,T]$ there exists a constant c_u^Y and a predictable process $(\theta_{u,s}^Y)_{s\in[0,T]} \in L^2(W)$, with $\theta_{u,s}^Y = 0$ if $s > u$, such that

$$
\begin{aligned}
E\left[e^{-ru} Y_u e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] &= c_u^Y + \int_0^{t\wedge u} \theta_{u,s}^Y \, dW_s \\
&= c_u^Y + \int_0^t \theta_{u,s}^Y \mathbb{I}_{[0,u]}(s)\, dW_s. \qquad (13)
\end{aligned}
$$

We set $c^Y = \int_0^T c_u^Y \, du < \infty$. Note that the $\int_0^\cdot \theta_{u,s}^Y \mathbb{I}_{[0,u]}(s)\, dW_s$ are bounded martingales, uniformly in u.

Proposition 5 *The GKW-decomposition of a* ***general annuity*** C^Y *is*

$$
E\left[\int_0^T e^{-ru}\,(1-H_u)\,Y_u\,du \,\middle|\, \mathcal{G}_t\right] = c^Y + \int_0^t \rho_s^W \, dW_s + \int_{0+}^t \rho_s^M \, dM_s\,, \qquad (14)
$$

where the predictable integrands are given as

$$
\begin{aligned}
\rho_s^W &= L_{s-} \int_s^T \theta_{u,s}^Y \, du, \qquad (15)\\
\rho_s^M &= -L_{s-} \int_s^T \left(c_u^Y + \int_0^s \theta_{u,v}^Y \, dW_v \right) du\,.
\end{aligned}
$$

Proof. By Equation (5.13) of Bielecki and Rutkowski [6], we have for $u \in (t,T]$

$$
\begin{aligned}
E\left[e^{-ru}\,(1-H_u)\,Y_u \,\middle|\, \mathcal{G}_t\right] &= (1-H_t)\, E\left[e^{-ru} Y_u e^{\Gamma_t - \Gamma_u} \,\middle|\, \mathcal{F}_t\right] \\
&= L_t E\left[e^{-ru} Y_u e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

For $u \in [0,t]$ we have

$$
\begin{aligned}
E\left[e^{-ru}\,(1-H_u)\,Y_u \,\middle|\, \mathcal{G}_t\right] &= (1-H_u)\, E\left[e^{-ru} Y_u \,\middle|\, \mathcal{F}_t\right] \\
&= L_u E\left[e^{-ru} Y_u e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

Hence, for every $u \in [0,T]$ we have

$$
\begin{aligned}
E\left[e^{-ru}\,(1-H_u)\,Y_u \,\middle|\, \mathcal{G}_t\right] &= L_{u\wedge t} E\left[e^{-ru} Y_u e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] \\
&= L_t^u E\left[e^{-ru} Y_u e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right],
\end{aligned}
$$

where L^u is the process L stopped at time $u \in [0,T]$. By integration by parts and (13),

$$L_t^u E\left[e^{-ru} Y_u e^{-\Gamma_u} \mid \mathcal{F}_t\right] = c_u^Y + \int_0^t \phi_{u,s}\, dW_s + \int_{0+}^t \nu_{u,s}\, dM_s\,, \tag{16}$$

where

$$\begin{aligned}
\phi_{u,s} &= L_{s-}\, \theta_{u,s}^Y \mathbb{I}_{[0,u]}(s), \\
\nu_{u,s} &= -L_{s-} \left(c_u^Y + \int_0^s \theta_{u,v}^Y \mathbb{I}_{[0,u]}(v)\, dW_v \right) \mathbb{I}_{[0,u]}(s).
\end{aligned}$$

By Fubini, as well as the Itô-isometry,

$$\begin{aligned}
E\left[\int_0^T \int_0^T \phi_{u,s}^2\, du\, ds\right] &= \int_0^T E\left[\int_0^T \phi_{u,s}^2\, ds\right] du \\
&= \int_0^T E\left[\left(\int_0^T \phi_{u,s}\, dW_s\right)^2\right] du \\
&\leq C_1 T,
\end{aligned}$$

where, by Lemma 10 (see Appendix),

$$C_1 = \sup_{0 \leq u \leq T} \left\| \int_0^T \phi_{u,s}\, dW_s \right\|_{L^2}^2 < \infty.$$

Moreover,

$$\begin{aligned}
E\left[\int_0^T \int_0^T \nu_{u,s}^2\, du\, d\Gamma_s\right] &= \int_0^T E\left[\int_0^T \nu_{u,s}^2\, d\Gamma_s\right] du \\
&= \int_0^T E\left[\left(\int_0^T \nu_{u,s}\, dM_s\right)^2\right] du \\
&\leq C_2 T,
\end{aligned}$$

where, by Lemma 11 (see Appendix),

$$C_2 = \sup_{0 \leq u \leq T} \left\| \int_0^T \nu_{u,s}\, dM_s \right\|_{L^2}^2 < \infty.$$

Hence we may apply the stochastic Fubini theorem (see Protter [23], Theorem IV.65) to get from (16)

$$
\begin{aligned}
E\left[\int_0^T e^{-ru}\,(1-H_u)\,Y_u\,du\,\middle|\,\mathcal{G}_t\right] =& \int_0^T E\left[e^{-ru}\,(1-H_u)\,Y_u \,\middle|\, \mathcal{G}_t\right]\,du \\
=& \int_0^T \left(c_u + \int_0^t \phi_{u,s}\,dW_s + \int_{0+}^t \nu_{u,s}\,dM_s\right)\,du \\
=& c^Y + \int_0^t \int_0^T \phi_{u,s}\,du\,dW_s + \int_{0+}^t \int_0^T \nu_{u,s}\,du\,dM_s \\
=& c^Y + \int_0^t \rho_s^W\,dW_s + \int_{0+}^t \rho_s^M\,dM_s\,,
\end{aligned}
$$

where the predictable integrands ρ^W, ρ^M are as desired. ■

We have already introduced a new type of insurance product, namely a gratification annuity, which we think of as an interesting insurance product for the life insurance market. In order to compare this product to an existing annuity, we also derive the GKW-decomposition of a simple life annuity. The results are given in the following corollary.

Corollary 6 *The GKW-decompositions of a* gratification annuity

$$C^{ga} = \int_0^T e^{-ru}(1-H_u)(1-G_u)\,du$$

and a simple life annuity

$$C^{la} = \int_0^T e^{-ru}(1-H_u)\,du$$

are

$$E\left[\int_0^T e^{-ru}\,(1-H_u)\,(1-G_u)\,du\,\middle|\,\mathcal{G}_t\right] = c^{ga} + \int_0^t \gamma_s^W\,dW_s + \int_{0+}^t \gamma_s^M\,dM_s\,, \tag{17}$$

$$E\left[\int_0^T e^{-ru}\,(1-H_u)\,du\,\middle|\,\mathcal{G}_t\right] = c^{la} + \int_0^t \delta_s^W\,dW_s + \int_{0+}^t \delta_s^M\,dM_s\,, \tag{18}$$

where the predictable integrands are given as

$$\gamma_s^W = L_{s-}\int_s^T \theta_{u,s}^{ga}\,du, \tag{19}$$

$$\gamma_s^M = -L_{s-}\int_s^T \left(c_u^{ga} + \int_0^s \theta_{u,v}^{ga}\,dW_v\right)\,du \tag{20}$$

and

$$\delta_s^W = L_{s-} \int_s^T \theta_{u,s}^{la} \, du, \tag{21}$$

$$\delta_s^M = -L_{s-} \int_s^T \left(c_u^{la} + \int_0^s \theta_{u,v}^{la} \, dW_v \right) du \tag{22}$$

and the processes θ_u^{ga} *and* θ_u^{la} *as well as the constants* c^{ga} *and* c^{la} *are given through the martingale representations* (13) *for the respective choice of* Y.

Proof. The results are straightforward applications of Proposition 5 with the positive, bounded and $\mathbb{F}$-adapted processes Y with $Y_t = 1 - G_t$ and $Y_t = 1$, $t \in [0,T]$, respectively. ■

Finally, we turn to the longevity bond. By martingale representation, for each $u \in [0,T]$ there exists a constant k_u and a predictable process $(\xi_{u,s})_{s\in[0,T]}$, with $\xi_{u,s} = 0$ for $s > u$, such that

$$\begin{aligned} E\left[e^{-ru} G_u \,\middle|\, \mathcal{F}_t \right] = E\left[e^{-ru} e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t \right] &= k_u + \int_0^{u\wedge t} \xi_{u,s} \, dW_s \\ &= k_u + \int_0^t \xi_{u,s} \mathbb{I}_{[0,u]}(s) \, dW_s. \end{aligned} \tag{23}$$

We set $c = \int_0^T k_u \, du$.

Proposition 7 *The GKW-decomposition of the* ***longevity bond*** *is*

$$V_t = E\left[\int_0^T e^{-ru} G_u \, du \,\middle|\, \mathcal{G}_t \right] = c + \int_0^t \xi_s \, dW_s,$$

where the predictable integrand ξ *is given as*

$$\xi_s = \int_s^T \xi_{u,s} \, du. \tag{24}$$

Proof. The discounted survival probability $e^{-ru} G_u$ is bounded and $\mathcal{F}_u$-measurable for every $u \in [0,T]$. Due to Hypothesis (H) we then get

$$\begin{aligned} E\left[e^{-ru} G_u \,\middle|\, \mathcal{G}_t \right] = E\left[e^{-ru} G_u \,\middle|\, \mathcal{F}_t \right] &= k_u + \int_0^{u\wedge t} \xi_{u,s} \, dW_s \\ &= k_u + \int_0^t \xi_{u,s} \mathbb{I}_{[0,u]}(s) \, dW_s. \end{aligned}$$

Since G is bounded by one, we have that the $\int_0^{\cdot} \xi_{u,s} \mathbb{I}_{[0,u]}(s)\, dW_s$ are bounded martingales, uniformly in u. We can again apply stochastic Fubini to get

$$E\left[\int_0^T e^{-ru} G_u\, du \,\middle|\, \mathcal{G}_t\right] = \int_0^T E\left[e^{-ru} G_u \mid \mathcal{G}_t\right] du$$
$$= c + \int_0^t \xi_s\, dW_s$$

where the predictable integrand ξ is given as

$$\xi_s = \int_s^T \xi_{u,s}\, du\,.$$

This ends the proof. ■

Summing up, the various discounted claim payoffs allow for a representation

$$C = c^C + \int_0^T \epsilon_s^{C,W}\, dW_s + \int_{0+}^T \epsilon_s^{C,M}\, dM_s\,,$$

where the integrands $\epsilon^{C,W}$, $\epsilon^{C,M}$ as well as the constant c^C are claim-specific and have been obtained in the foregoing propositions. Moreover, the longevity bond, which serves as hedging instrument, has representation

$$V_t = c + \int_0^t \xi_s\, dW_s\ .$$

As we have seen, the integrands $\epsilon^{C,W}$, $\epsilon^{C,M}$, ξ can all be computed and therefore be considered as known quantities. Our goal is now to find the Galtchouk-Kunita-Watanabe decomposition

$$E\left[C \mid \mathcal{G}_t\right] = c^C + \int_0^t \vartheta_u^{*,C}\, dV_u + V_t^{\perp}, \tag{25}$$

where $V^{\perp}$ is a square-integrable martingale, strongly orthogonal to V with decomposition

$$V_t^{\perp} = \int_{0+}^t \epsilon_s^{C,M}\, dM_s.$$

Here $c^C + \int_0^t \vartheta_u^{*,C}\, dV_u$ can be interpreted as the part of the risk that can be perfectly replicated by means of our optimal hedging strategy $\vartheta^{*,C}$, and $V_t^{\perp}$ as the part of the risk that is totally unhedgeable.
The integrand $\vartheta^{*,C}$ in the Galtchouk-Kunita-Watanabe decomposition (25) is determined uniquely by the equation

$$\vartheta^{*,C} \xi = \epsilon^{C,W}\,. \tag{26}$$

Here, uniqueness is understood modulo the following equivalence relation: if $\vartheta, \psi \in \Theta$, then

$$\vartheta \sim \psi \quad \text{if} \qquad \int_0^T (\vartheta_t - \psi)^2 \, d[V]_t = 0.$$

In particular, the predictable process $\vartheta^* \in \Theta$ gives the unique mean-variance hedging strategy of the claim by trading in the underlying longevity bond.

3.2 Insurance Portfolio

For an insurance company it is important to hedge the risk of a whole insurance portfolio rather than the risk of a single insurance contract. Following ideas of Biffis and Millossovich [8], we extend the results of the previous subsection to hedging strategies for an insurance portfolio.
Let $I^{pe} = \{x_1, ..., x_n\}, I^{ti} = \{y_1, ..., y_m\}, I^Y = \{z_1, ..., z_k\}$ denote the set of insured persons having purchased coverage through pure endowment, term insurance and/or general annuity respectively. For either of those sets we consider a finite counting measure ϱ^{pe}, ϱ^{ti}, ϱ^Y on $(I^{pe}, \mathcal{P}(I^{pe}))$, $(I^{ti}, \mathcal{P}(I^{ti}))$, $(I^Y, \mathcal{P}(I^Y))$, respectively, allowing the insurance company to weight the risk exposures of the different insured persons to the overall portfolio risks differently.
For every $x \in I^{\cdot}$, we consider its random time of decease τ^x with distribution driven by the continuous, increasing and $\mathbb{F}$-adapted hazard process $(\Gamma_t^x)_{t\in[0,T]}$, see Section 2. We write $H_t^x = \mathbb{I}_{\{\tau^x \le t\}}$, $G_t^x = \mathbb{P}(\tau^x > t \,|\, \mathcal{F}_t) = e^{-\Gamma_t^x}$ as well as $L_t^x = (1 - H_t^x)e^{\Gamma_t^x}$ and $M_t^x = H_t^x - \int_0^t (1 - H_s^x) d\Gamma_s^x$.
Of course, the insurance company is aware of each single life status $x \in I^{\cdot}$ of its portfolios and we have to expand the filtration setting of our probability space. Denoting by $\mathbb{H}^x$ the natural filtration, generated by the processes $(H_t^x)_{t\in[0,T]}$, we assume the insurance company's complete "portfolio"-information to be represented by the filtrations $\mathbb{G}^{\cdot} = \mathbb{F} \vee \bigvee_{x\in I^{\cdot}} \mathbb{H}^x$. In this context we extend the martingale invariance property (Hypothesis (H)) to the filtrations $\mathbb{G}^{\cdot}$, i.e. we assume every $\mathbb{F}$-local martingale to be also a $\mathbb{G}^{\cdot}$-local martingale.
By $C^{\cdot,x}$ we denote the single life (discounted) payoffs of pure endowment, term insurance and general annuity, associated with $x \in I^{\cdot}$. The weighted, discounted portfolio payoffs

$C^{P,pe}$, $C^{P,ti}$ and $C^{P,Y}$ up to time T are then given as

$$
\begin{aligned}
C^{P,pe} &= \sum_{i=1}^{n} C^{pe,x_i}\, \varrho^{pe}(x_i) = \sum_{i=1}^{n} e^{-rT}(1 - H_T^{x_i})\, \varrho^{pe}(x_i)\;, \\
C^{P,ti} &= \sum_{j=1}^{m} C^{ti,y_j}\, \varrho^{ti}(y_j) = \sum_{j=1}^{m} e^{-r\tau^{y_j}} H_T^{y_j}\, \varrho^{ti}(y_j)\;, \\
C^{P,Y} &= \sum_{l=1}^{k} C^{Y,z_l}\, \varrho^{Y}(Z_l) = \sum_{l=1}^{k} \int_0^T e^{-ru}(1 - H_u^{z_l}) Y_u^{z_l}\, du\, \varrho^{Y}(z_l)\;.
\end{aligned}
$$

In order to apply the results of Section 3.1 for a single life status to the weighted, discounted portfolio payoffs, we assume the following conditional independence relation.

Assumption 8 *We assume that the family $(\tau^x)_{x \in I^{\cdot}}$ is conditionally independent given $\mathcal{F}_T$.*

With the presence of the general weighting functions $\varrho^{\cdot}$, we also have to adopt additional integrability conditions. For every $x \in I^{\cdot}$ we denote by ψ^x, χ^x and θ_u^x the predictable processes of the respective martingale representations (4), (7) and (13), related to x. Analogously we write $c^{x,pe}$, $c^{x,ti}$, $c^{x,Y}$, $\alpha^{x,W}$, $\alpha^{x,M}$, $\beta^{x,W}$, $\beta^{x,M}$, $\rho^{x,W}$ and $\rho^{x,M}$ for the constants and integrands in the Galtchouk-Kunita-Watanabe decompositions (5), (8) and (14), related to x.
Now we are ready to provide the Galtchouk-Kunita-Watanabe decompositions in analogy to the previous subsection.

Proposition 9 *The Galtchouk-Kunita-Watanabe decompositions of the weighted, discounted portfolio payoffs of pure endowments, term insurances or general annuities are given as*

$$
\begin{aligned}
E\left[C^{P,pe} \,\middle|\, \mathcal{G}_t^{pe}\right] &= c^{P,pe} + \int_0^t \alpha_s^{P,W}\, dW_s + \sum_{i=1}^{n} \int_{0+}^t \alpha_s^{x_i,M}\, \varrho^{pe}(x_i)\, dM_s^{x_i}\;, \\
E\left[C^{P,ti} \,\middle|\, \mathcal{G}_t^{ti}\right] &= c^{P,ti} + \int_0^t \beta_s^{P,W}\, dW_s + \sum_{j=1}^{m} \int_{0+}^t \beta_s^{y_j,M}\, \varrho^{ti}(y_j)\, dM_s^{y_j}\;, \\
E\left[C^{P,Y} \,\middle|\, \mathcal{G}_t^{Y}\right] &= c^{P,Y} + \int_0^t \rho_s^{P,W}\, dW_s + \sum_{l=1}^{k} \int_{0+}^t \rho_s^{z_l,M}\, \varrho^{Y}(z_l)\, dM_s^{z_l}\;,
\end{aligned}
$$

where $c^{P,pe} = \sum_{i=1}^{n} c^{x_i,pe}$, $c^{P,ti} = \sum_{j=1}^{m} c^{y_j,ti}$, $c^{P,pe} = \sum_{l=1}^{k} c^{z_l,Y}$ and the predictable inte-

grands $\alpha^{P,W}$*,* $\beta^{P,W}$ *and* $\rho^{P,W}$ *are given as*

$$\alpha_s^{P,W} = \sum_{i=1}^{n} \psi_s^{x_i} L_{s-}^{x_i}\, \varrho^{pe}(x_i)\,,$$
$$\beta_s^{P,W} = \sum_{j=1}^{m} L_{s-}^{y_j} \chi_s^{y_j}\, \varrho^{ti}(y_j)\,,$$
$$\rho_s^{P,W} = \sum_{l=1}^{k} L_{s-}^{z_l} \int_s^T \theta_{u,s}^{z_l}\, du\, \varrho^{Y}(z_l)\,,$$

respectively.

Proof. We illustrate the proof only for the weighted, discounted portfolio payoff of pure endowments, as the proofs for term insurances and general annuities are identical. We have

$$\begin{aligned}
E\left[C^{P,pe} \,\middle|\, \mathcal{G}_t^{pe}\right] &= E\left[\sum_{i=1}^{n} C^{x_i,pe}\, \varrho^{pe}(x_i) \,\middle|\, \mathcal{G}_t^{pe}\right] \\
&= \sum_{i=1}^{n} E\left[C^{x_i,pe} \mid \mathcal{G}_t^{pe}\right] \varrho^{pe}(x_i) \\
&= \sum_{i=1}^{n} E\left[C^{x_i,pe} \mid \mathcal{F}_t \vee \mathcal{H}_t^{x_i}\right] \varrho^{pe}(x_i) \qquad (27) \\
&= \sum_{i=1}^{n} \left(c^{x_i} + \int_0^t \alpha_s^{x_i,W}\, dW_s + \int_{0+}^t \alpha_s^{x_i,M}\, dM_s^{x_i}\right) \varrho^{pe}(x_i) \\
&= c_4 + \int_0^t \sum_{i=1}^{n} \alpha_s^{x_i,W}\, \varrho^{pe}(x_i)\, dW_s + \sum_{i=1}^{n} \int_{0+}^t \alpha_s^{x_i,M}\, dM_s^{x_i}\, \varrho^{pe}(x_i) \\
&= c_4 + \int_0^t \alpha_s^{P,W}\, dW_s + \sum_{i=1}^{n} \int_{0+}^t \alpha_s^{x_i,M}\, dM_s^{x_i}\, \varrho^{pe}(x_i)\,,
\end{aligned}$$

where (27) follows by Assumption 8. Note that as M^x and W are orthogonal martingales for all $x \in I^{pe}$, so are $\sum_{i=1}^{n} \int_{0+}^{t} \alpha_s^{x_i,M}\, \varrho^{pe}(x_i)\, dM_s^{x_i}$ and W. ■

4 Affine Models

In this section we assume the hazard process Γ to be absolutely continuous with respect to the Lebesgue measure, i.e. to be of the form $\Gamma_t = \int_0^t \mu_s\, ds$. The stochastic intensity process $\mu = (\mu_t)_{t\in[0,T]}$ is assumed to be $\mathbb{F}$-progressively measurable, non-negative and affine. Moreover, we assume

$$C := \sup_{u\in[0,T]} E\left[\mu_u^2\right] \;<\; \infty \,. \tag{28}$$

The derivation of the hedging strategies then boils down to solving well known Riccati ODEs.
In more detail let μ follow the dynamics

$$\begin{cases} d\mu_t = \delta(t,\mu_t)dt + \sigma(t,\mu_t)dW_t \\ \mu_0 = \overline{\mu} \end{cases}$$

for some $\overline{\mu} > 0$, where the drift function δ as well as the instantaneous variance function σ^2 are assumed to have affine dependence on μ, i.e.

$$\begin{aligned} \delta(t,\mu_t) &= d_0(t) + d_1(t)\mu_t \,, \\ \sigma^2(t,\mu_t) &= v_0(t) + v_1(t)\mu_t \,, \end{aligned}$$

with the deterministic functions d_0, d_1, v_0 and v_1 being bounded and continuous.
It is then a well known fact, see Biffis [7], that for $u \in (t,T]$ we have

$$E\left[e^{-\int_t^u \mu_s\, ds} \,\middle|\, \mathcal{F}_t \right] = e^{\alpha_u(t)+\beta_u(t)\mu_t} \,,$$

where the functions α_u and β_u solve the following ODEs

$$\begin{aligned} &\begin{cases} \frac{d\beta_u}{dt}(t) = 1 - d_1(t)\beta_u(t) - \frac{1}{2}v_1(t)\beta_u^2(t) \\ \beta_u(u) = 0 \,, \end{cases} \\ &\begin{cases} \frac{d\alpha_u}{dt}(t) = -d_0(t)\beta_u(t) - \frac{1}{2}v_0(t)\beta_u^2(t) \\ \alpha_u(u) = 0 \,. \end{cases} \end{aligned} \tag{29}$$

Similarly, for $u \in (t,T]$ we have

$$E\left[e^{-2\int_t^u \mu_s\, ds} \,\middle|\, \mathcal{F}_t \right] = e^{\tilde{\alpha}_u(t)+\tilde{\beta}_u(t)\mu_t} \,,$$

where the functions $\tilde{\alpha}_u$ and $\tilde{\beta}_u$ solve the ODEs of the following form

$$\begin{aligned} &\begin{cases} \frac{d\tilde{\beta}_u}{dt}(t) = 2 - d_1(t)\tilde{\beta}_u(t) - \frac{1}{2}v_1(t)\tilde{\beta}_u^2(t) \\ \tilde{\beta}_u(u) = 0 \,, \end{cases} \\ &\begin{cases} \frac{d\tilde{\alpha}_u}{dt}(t) = -d_0(t)\tilde{\beta}_u(t) - \frac{1}{2}v_0(t)\tilde{\beta}_u^2(t) \\ \tilde{\alpha}_u(u) = 0 \,. \end{cases} \end{aligned}$$

Finally for $u \in (t,T]$ we have

$$E\left[e^{-\int_t^u \mu_s\, ds}\mu_u \,\middle|\, \mathcal{F}_t\right] = e^{\alpha_u(t)+\beta_u(t)\mu_t}\left(\hat{\alpha}_u(t) + \hat{\beta}_u(t)\mu_t\right) .$$

Here the functions α_u, β_u are again solutions to (29) and $\hat{\alpha}_u$ and $\hat{\beta}_u$ are derived by differentiating (29) with respect to u and hence solve the following ODEs.

$$\begin{cases} \frac{d\hat{\beta}_u}{dt}(t) = -d_1(t)\hat{\beta}_u(t) - v_1(t)\beta_u(t)\hat{\beta}_u(t) \\ \hat{\beta}_u(u) = 1 \ , \end{cases}$$

$$\begin{cases} \frac{d\hat{\alpha}_u}{dt}(t) = -d_0(t)\hat{\beta}_u(t) - v_0(t)\beta_u(t)\hat{\beta}_u(t) \\ \hat{\alpha}_u(u) = 0 \ . \end{cases}$$

Note that the non-negativity of μ and assumption (28) depend on the model parameters. In particular they are satisfied for the Cox-Ingersoll-Ross process. We refer to Duffie et al. [17] for an extensive study of affine models.

Based on this insight, we get for every $u \in (t,T]$

$$\begin{aligned} &E\left[e^{-ru}e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] = e^{-ru}e^{-\Gamma_t}E\left[e^{-\int_t^u \mu_s ds} \,\middle|\, \mathcal{F}_t\right] = e^{-ru}e^{-\Gamma_t}e^{\alpha_u(t)+\beta_u(t)\mu_t} \\ =&e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} + \int_0^t e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\sigma(s,\mu_s)dW_s + X_t^3 \ , \end{aligned}$$

where

$$X_t^3 = e^{-ru}\int_0^t e^{\alpha_u(s)+\beta_u(s)\mu_s}e^{-\Gamma_s}\left(\partial_s\alpha_u(s) + \mu_s\partial_s\beta_u(s) + \beta_u(s)\delta(s,\mu_s) + \frac{1}{2}\beta_s^2(u)\sigma^2(s,\mu_s) - \mu_s\right)ds$$

is of finite variation and has to vanish as the conditional expectation on the l.h.s is a square integrable continuous martingale.

For $u \in [0,t]$ we note that

$$\begin{aligned} &E\left[e^{-ru}e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] = e^{-ru}e^{-\Gamma_u} = \lim_{v\nearrow u} E\left[e^{-ru}e^{-\Gamma_u} \,\middle|\, \mathcal{F}_v\right] \\ =&\lim_{v\nearrow u} e^{-ru}\left(e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} + \int_0^v e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\sigma(s,\mu_s)dW_s\right) \\ =&e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} + \int_0^u e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\sigma(s,\mu_s)dW_s \ , \end{aligned}$$

where we have used the fact that $\mathbb{F}$-martingales are continuous. This shows that for arbitrary $u \in [0,T]$ we have

$$\begin{aligned} E\left[e^{-ru}e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] &= e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} + \int_0^{t\wedge u} e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\sigma(s,\mu_s)dW_s \\ &= e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} + \int_0^t e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\sigma(s,\mu_s)\mathbb{I}_{[0,u]}(s)dW_s \ . \end{aligned}$$

From this we can directly infer the processes ψ, θ_u^{la} and ξ_u of the martingale representations (4) (for the special case $Z = e^{-rT}$), (13) (for the case of a simple life annuity with $Y_t = 1, t \in [0,T]$) and (23) respectively to be

$$\psi_s = e^{-rT} e^{-\Gamma_s} \sigma(s,\mu_s) e^{\alpha_T(s)+\beta_T(s)\mu_s} \beta_T(s) \tag{30}$$

and

$$\theta_{u,s}^{la} = \xi_{u,s} = e^{-ru} e^{-\Gamma_s} \sigma(s,\mu_s) e^{\alpha_u(s)+\beta_u(s)\mu_s} \beta_u(s) \mathbb{I}_{[0,u]}(s)\,. \tag{31}$$

Similarly we get for $u \in [0,T]$

$$\begin{aligned} E\left[e^{-ru}e^{-\Gamma_u}\mu_u \,\middle|\, \mathcal{F}_t\right] &= \underbrace{e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}}\left(\hat{\alpha}_u(0)+\hat{\beta}_u(0)\overline{\mu}\right)}_{=:c_u} \\ &+ \int_0^t \underbrace{e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\sigma(s,\mu_s)\left(\hat{\beta}_u(s)+\left(\hat{\alpha}_u(s)+\hat{\beta}_u(s)\mu_s\right)\beta_u(s)\right)\mathbb{I}_{[0,u]}(s)}_{=:\eta(u,s)}\,dW_s\,. \end{aligned} \tag{32}$$

Note that for all $u \in [0,T]$ and all $t \in [0,T]$, we get by (28) that

$$\begin{aligned} E\left[E\left[e^{-ru}e^{-\Gamma_u}\mu_u \,\middle|\, \mathcal{F}_t\right]^2\right] &\leq E\left[E\left[e^{-2ru}e^{-2\Gamma_u}\mu_u^2 \,\middle|\, \mathcal{F}_t\right]\right] \\ &= E\left[e^{-2ru}e^{-2\Gamma_u}\mu_u^2\right] \leq E\left[\mu_u^2\right] \leq C\,. \end{aligned}$$

Hence, $\left(\int_0^t \eta(u,s)dW_s\right)$ is a square-integrable martingale. Moreover note that due to the Itô isometry and Fubini's theorem, we have by (28) that

$$\begin{aligned} E\left[\int_0^T\int_0^T \eta(u,s)^2 du ds\right] &= \int_0^T E\left[\int_0^T \eta(u,s)^2 ds\right] du \\ &= \int_0^T E\left[\left(\int_0^T \eta(u,s)dW_s\right)^2\right] du \\ &= \int_0^T E\left[\left(E\left[e^{-ru}e^{-\Gamma_u}\mu_u \,\middle|\, \mathcal{F}_T\right] - c_u\right)^2\right] du \\ &\leq \int_0^T E\left[\mu_u^2\right] du \\ &\leq CT < \infty\,, \end{aligned}$$

since $c_u \geq 0$ for all $u \in [0,T]$.
Hence, we may apply the stochastic Fubini theorem and (32) to obtain

$$
\begin{aligned}
&E\left[\int_0^T e^{-ru}e^{-\Gamma_u}d\Gamma_u \,\middle|\, \mathcal{F}_t\right] = \int_0^T E\left[e^{-ru}e^{-\Gamma_u}\mu_u \,\middle|\, \mathcal{F}_t\right] du \\
&= \int_0^T e^{-ru}\bigg\{ e^{\alpha_u(0)+\beta_u(0)\overline{\mu}}\left(\hat{\alpha}_u(0)+\hat{\beta}_u(0)\overline{\mu}\right) \\
&\quad + \int_0^t e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\sigma(s,\mu_s)\left(\hat{\beta}_u(s)+\left(\hat{\alpha}_u(s)+\hat{\beta}_u(s)\mu_s\right)\beta_u(s)\right)\mathbb{I}_{[0,u]}(s)dW_s\bigg\}du \\
&= \int_0^T e^{-ru}e^{\alpha_u(0)+\beta_u(0)\overline{\mu}}\left(\hat{\alpha}_u(0)+\hat{\beta}_u(0)\overline{\mu}\right)du \\
&\quad + \int_0^t\int_s^T e^{-ru}e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\sigma(s,\mu_s)\left(\hat{\beta}_u(s)+\left(\hat{\alpha}_u(s)+\hat{\beta}_u(s)\mu_s\right)\beta_u(s)\right)du dW_s\,.
\end{aligned}
$$

From this we can infer the process χ of the martingale representation (7) to equal

$$
\chi_s = e^{-\Gamma_s}\sigma(s,\mu_s)\int_s^T e^{-ru}e^{\alpha_u(s)+\beta_u(s)\mu_s}\left(\hat{\beta}_u(s)+\left(\hat{\alpha}_u(s)+\hat{\beta}_u(s)\mu_s\right)\beta_u(s)\right)du\,. \tag{33}
$$

Finally, we have for $u \in (t,T]$:

$$
\begin{aligned}
&E\left[e^{-ru}\left(1-G_u\right)e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] = e^{-ru}e^{-\Gamma_t}E\left[e^{-\int_t^u \mu_v dv} \,\middle|\, \mathcal{F}_t\right] - e^{-ru}e^{-2\Gamma_t}E\left[e^{-2\int_t^u \mu_v dv} \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-ru}e^{-\Gamma_t}e^{\alpha_u(t)+\beta_u(t)\mu_t} - e^{-ru}e^{-2\Gamma_t}e^{\tilde{\alpha}_u(t)+\tilde{\beta}_u(t)\mu_t} \\
&= e^{-ru}\left(e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} - e^{\tilde{\alpha}_u(0)+\tilde{\beta}_u(0)\overline{\mu}}\right) \\
&\quad + \int_0^t e^{-ru}e^{-\Gamma_s}\sigma(s,\mu_s)\left(e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s) - e^{-\Gamma_s}e^{\tilde{\alpha}_u(s)+\tilde{\beta}_u(s)\mu_s}\tilde{\beta}_u(s)\right)dW_s + X_t^4\,,
\end{aligned}
$$

where

$$
\begin{aligned}
X_t^4 = &\int_0^t e^{-\Gamma_s}e^{\alpha_u(s)+\beta_u(s)\mu_s}\left(\partial_s\alpha_u(s)+\mu_s\partial_s\beta_u(s)+\beta_u(s)\delta(s,\mu_s)+\frac{1}{2}\beta_s^2(u)\sigma^2(s,\mu_s)-\mu_s\right)ds \\
&- \int_0^t e^{-2\Gamma_s}e^{\tilde{\alpha}_u(s)+\tilde{\beta}_u(s)\mu_s}\left(\partial_s\tilde{\alpha}_u(s)+\mu_s\partial_s\tilde{\beta}_u(s)+\tilde{\beta}_u(s)\delta(s,\mu_s)+\frac{1}{2}\tilde{\beta}_u^2(s)\sigma^2(s,\mu_s)-2\mu_s\right)ds
\end{aligned}
$$

is of finite variation and has to vanish.
For $u \in [0,t]$ we get by the same limit arguments as above

$$
\begin{aligned}
e^{-ru}\left(1-G_u\right)e^{-\Gamma_u} &= e^{-ru}\left(e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} - e^{\tilde{\alpha}_u(0)+\tilde{\beta}_u(0)\overline{\mu}}\right) \\
&+ \int_0^u e^{-ru}e^{-\Gamma_s}\sigma(s,\mu_s)\left(e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s) - e^{-\Gamma_s}e^{\tilde{\alpha}_u(s)+\tilde{\beta}_u(s)\mu_s}\tilde{\beta}_u(s)\right)dW_s\,.
\end{aligned}
$$

Hence we get for arbitrary $u \in [0,T]$:

$$E\left[e^{-ru}\left(1-G_u\right)e^{-\Gamma_u} \,\middle|\, \mathcal{F}_t\right] = e^{-ru}\left(e^{\alpha_u(0)+\beta_u(0)\overline{\mu}} - e^{\tilde{\alpha}_u(0)+\tilde{\beta}_u(0)\overline{\mu}}\right)$$
$$+ \int_0^t e^{-ru}e^{-\Gamma_s}\sigma(s,\mu_s)\left(e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s) - e^{-\Gamma_s}e^{\tilde{\alpha}_u(s)+\tilde{\beta}_u(s)\mu_s}\tilde{\beta}_u(s)\right)\mathbb{I}_{[0,u]}(s)dW_s\,,$$

from which we infer the process θ_u^{ga} in the martingale representation (13) (for the special case of a gratification annuity with $Y_t = 1 - G_t$, $t \in [0,T]$) to be

$$\theta_{u,s}^{ga} = e^{-ru}e^{-\Gamma_s}\sigma(s,\mu_s)\left(e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s) - e^{-\Gamma_s}e^{\tilde{\alpha}_u(s)+\tilde{\beta}_u(s)\mu_s}\tilde{\beta}_u(s)\right)\mathbb{I}_{[0,u]}(s)\,. \tag{34}$$

By (30), (31), (33), (34) as well as (6), (9), (19), (21) (24) and (26), we hence obtain the optimal hedging strategies $\vartheta^{*,pe}$, $\vartheta^{*,ti}$, $\vartheta^{*,ga}$ and $\vartheta^{*,la}$ for pure endowment, term insurance, gratification annuity and simple life annuity respectively as

$$\vartheta_s^{*,pe} = \frac{e^{-rT}L_{s-}e^{\alpha_T(s)+\beta_T(s)\mu_s}\beta_T(s)}{\int_s^T e^{-ru}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\,du}\,,$$

$$\vartheta_s^{*,ti} = \frac{L_{s-}\int_s^T e^{-ru}e^{\alpha_u(s)+\beta_u(s)\mu_s}\left(\hat{\beta}_u(s) + \left(\hat{\alpha}_u(s) + \hat{\beta}_u(s)\mu_s\right)\beta_u(s)\right)du}{\int_s^T e^{-ru}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\,du}\,,$$

$$\vartheta_s^{*,ga} = \frac{L_{s-}\int_s^T e^{-ru}\left(e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s) - e^{-\Gamma_s}e^{\widetilde{\alpha}_u(s)+\widetilde{\beta}_u(s)\mu_s}\widetilde{\beta}_u(s)\right)du}{\int_s^T e^{-ru}e^{\alpha_u(s)+\beta_u(s)\mu_s}\beta_u(s)\,du}\,, \tag{35}$$

$$\vartheta_s^{*,la} = L_{s-}\,. \tag{36}$$

5 Risk study

In this section we perform a risk study for gratification and simple life annuities. Based on numerical simulations we first compare exemplary paths of the optimal mean-variance hedging strategies as well as surfaces for their residual hedging error. Then we compare the systematic risk parts of both annuities. Remember that both annuities are general annuities with $Y_t = 1 - G_t$ and $Y_t = 1$, $t \in [0,T]$, in (1), respectively.

As in the previous section, we assume the hazard process to be absolutely continuous with respect to the Lebesgue measure and the implied intensity to follow an affine process. There exist several works which estimate different types of affine processes to existing life tables, see e.g. Biffis [7], Dahl and Møller [15] or Luciano and Vigna [18]. For our risk study we particularly focus on affine mortality intensities, following a non mean-reverting Ornstein-Uhlenbeck process and a non mean-reverting Feller process, respectively. Both processes are introduced and suggested to be suitable for mortality intensities in Luciano and Vigna [18]. More explicitly, for the Ornstein-Uhlenbeck process we set $d_0 = v_1 = 0$ and for the Feller process $d_0 = v_0 = 0$. In both cases, we find explicit solutions of the Riccati-ODEs, given in the previous section. For the non-mean-reverting Ornstein-Uhlenbeck process, we get for $t \in [0, u]$:

$$\begin{aligned}
\beta_u(t) &= \frac{1}{d_1}\left(1 - e^{d_1(u-t)}\right), & \alpha_u(t) &= \frac{v_0(3 + 2d_1(u-t) + e^{2d_1(u-t)} - 4e^{d_1(u-t)})}{4d_1^3}, \\
\tilde{\beta}_u(t) &= \frac{2}{d_1}\left(1 - e^{d_1(u-t)}\right), & \tilde{\alpha}_u(t) &= \frac{v_0(3 + 2d_1(u-t) + e^{2d_1(u-t)} - 4e^{d_1(u-t)})}{d_1^3}, \\
\hat{\beta}_u(t) &= e^{d_1(u-t)}, & \hat{\alpha}_u(t) &= \frac{v_0(2e^{d_1(u-t)} - e^{2d_1(u-t)} - 1)}{2d_1^2}\ .
\end{aligned}$$

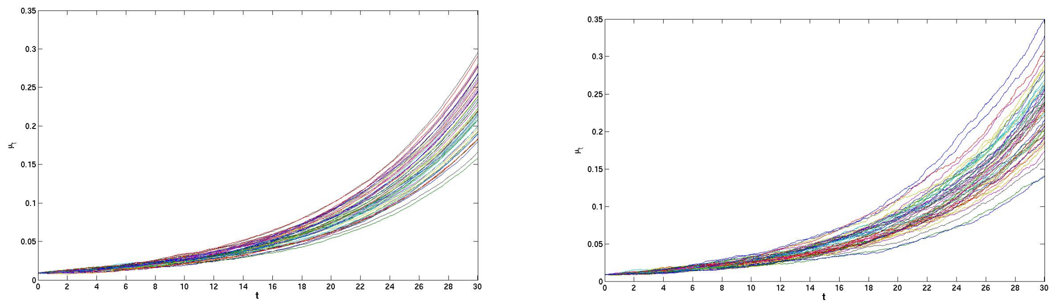
For the non-mean-reverting Feller process, we get for $t \in [0, u]$:

$$\begin{aligned}
\beta_u(t) &= \frac{2(e^{\gamma(u-t)} - 1)}{(d_1 - \gamma)(e^{\gamma(u-t)} - 1) - 2\gamma}, & \alpha_u(t) &= 0, \\
\tilde{\beta}_u(t) &= \frac{4(e^{\tilde{\gamma}(u-t)} - 1)}{(d_1 - \tilde{\gamma})(e^{\tilde{\gamma}(u-t)} - 1) - 2\tilde{\gamma}}, & \tilde{\alpha}_u(t) &= 0, \\
\hat{\beta}_u(t) &= \frac{4\gamma^2 e^{\gamma(u-t)}}{\left((\gamma - d_1)(e^{\gamma(u-t)} - 1) + 2\gamma\right)^2}, & \hat{\alpha}_u(t) &= 0,
\end{aligned}$$

where $\gamma = \sqrt{d_1^2 + 2v_1}$ and $\tilde{\gamma} = \sqrt{d_1^2 + 4v_1}$.

Note that with the lack of the mean-reversion property, both processes, in contrast to their mean-reverting analogues (the Vasicek and the Cox Ingersoll Ross model), are of exponential structure as is illustrated in Figure 1. Here and for the following illustrations, the parameters are taken from Luciano and Vigna [18]. Note that the non-mean-reverting Ornstein-Uhlenbeck process does a priori not show the property of non-negativity. Yet, with an appropriate choice of the model parameters, one can set the probability that the process reaches negative values very small. In particular this is true for the parameters found in Luciano and Vigna [18]. We also respect this issue when we vary some of the model parameters for the illustrations. This way, we still consider the non-mean-reverting Ornstein-Uhlenbeck process as suitable for our results, a common assumption in the literature, see e.g. Schrager [24] or Luciano and Vigna [18].

Based on the simulated paths of the mortality intensity and the affine model parameters, we have numerically generated the optimal mean-variance hedging strategies according to the formulas (35) and (36) respectively for the Ornstein-Uhlenbeck and the Feller process. Figures 2 and 3 show exemplary paths of the strategies for gratification and simple life annuities with maturities $T = 5$ and $T = 30$. Note that the strategies which jump to zero before the maturity show that the insured person died at that time. Hence, the



(a) Non-mean-reverting Ornstein-Uhlenbeck process (b) Non-mean-reverting Feller process

Figure 1: Exemplary paths of a non-mean-reverting Ornstein-Uhlenbeck and a non-mean-reverting Feller process.

optimal hedging strategies intrinsically offer a reasonable property: if the insured dies before maturity, there is no further necessity to keep a position in the hedging instrument for this contract.

A remarkable difference between the gratification annuity and the simple life annuity is that for both maturities, the insurance company has initially to go short in the longevity bond in order to hedge the risk exposure of a gratification annuity, whereas it has to go long in the longevity bond to hedge the risk exposure of a simple life annuity. This is due to the fact that every rate-payment of the gratification annuity is inferred from the mortality intensity, too.

More explicitly, we remark that selling an insurance product means to have a short position in the respective instrument for the insurance company. The rate payments of a single life annuity only depend on the individual survival process $1 - H$ whereas the rate payments of a gratification annuity depend on both the individual survival process $1 - H$ and the mortality rate $1 - G$. The short position in the life annuity therefore yield high overall rate payments with a high realized survival of the insured person. That is why the insurance company has to go long in a longevity bond, as this means to receive higher rate payments with a higher survival rate in the reference portfolio of the longevity index, which can be assumed to be a good proxy for the realized survival process of the insured person. On the contrary, the short position in a gratification annuity means to suffer from both a lower survival rate of the reference portfolio and a higher realized survival of the insured. For a young insured, i.e. at the beginning of the insurance contract, the suffering from a lower survival rate in the reference portfolio dominates the suffering from a higher realized survival of the insured and the insurance company has to go short in a longevity bond in order to cover these rate payments[1]. Only for large maturities and when the insured person gets older, the suffering from her higher realized survival dominates the suffering from lower survival rates in the reference portfolio and a long position in the longevity bond has to cover these long term rate payments.

Another important issue besides the determination of the optimal hedging strategies is the quantification of the residual hedging error. With the mean-variance hedging approach we have found self-financing trading strategies, which do not perfectly replicate the insurance claim C, but yield a value process whose final outcome is optimally close to C in the L^2-norm. However, this value process, although optimal, could still be too far away from the claim and the hedging strategy therefore less reasonable for the insurance company. In order to measure the residual hedging risk of the optimal strategy $\vartheta^{*,C}$, we introduce the (residual) hedging error process R^C, given by

$$R_t^C = E\left[\left(C - c^C - \int_0^T \vartheta_s^{*,C} dV_s\right)^2 \,\middle|\, \mathcal{G}_t\right] , \quad t \in [0,T] ,$$

where $c^C = E\left[C\right]$ is the necessary amount to initiate the hedging scheme.

For our simulations, we only consider R_0^C. By using the results of Section 3, we hence need to simulate

$$\begin{aligned} R_0^C &= E\left[\left(C - c^C - \int_0^T \vartheta_s^{*,C} dV_s\right)^2\right] = E\left[\left(\int_0^T \epsilon_s^{C,M} dM_s\right)^2\right] = E\left[\int_0^T (\epsilon_s^{C,M})^2 d[M]_s\right] \\ &= E\left[\int_0^{T\wedge\tau} (\epsilon_s^{C,M})^2 d\Gamma_s\right] = E\left[\int_0^{T\wedge\tau} (\epsilon_s^{C,M})^2 \mu_s ds\right] , \end{aligned}$$

where $\epsilon^{C,M}$ denotes the integrand with respect to M in the GKW-decompositions of C.

Figure 4 and Figure 5 show the residual hedging errors R_0 for a gratification annuity and a simple life annuity, where the maturity T and the initial mortality intensity level μ_0 are varying. For both products, the results are again calculated with a mortality intensity following a non-mean-reverting Ornstein-Uhlenbeck process or a non-mean-reverting Feller process, respectively.

For both insurance products, the hedging error increases with increasing maturity, which is not surprising. The remarkable feature, however, is that the residual hedging error of a gratification annuity is considerably lower than the hedging error of a simple life annuity. The levels of R_0 are lower for all considered combinations of maturity and initial mortality intensity under both affine specifications of the mortality intensity. This is due to the fact that the rate payments of the gratification annuity and the longevity bond both depend on the survival rate G, whereas the rate payments of the single life annuity only depend

[1] Note that the longevity bond offers rate payments G.

on the individual survival process $1-H$. Hence there is a higher correlation between the rate payments of the gratification annuity and the longevity bond than between the rate payments of the single life annuity and the longevity bond. As parts of the mortality risk are forwarded to the insured person through the gratification annuity's rate payments hence yields a good performance of the gratification annuity's optimal hedging scheme. On the contrary, the residual hedging error for existing insurance products like a simple life annuity suggests to consider their optimal hedging strategy rather carefully, especially for longer maturities.

Another point of interest in the context of an insurance claim's risk is the investigation of its systematic and unsystematic parts, see e.g. Norberg [21]. The systematic part of an insurance claim's risk can be understood as the part which is due to common risk drivers and its consequences for the insurance company cannot be reduced through diversification. The unsystematic part of an insurance claim's risk can be understood as the part that is due to the insured person's individual characteristics. Its consequences for the insurance company can be reduced through diversification.

Note that in the setting of the present paper, the GKW-decompositions of the different insurance claims intrinsically cover the separation of systematic and unsystematic risk: as we have $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ and $\mathbb{F}$ is generated by W, every claim C can be represented as

$$C = c^C + \underbrace{\int_0^T \epsilon_s^{C,W}\, dW_s}_{\text{systematic risk}} + \underbrace{\int_{0+}^T \epsilon_s^{C,M}\, dM_s}_{\text{unsystematic risk}} .$$

As the Brownian motion W is the unique "external" risk driver for all insurance claims, the stochastic integral with respect to W can be considered as the systematic part. The martingales M, however, vary for different insured persons and the integrals with respect to M can therefore be considered as the unsystematic part.

As the effects of the unsystematic part diversify through pooling, we now want to compare the systematic risk of a gratification annuity and a simple life annuity. In particular, we can measure the systematic risk SR through

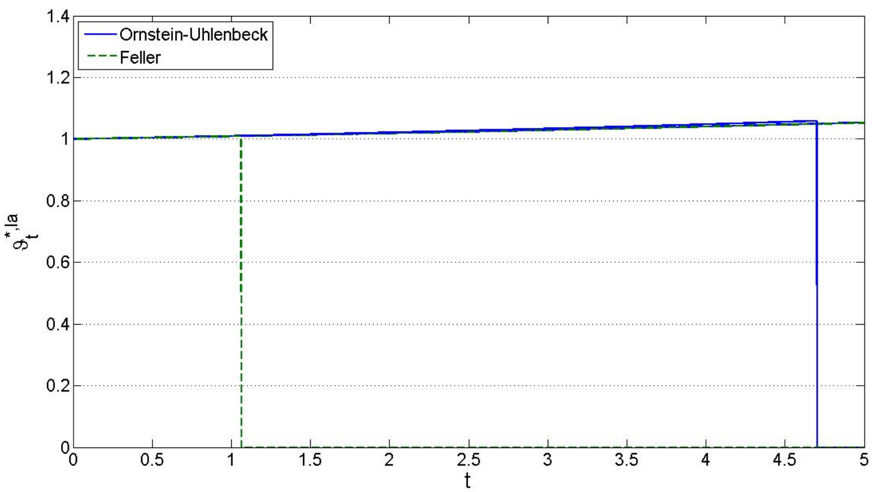
$$SR = \mathbb{E}\left[\left(\int_0^T \epsilon_s^{C,W}\, dW_s\right)^2\right] = \mathbb{E}\left[\int_0^T \left(\epsilon_s^{C,W}\right)^2\, ds\right].$$

In our particular affine framework of this section, equations (31) and (34) show that the systematic risk of the simple life annuity is lower or equal than that of the gratification annuity, if $\tilde{\beta}_u(s) \le 0, \forall s \in [0,T], u \in [s,T]$. This is particularly the case for the mortality intensity μ, following an Ornstein-Uhlenbeck or a Feller process, as well as for most models of practical interest. This is due to the fact, that the gratification annuity is exposed to systematic risk in both directions: a structural change in the systematic risk drivers affects both, the insurance company's pool of policyholders and the age cohort from which the

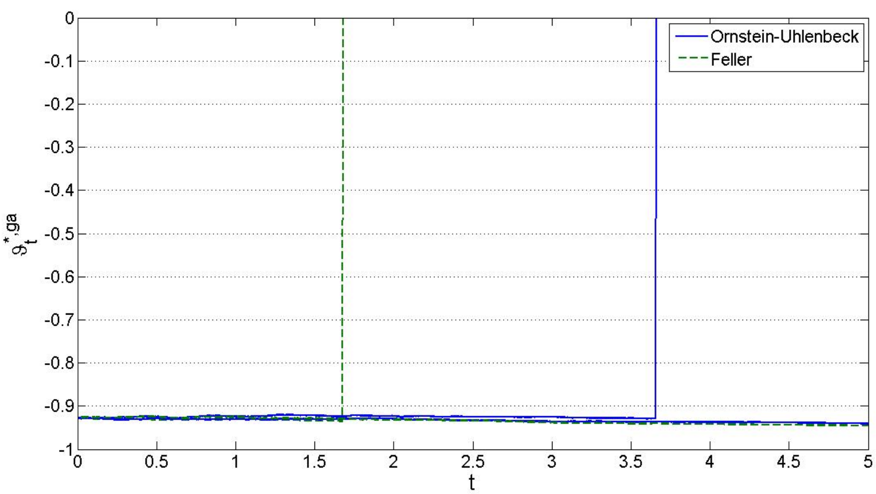
rate payments are inferred. A structural decrease in the underlying mortality intensity would e.g. lead to lower claim payments with respect to the insurance portfolio on the one hand, but also to higher annuity rates on the other. While a portfolio of simple life annuities would benefit from a structural decrease in the mortality intensity, a portfolio of gratification annuities could also suffer from it. Still, the gratification annuity inherits an advantageous feature from its payout structure: the most common systematic risk exposures of life insurance companies or pension funds are due to increasing longevity. Here, a gratification annuity relaxes the exposure, as increasing longevity leads to lower rates.
The results hence show, that the systematic risk of a gratification annuity is higher than that of a simple life annuity, existing on the markets, because the gratification annuity is exposed to risk in any direction. Yet, for the most important systematic risk exposure, increasing longevity, the gratification annuity transfers parts of the systematic risk to the policyholders. For a more thorough investigation of systematic risk in an even more general setting, we refer to Biagini and Schreiber [5].
The investigation of the systematic risk is important under the assumption that no longevity bond is available. With the presence of longevity bonds on the market, however, we have seen that the complete systematic risk of the insurance claims can be secured. Here, the remaining risk due to hedging errors is considerably lower for the gratification annuity than for a simple life annuity.
Besides the nice "marketing" feature that the insured person gets gratified if she is healthier as was originally expected, the gratification annuity therefore shows a better risk behavior than other insurance products, already existing on the life market, given the presence of longevity bonds. Moreover we have seen that the insurance company must initially hold a short position in the longevity bond in order to hedge a gratification annuity. This, however, means lower initial overall costs to hedge all longevity products of an insurance company.
All these advantageous features can constitute incentives for insurance companies to introduce gratification annuities as a new life insurance product.

Acknowledgment

We would like to thank Irene Schreiber for interesting comments and remarks and Albena Hristova for her help with the simulations.

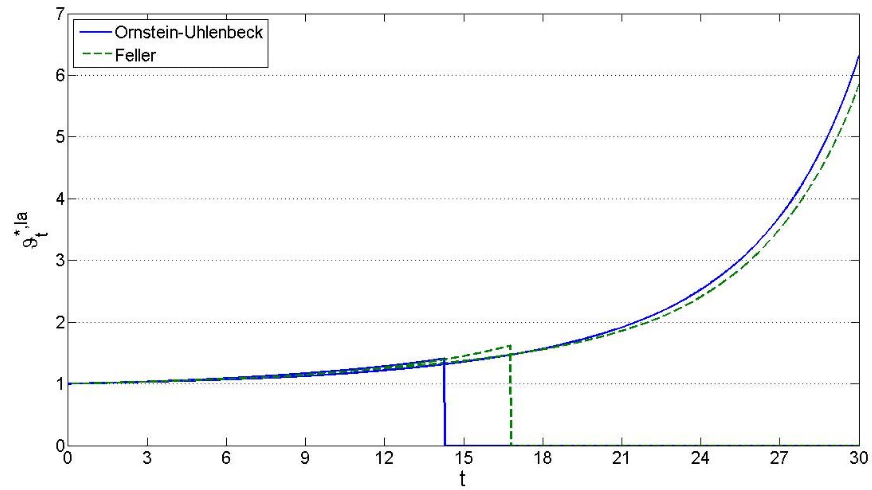


(a) Simple Life Annuity

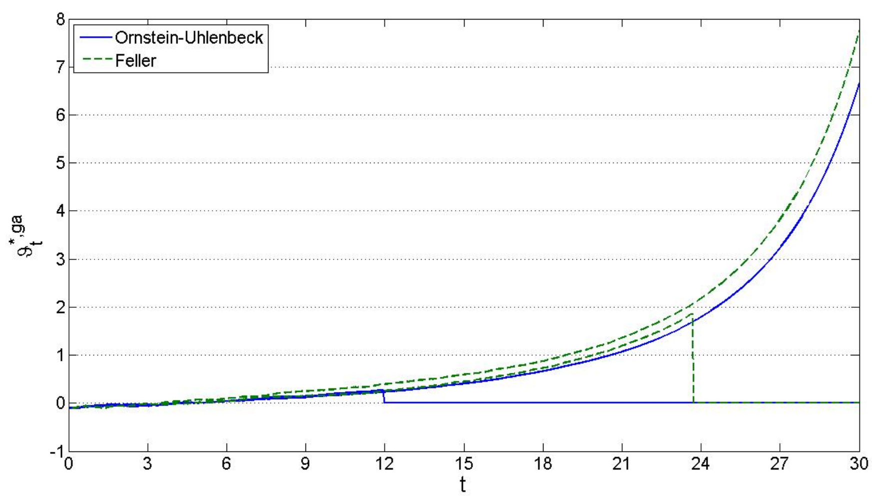


(b) Gratification Annuity

Figure 2: Exemplary paths of the optimal hedging strategies for a simple life annuity and a gratification annuity with maturity $T = 5$.

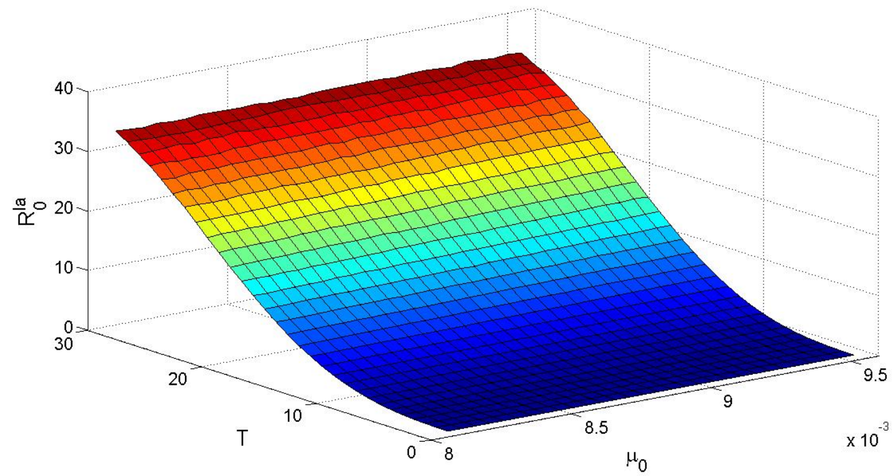


(a) Simple Life Annuity

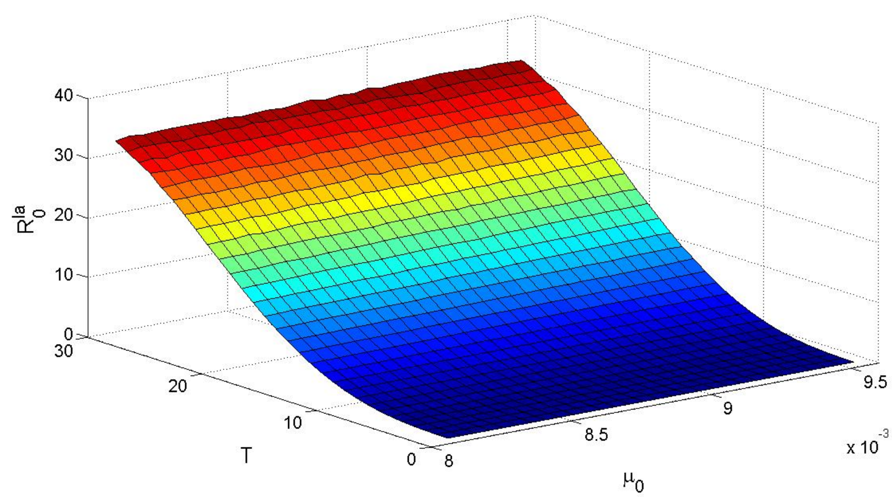


(b) Gratification Annuity

Figure 3: Exemplary paths of the optimal hedging strategies for a simple life annuity and a gratification annuity with maturity $T = 30$.

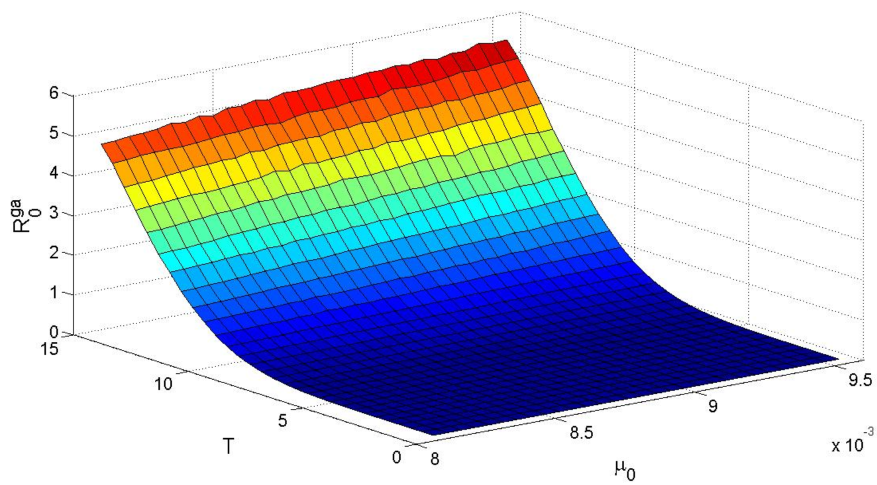


(a) R_0 of a simple life annuity, based on an Ornstein-Uhlenbeck process

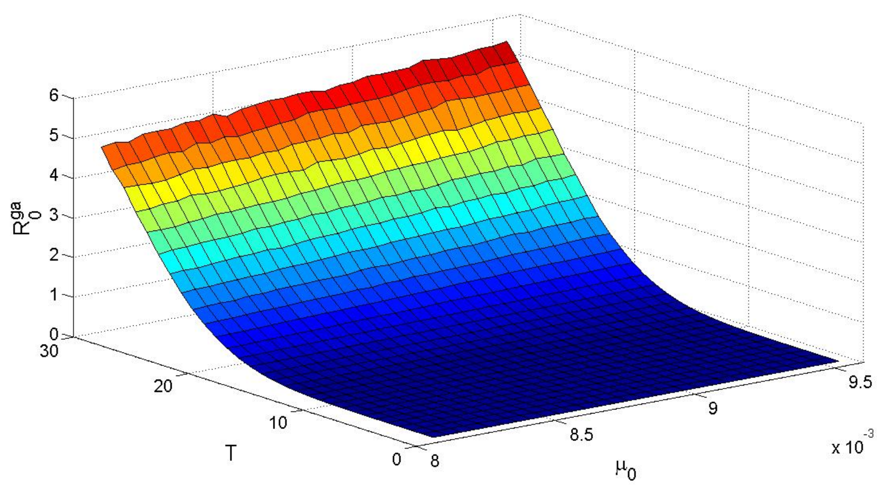


(b) R_0 of a simple life annuity, based on a Feller process

Figure 4: Residual hedging error R_0 for a simple life annuity with mortality intensity, simulated with a non-mean-reverting Ornstein-Uhlenbeck process and a non-mean-reverting Feller process.



(a) R_0 of a gratification annuity, based on an Ornstein-Uhlenbeck process



(b) R_0 of a gratification annuity, based on a Feller process

Figure 5: Residual hedging error R_0 for a gratification annuity with mortality intensity, simulated with a non-mean-reverting Ornstein-Uhlenbeck process and a non-mean-reverting Feller process.

References

[1] J. Barbarin. Heath-Jarrow-Morton modelling of longevity bonds and the risk minimization of life insurance portfolios. *Insurance: Mathematics and Economics*, 43(1): 41–55, 2008.

[2] J. Barbarin. *Valuation, Hedging and the Risk Management of Insurance Contracts*. PhD thesis, Université Catholique de Louvain, 2008.

[3] P. Barrieu and L. Albertini, editors. *The Handbook of Insurance-Linked Securities*. Wiley, Chichester, 2009.

[4] F. Biagini and A. Cretarola. Local risk-minimization for defaultable claims with recovery process. *Applied Mathematics & Optimization*, 65:293–314, 2012.

[5] F. Biagini and I. Schreiber. Risk minimization for life insurance liabilities. Working Paper, University of Munich, 2012.

[6] T.R. Bielecki and M. Rutkowski. *Credit Risk: Modelling, Valuation and Hedging*. Springer-Finance. Springer, second edition, 2004.

[7] E. Biffis. Affine processes for dynamic mortality and actuarial valuation. *Insurance: Mathematics and Economics*, 37(3):443–468, 2005.

[8] E. Biffis and P. Millossovich. A bidimensional approach to mortality risk. *Decisions in Economics and Finance*, 29(2):71–94, 2006.

[9] D. Blake and W. Burrows. Survivor bonds: helping to hedge mortality risk. *Journal of Risk and Insurance*, 68(2):336–348, 2001.

[10] D. Blake, A. J. G. Cairns, and K. Dowd. Pricing death: frameworks for the valuation and securitization of mortality risk. *ASTIN Bulletin*, 36(1):79–120, 2006.

[11] D. Blake, A. J. G. Cairns, and K. Dowd. The birth of the life market. *Asia-Pacific Journal of Risk and Insurance*, 3(1):6–36, 2008.

[12] D. Blake, T. Boardman, and A. J. G. Cairns. Longevity risk: Why governments should issue longevity bonds. Discussion Paper. Pensions Institute, 2010.

[13] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance & Stochastics*, 8(1):145–159, 2004.

[14] D. Coculescu, M. Jeanblanc, and A. Nikeghbali. Default times, non arbitrage conditions and change of probability measures. Working Paper, University of Zurich, Université d'Evry, 2008.

[15] M. Dahl and T. Møller. Valuation and hedging of life insurance liabilities with systematic mortality risk. *Mathematics and Economics*, 39(2):193–217, 2006.

[16] M. Dahl, M. Melchior, and T. Møller. On systematic mortality risk and risk-minimization with survivor swaps. *Scandinavian Actuarial Journal*, 2(2):114–146, 2008.

[17] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Annals of Applied Probability*, 13(3):984–1053, 2003.

[18] E. Luciano and E. Vigna. Mortality risk via affine stochastic intensities: calibration and empirical relevance. *Belgian Actuarial Bulletin*, 8:5–16, 2008.

[19] T. Møller. Risk-minimizing hedging strategies for unit-linked life insurance contracts. *ASTIN Bulletin*, 28(1):17–47, 1998.

[20] T. Møller. Risk-minimizing hedging strategies for insurance payment processes. *Finance and Stochastics*, 5(4):419–446, 2001.

[21] R. Norberg. The pension crisis: its causes, possible remedies, and the role of the regulator. *Erfaringer og utfordringer, 20 years Jubilee Volume of Kredittilsynet, the Financial Supervisory Authority of Norway*, 2006.

[22] R. Norberg. Optimal hedging of demographic risk in life insurance and pension. *Finance and Stochastics*, 2012. Forthcoming.

[23] P. E. Protter. *Stochastic Integration and Differential Equations*. Springer, New York, second edition, 2005.

[24] D.F. Schrager. Affine stochastic mortality. *Insurance: Mathematics and Economics*, 38(1):81–97, 2006.

[25] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, Cambridge, UK, 2001.

[26] M. Wadsworth, A. Findlater, and T. Boardman. Reinventing annuities. Working Paper, Staple Inn Society, 2001.

6 Appendix

Lemma 10 *We have*

$$\sup_{0\le u\le T}\left\|\int_0^T \phi_{u,s}\,dW_s\right\|_{L^2} < \infty. \tag{37}$$

Proof. First note that since $(1-H_{s-})\le 1$,

$$E\left[\int_0^T L_{s-}^2\theta_{u,s}^2\mathbb{I}_{[0,u]}(s)\,ds\right] \le E\left[\int_0^T e^{2\Gamma_s}\theta_{u,s}^2\mathbb{I}_{[0,u]}(s)\,ds\right],$$

hence by the Itô-isometry, (37) holds if

$$E\left[\left(\int_0^T e^{\Gamma_s}\theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s\right)^2\right] < \infty. \tag{38}$$

Since by (13) the $\int_0^\cdot \theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s$ are bounded martingales, uniformly in u, we have by integration by parts that for each $u\in[0,T]$

$$\begin{aligned}\left|\int_0^T e^{\Gamma_s}\theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s\right| &= \left|e^{\Gamma_T}\int_0^T \theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s - \int_0^T\int_0^s \theta_{u,v}\mathbb{I}_{[0,u]}(v)\,dW_v\,de^{\Gamma_s}\right| \\ &\le 2Ce^{\Gamma_T},\end{aligned}$$

where the constant C is independent of u. Therefore (38) follows from assumption (2), namely that $e^{\Gamma_T}\in L^2(P)$. ■

Lemma 11 *We have*

$$\sup_{0\le u\le T}\left\|\int_0^T \nu_{u,s}\,dM_s\right\|_{L^2} < \infty. \tag{39}$$

Proof. As

$$\nu_{u,s} = -L_{s-}\left(c_u + \int_0^s \theta_{u,v}\mathbb{I}_{[0,u]}(v)\,dW_v\right)\mathbb{I}_{[0,u]}(s),$$

and $c_u + \int_0^\cdot \theta_{u,v}\mathbb{I}_{[0,u]}(v)\,dW_v$ are bounded martingales, uniformly in u, it follows that the $\nu_{u,\cdot}$ are bounded as well by some constant C independent of u. Therefore

$$E\left[\int_0^T \nu_{u,s}^2\,d\langle M\rangle_s\right] \le C^2E\left[\langle M\rangle_T\right],$$

and by the Itô-isometry and the definition of the angle bracket (i.e. predictable compensator), (39) holds if and only if

$$E\left[\langle M\rangle_T\right] = E\left[\Gamma_{T\wedge\tau}\right] \le E\left[\Gamma_T\right] < \infty,$$

which is implied by (2). ■