RANGE CONVEXITY: PROBABILITIES, RISK MEASURES, AND GAMES

MASSIMILIANO AMARANTE

Département de Sciences Économiques & CIREQ,
Université de Montréal, Canada

FELIX-BENEDIKT LIEBRICH

Workgroup Financial and Insurance Mathematics, Department of Mathematics,
University of Munich, Germany

COSIMO MUNARI

Center for Finance and Insurance & Swiss Finance Institute,
University of Zurich, Switzerland

Abstract. We revisit Marinacci's uniqueness theorem for convex-ranged probabilities and its applications. Our approach does away with both the countable additivity and the positivity of the charges involved. In the process, we uncover several new equivalent conditions, which lead to a novel set of applications. These include extensions of the classic Fréchet-Hoeffding bounds as well as of the automatic Fatou property of law-invariant functionals. We also generalize existing results of the "collapse to the mean"-type concerning capacities and α-MEU preferences.

1. Introduction

Two convex-ranged probabilities $\mathfrak{p}$ and $\mathfrak{q}$ on a λ-system are one and the same if there exists an event E with $0 < \mathfrak{p}(E) < 1$ and for every event A with $\mathfrak{p}(A) = \mathfrak{p}(E)$ we have $\mathfrak{q}(A) = \mathfrak{q}(E)$. This result was originally obtained by Marinacci [25] under the assumption that $\mathfrak{q}$ be countably additive, and has been recently extended by Svistula [32], who allows for both $\mathfrak{p}$ and $\mathfrak{q}$ to be only finitely additive.

Shortly after obtaining it, Marinacci gave two applications of his result. The first (Marinacci [24]) deals with symmetric coherent capacities. Introduced by Kadane & Wasserman in [20] with the purpose of providing a suitable analog of the uniform measure for use in robust statistics, symmetric coherent capacities are non-additive set functions defined as the upper

E-mail addresses: massimiliano.amarante@umontreal.ca, liebrich@math.lmu.de,
cosimo.munari@bf.uzh.ch.

envelope of sets of countably additive probabilities that satisfy a condition called $\mathfrak{p}$-invariance. This stipulates that the capacity of an event depends only on its $\mathfrak{p}$-probability, where $\mathfrak{p}$ is a given reference probability. Symmetric coherent capacities encode the notion that the probability of the various events may be fuzzy, in the sense that it could be pinned down to an interval but not necessarily to a point estimate. Nonetheless, Marinacci showed that, when $\mathfrak{p}$ has convex range and is countably additive, the idea of symmetric coherent capacities becomes vacuous as soon as there exists a single non-trivial event that admits such a point estimate. More precisely, if there exists one event to which all the measures in the defining set assign the same probability and this is strictly in between 0 and 1, then a symmetric coherent capacity can only be a measure.

A second application (Marinacci [26]) concerns the so-called α-MEU model. This is a decision-theoretic paradigm whereby uncertain prospects, represented as mappings from a measurable space to the reals, are evaluated by means of a functional of the form

$$V(f) = \alpha \max_{\mathfrak{q}\in\mathcal{Q}} \int f d\mathfrak{q} + (1-\alpha) \min_{\mathfrak{q}\in\mathcal{Q}} \int f d\mathfrak{q},$$

where $\alpha \in [0,1]$ and $\mathcal{Q}$ is a set of finitely additive probabilities. An α-MEU model is said to be $\mathfrak{p}$-*invariant* if there exists an underlying convex-ranged probability $\mathfrak{p}$ such that the value of the functional depends only on the $\mathfrak{p}$-distribution of its argument. Marinacci showed that, if all the probabilities in the set $\mathcal{Q}$ are countably additive (such α-MEU models are called *monotone continuous*), then the existence of a single non-trivial event on which all probabilities in $\mathcal{Q}$ agree implies that, for every prospect f, we must have $V(f) = \int f d\mathfrak{p}$. That is, once again, the typically non-linear functional under consideration must collapse to a linear one.

These two applications are the prototype of others that have followed, which all consisted in showing that, rather surprisingly, certain non-linear functionals (pricing functions, risk measures, cooperative games) under the seemingly mild condition of $\mathfrak{p}$-invariance must collapse to a linear form. Specifically, Castagnoli et al. [9] have shown that, in a context of financial markets with frictions, the existence of a single frictionless risky asset implies the linearity of the pricing function and an analogous result has been obtained by Frittelli and Rosazza Gianin [15] in a risk measure setting. Later, Bellini et al. [7] have obtained a general "collapse to the mean" result for convex functionals, which contains the previous results as special cases. Similar results, proved with different methods yet reducible to an application of Marinacci's theorem, have been obtained by Liebrich and Munari [22], who establish the "collapse to the mean" for a variety of functionals beyond the convex case, and by Amarante [4], who observes that the anticore of a submodular, nonatomic, law-invariant capacity is either a singleton or must span an infinite dimensional subspace.

With regard to this type of applications, it must be noted that Svistula's extension of Marinacci's result is significant as in many applications the assumption of countable additivity is not met. For instance, when dealing with the α-MEU model Marinacci had to introduce the extra assumption of monotone continuity; in the case of risk measures, discontinuities at 0 are common and require considerations of purely finitely additive charges. Similar considerations can be made for all other settings. Thus, by weakening the assumption that the probability $\mathfrak{q}$ in Marinacci's theorem be countably additive, Svistula's extension has considerably widened the range of applicability of the original result.

In the present paper, we are concerned both with the uniqueness result for convex-ranged probabilities *per se* and with its applications. Two sections, Section 3 and Section 8, are devoted to the uniqueness theorem. In Section 3 we provide a new proof for the uniqueness result for convex-ranged probabilities that are finitely additive. Notably, our proof is much shorter and, at the same time, much simpler than the existing ones. The main dividend is that not only does our proof clarify the nature of the result under consideration, but it also leads us to uncovering several equivalent conditions, thus easing the applicability of the result (Theorems 3.3 and 3.5). We complete Section 3 by presenting a strikingly short proof for the special case of countably additive probabilities defined on a σ-algebra (Corollary 3.6). Building on the approach of Section 3, in Section 8 we prove an altogether novel uniqueness result for signed charges (Theorem 8.3).

Sections 4 to 7 as well as the second part of Section 8 are devoted to applications. Those of Section 4 are "standard", in the sense that they concern the collapse to a linear form of various non-linear functionals. We prove a new result about the "collapse" of the (anti)core of a law-invariant capacity, which showcases the most fundamental implication of commonly made assumptions (Theorem 4.2). We, then, show that the results in the existing literature follow at once as simple corollaries. We round up Section 4 by re-considering the same problems under conditions milder than those imposed in the existing literature. In Section 5, we present an entirely novel application of the uniqueness theorem: an extension of the classical Fréchet-Hoeffding bounds to the case of finitely additive probabilities (Theorem 5.1). This extension, in turn, provides the grounds for the two other novel applications of the subsequent two sections. In Section 6, we extend the important result on the automatic Fatou property of convex, law-invariant risk measures to the case of an underlying probability which is only finitely additive (Theorem 6.2). Notably, our approach to this problem, informed as it is by the developments of the previous sections, provides a fresh perspective on the automatic Fatou property even in the countably additive case. In Section 7, we investigate under which conditions a capacity can be simultaneously law-invariant with respect to different probabilities and show that this can only occur for very special capacities (Proposition 7.3). Finally, in the second part of Section 8, after having established the uniqueness result for signed measures, we provide extended versions of the Fréchet-Hoeffding bounds and of the automatic Fatou property.

2. Preliminaries

In this section we collect some basic definitions about set functions. We refer the reader to [11, 27] for more on the subject matter. Throughout the paper, the basic environment is a pair (Ω, Σ), where Ω is a nonempty set and Σ a family of subsets of Ω. In most applications, the set Ω is interpreted as a set of states of nature and Σ as the class of measurable events, but other interpretations are possible. For instance, Ω can be interpreted as the set of players in a cooperative game and Σ as the class of possible coalitions; Ω can be regarded as a set of possible criteria for evaluating a complex system, etc. We stipulate the following minimal assumption on Σ that we maintain throughout the paper. Notice that this covers most modelling situations where it is typically assumed that either Σ is a λ-system or an algebra or a σ-algebra.

Assumption 2.1. Σ contains Ω and is closed under complementation and finite unions of pairwise disjoint sets.

A set function $v\colon \Sigma \to \mathbb{R}$ is a *(cooperative) game* if $v(\varnothing) = 0$. It is a *capacity* if it is *monotone*, i.e., $v(A) \leq v(B)$ for all $A, B \in \Sigma$ with $A \subset B$. Throughout the paper, capacities are assumed to be finite, i.e., $v(\Omega) < +\infty$, and are *normalized* so that $v(\Omega) = 1$.
We denote by $\mathbf{ba}(\Sigma)$ the normed space of *(signed) charges*, i.e., finitely additive set functions $\mu : \Sigma \to \mathbb{R}$ with bounded variation, equipped with the total variation norm, and by $\mathbf{ca}(\Sigma)$ its subspace consisting of countably additive functions. We denote by $\mathcal{P}(\Sigma)$ the subset of *probability* charges. Normally, we will use gothic letters like $\mathfrak{p}$ and $\mathfrak{q}$ to indicate finitely additive probabilities, while we will use capital boldface letters like $\mathbb{P}$ and $\mathbb{Q}$ for countably additive probabilities. For $\mathfrak{p} \in \mathcal{P}(\Sigma)$ and $\mu \in \mathbf{ba}(\Sigma)$ we say that:

(1) μ is *absolutely continuous* with respect to $\mathfrak{p}$, written $\mu \ll \mathfrak{p}$, if, for every $N \in \Sigma$ with $\mathfrak{p}(N) = 0$, we have $\mu(N) = 0$.
(2) μ is *strongly absolutely continuous* with respect to $\mathfrak{p}$, written $\mu \lll \mathfrak{p}$, if, for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $A \in \Sigma$, $\mathfrak{p}(A) \leq \delta$ implies $|\mu(A)| \leq \varepsilon$.
(3) $\mathfrak{p}$ is *nonatomic* if, for every $A \in \Sigma$ with $\mathfrak{p}(A) > 0$, there exists $B \in \Sigma$ such that $B \subset A$ and $0 < \mathfrak{p}(B) < \mathfrak{p}(A)$.
(4) $\mathfrak{p}$ has *convex range* or is *strongly nonatomic* if, for all $A \in \Sigma$ and $0 \leq \varepsilon \leq \mathfrak{p}(A)$, there exists $B \in \Sigma$ such that $B \subset A$ and $\mathfrak{p}(B) = \varepsilon$.

The set of signed charges, resp. finitely additive probabilities, that are absolutely continuous with respect to $\mathfrak{p}$ is denoted by $\mathbf{ba}_{\mathfrak{p}}(\Sigma)$, resp. $\mathcal{P}_{\mathfrak{p}}(\Sigma)$. Clearly, a convex range implies nonatomicity. The converse implication fails in general but it holds for countably additive probabilities if Σ is a σ-algebra; see [8, Theorem 5.1.6].
A game $v : \Sigma \to \mathbb{R}$ is *law-invariant with respect to* $\mathfrak{p}$, in short $\mathfrak{p}$*-invariant*, if it is constant on the equilikelihood classes under $\mathfrak{p}$, i.e., if, for all $A, B \in \Sigma$, we have

$$\mathfrak{p}(A) = \mathfrak{p}(B) \implies v(A) = v(B).$$

The *core*, resp. *anticore*, of a game v is the (possibly empty) set of all signed charges that dominate, resp. are dominated by, v and agree with v at Ω, i.e.,

$$\operatorname{core}(v) := \{\mu \in \mathbf{ba}(\Sigma)\,;\ \mu(\Omega) = v(\Omega),\ \forall A \in \Sigma,\ \mu(A) \geq v(A)\},$$
$$\operatorname{acore}(v) := \{\mu \in \mathbf{ba}(\Sigma)\,;\ \mu(\Omega) = v(\Omega),\ \forall A \in \Sigma,\ \mu(A) \leq v(A)\}.$$

If v is $\mathfrak{p}$-invariant and $\mu \in \operatorname{core}(v)$ or $\mu \in \operatorname{acore}(v)$, then $\mu \ll \mathfrak{p}$. A game v is called *submodular* if, for all $A, B \in \Sigma$ with $A \cup B$, $A \cap B \in \Sigma$, we have

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B). \tag{2.1}$$

It is *supermodular* if the reverse inequality holds. The core (anticore) of a supermodular (submodular) capacity is always nonempty. A capacity is *exact* if it is either the lower envelope of its core or the upper envelope of its anticore; see [29].

We denote by Π the set of finite partitions of Ω consisting of elements of Σ. We say that $\pi \in \Pi$ has *size* $n \in \mathbb{N}$ if it contains n elements. For a given $\mathfrak{p} \in \mathcal{P}(\Sigma)$, we say that π is a $\mathfrak{p}$-*equipartition* (of Ω) if the elements of π have the same $\mathfrak{p}$-probability. The following lemma gives a simple but very useful result concerning the existence of certain $\mathfrak{p}$-equipartitions when $\mathfrak{p}$ has convex range.

Lemma 2.2. *Let $\mathfrak{p}$ be a finitely additive probability with convex range. If $A \in \Sigma$ satisfies $\mathfrak{p}(A) = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ with $m < n$, then there is a $\mathfrak{p}$-equipartition $\pi \in \Pi$ of size n such that A can be written as the union of m elements of π.*

Proof. As $\mathfrak{p}$ has convex range, there is $A_1 \in \Sigma$ such that $A_1 \subset A$ and $\mathfrak{p}(A_1) = \frac{1}{n}$. If $m = 1$, we can take $A_1 = A$. Else, $A \setminus A_1 = (A_1 \cup A^c)^c \in \Sigma$ satisfies $\mathfrak{p}(A \setminus A_1) = \frac{m-1}{n}$. Similarly, we find $A_2 \in \Sigma$ such that $A_2 \subset A \setminus A_1$ and $\mathfrak{p}(A_2) = \frac{1}{n}$. If $m = 2$, we may take $A_2 = A \setminus A_1$. Else, $A \setminus (A_1 \cup A_2) = (A_1 \cup A_2 \cup A^c)^c \in \Sigma$ satisfies $\mathfrak{p}(A \setminus (A_1 \cup A_2)) = \frac{m-2}{n}$. Continuing in the same manner, we find pairwise disjoint events $A_1, ..., A_m \in \Sigma$ with $\mathfrak{p}(A_i) = \frac{1}{n}$ for every $i = 1, \dots, m$ that form a partition of A. Applying the same reasoning to A^c, which satisfies $\mathfrak{p}(A^c) = \frac{n-m}{n}$, we may enlarge this to a $\mathfrak{p}$-equipartition of Ω of size n. ☐

We denote by $B_0(\Sigma)$ the pointed cone of all functions $X : \Omega \to \mathbb{R}$ of shape $X = \sum_{i=1}^n x_i \mathbf{1}_{A_i}$, where $n \in \mathbb{N}$, $x_1 < ... < x_n$ are real numbers, and $A_1, ..., A_n \in \Sigma$ form a partition of Ω into pairwise disjoint events. Similarly, $B(\Sigma)$ is the pointed cone of all bounded functions $X : \Omega \to \mathbb{R}$ with the property that $X^{-1}(B) \in \Sigma$ holds for every Borel set $B \subset \mathbb{R}$. One verifies that $B_0(\Sigma) \subset B(\Sigma)$. If Σ is an algebra, $B_0(\Sigma)$ is a linear space and normed when equipped with the supremum norm $\|\cdot\|$. Similarly, $B(\Sigma)$ is a Banach space with respect to the supremum norm $\|\cdot\|$ when Σ is a σ-algebra. In this case, we have the duality pairing $(B(\Sigma), \mathbf{ba}(\Sigma))$. The expectation (the Dunford-Schwartz integral) of $X \in B(\Sigma)$ with respect to $\mu \in \mathbf{ba}(\Sigma)$ is denoted by $\mathbb{E}_\mu[X]$. Convergence of a sequence $(X_n) \subset B(\Sigma)$ to an element $X \in B(\Sigma)$ is denoted by $X_n \to X$.

When Σ is an algebra, for any $\mathfrak{p} \in \mathcal{P}(\Sigma)$ and $A \in \Sigma$ with $\mathfrak{p}(A) > 0$ we define the finitely additive probability $\mathfrak{p}^A \in \mathcal{P}(\Sigma)$ by setting, for every $B \in \Sigma$,

$$\mathfrak{p}^A(B) := \tfrac{\mathfrak{p}(A \cap B)}{\mathfrak{p}(A)}.$$

Clearly, $\mathfrak{p}^A \in \mathcal{P}_\mathfrak{p}(\Sigma)$. Given a finite partition $\pi \in \Pi$, for every $X \in B(\Sigma)$ we define

$$\mathbb{E}_\mathfrak{p}[X|\pi] := \sum_{A \in \pi,\, \mathfrak{p}(A) > 0} \mathbb{E}_{\mathfrak{p}^A}[X] \mathbf{1}_A = \sum_{A \in \pi,\, \mathfrak{p}(A) > 0} \tfrac{\mathbb{E}_\mathfrak{p}[X \mathbf{1}_A]}{\mathfrak{p}(A)} \mathbf{1}_A.$$

For all $X, Y \in B(\Sigma)$ we write $X \sim_\mathfrak{p} Y$ to indicate that X and Y have the same distribution under $\mathfrak{p}$, i.e., for each interval $I \subset \mathbb{R}$, we have $\mathfrak{p}(X \in I) = \mathfrak{p}(Y \in I)$. We shall say that X is $\mathfrak{p}$-*nonconstant* if there is no $r \in \mathbb{R}$ such that $X \sim_\mathfrak{p} r$.[1]

[1] Equivalently, we can find two disjoint intervals $I, J \subset \mathbb{R}$ such that both $\mathfrak{p}(X \in I)$ and $\mathfrak{p}(X \in J)$ are positive.

When $B(\Sigma)$ is a linear space, for $\mathcal{X} = B(\Sigma)$ or $\mathcal{X} = B_0(\Sigma)$, we say that a functional $\varphi\colon \mathcal{X} \to \mathbb{R}$ is *convex* if, for all $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$\varphi(\lambda X + (1-\lambda)Y) \le \lambda\varphi(X) + (1-\lambda)\varphi(Y),$$

and *lower semicontinuous* if, for every sequence $(X_n) \subset \mathcal{X}$ and every $X \in \mathcal{X}$,

$$X_n \to X \quad \Longrightarrow \quad \varphi(X) \le \liminf_{n\in\mathbb{N}} \varphi(X_n).$$

We say that φ is $\mathfrak{p}$-*invariant* if, for all $X, Y \in \mathcal{X}$ with $X \sim_{\mathfrak{p}} Y$, we have $\varphi(X) = \varphi(Y)$. The conjugate function of φ is the map $\varphi^*\colon \mathbf{ba}(\Sigma) \to \mathbb{R} \cup \{\infty\}$ defined by

$$\varphi^*(\mu) := \sup_{X\in B(\Sigma)} \{\mathbb{E}_\mu[X] - \varphi(X)\}.$$

If φ is convex and lower semicontinuous, then for all $X \in B(\Sigma)$

$$\varphi(X) = \sup_{\mu\in\mathbf{ba}(\Sigma)} \{\mathbb{E}_\mu[X] - \varphi^*(\mu)\} = \sup_{\mu\in\mathrm{dom}(\varphi^*)} \{\mathbb{E}_\mu[X] - \varphi^*(\mu)\}. \tag{2.2}$$

where $\mathrm{dom}(\varphi^*) := (\varphi^*)^{-1}(\mathbb{R})$; cf. [12, Propositions 3.1 & 4.1].

3. Uniqueness of convex-ranged probabilities

The present section is divided into three subsections. In the first, we consider the general case where Σ is required to satisfy only Assumption 2.1 and probabilities are only required to be finitely additive, as in Svistula's extension of Marinacci's theorem. We provide two reformulations of the uniqueness result, each unveiling a new necessary and sufficient condition. The condition in the first reformulation yields, in our opinion, the key structural insight and leads to a proof that is more direct and compact than the existing proofs. The condition in the second reformulation, while obviously a recasting of the previous one, provides a new perspective and sets the grounds for the applications of Sections 5 to 7.

3.1. The general case.

The general result on the uniqueness of convex-ranged probabilities is stated in Theorem 3.1 below. With respect to the formulations of Marinacci and Svistula – which both entail only the equivalence of the conditions (i) and (iii) in Theorem 3.1 – our formulation adds condition (ii) that, in our opinion, provides the key insight on what drives the uniqueness result.

Theorem 3.1. *Suppose Σ satisfies Assumption 2.1 and let $\mathfrak{p}$ and $\mathfrak{q}$ be finitely additive probabilities such that $\mathfrak{p}$ has convex range. The following statements are equivalent:*

(i) There exists $E \in \Sigma$ such that $\mathfrak{p}(E) \in (0, 1)$ and

$$\mathfrak{p}(A) = \mathfrak{p}(E) \qquad \Longrightarrow \qquad \mathfrak{q}(A) = \mathfrak{q}(E)$$

(ii) Every $\mathfrak{p}$-equipartition is a $\mathfrak{q}$-equipartition.

(iii) $\mathfrak{p} = \mathfrak{q}$.

Proof. (i) $\Longrightarrow$ (ii): As E^c satisfies the same condition as E, we can assume without loss of generality that $\mathfrak{p}(E) \le 1/2$. Choose $k \in \mathbb{N}$ so that $k > 1/\mathfrak{p}(E)$ and take a $\mathfrak{p}$-equipartition

$\{A_1, ..., A_k\}$. Thus, $\mathfrak{p}(A_i) = 1/k$ for $i = 1, 2, \ldots, k$. For $i \neq 1$, we have $\mathfrak{p}((A_1 \cup A_i)^c) \geq \mathfrak{p}(E) - \mathfrak{p}(A_1)$. As $\mathfrak{p}$ is convex ranged, there exists $B \in \Sigma$ such that $B \subset (A_1 \cup A_i)^c$ and $\mathfrak{p}(B) = \mathfrak{p}(E) - \mathfrak{p}(A_1) = \mathfrak{p}(E) - \mathfrak{p}(A_i)$, whence $\mathfrak{p}(E) = \mathfrak{p}(A_1 \cup B) = \mathfrak{p}(A_i \cup B)$. By (i),

$$\mathfrak{q}(A_1) + \mathfrak{q}(B) = \mathfrak{q}(A_1 \cup B) = \mathfrak{q}(A_i \cup B) = \mathfrak{q}(A_i) + \mathfrak{q}(B).$$

As a result, $\mathfrak{q}(A_1) = \mathfrak{q}(A_i)$. Thus, every $\mathfrak{p}$-equipartition is a $\mathfrak{q}$-equipartition.
(ii) $\Longrightarrow$ (iii): To begin, take $A \in \Sigma$ so that $\mathfrak{p}(A) \in \mathbb{Q}$ and $0 < \mathfrak{p}(A) < 1$. Then, $\mathfrak{p}(A) = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ with $m < n$ and we can take a $\mathfrak{p}$-equipartition $\{A_1, \ldots, A_n\}$ as in Lemma 2.2. Without loss of generality, $A = A_1 \cup \cdots \cup A_m$. By (ii),

$$\mathfrak{p}(A) = \sum_{i=1}^{m} \mathfrak{p}(A_i) = \frac{m}{n} = \sum_{i=1}^{m} \mathfrak{q}(A_i) = \mathfrak{q}(A).$$

Next, let $\mathfrak{p}(A) \in (0, 1]$. As $\mathfrak{p}$ has convex range, we find $B \in \Sigma$ such that $B \subset A$, $\mathfrak{p}(B) \in \mathbb{Q}$, and $0 < \mathfrak{p}(B) < 1$. Note that

$$\begin{aligned}\mathfrak{q}(A) &\geq \sup\{\mathfrak{q}(B)\,;\ B \in \Sigma,\ B \subset A,\ \mathfrak{p}(B) \in (0,1) \cap \mathbb{Q}\} \\ &= \sup\{\mathfrak{p}(B)\,;\ B \in \Sigma,\ B \subset A,\ \mathfrak{p}(B) \in (0,1) \cap \mathbb{Q}\} = \mathfrak{p}(A),\end{aligned}$$

where the first equality follows from what we have proved above and the second one from convex rangedness of $\mathfrak{p}$. If $\mathfrak{p}(A) = 1$, then also $\mathfrak{q}(A) = 1$ must hold. If $\mathfrak{p}(A) < 1$, the same reasoning additionally implies $\mathfrak{q}(A^c) \geq \mathfrak{p}(A^c)$, whence $\mathfrak{q}(A) = \mathfrak{p}(A)$ follows; the desired implication is established.
(iii) $\Longrightarrow$ (i): Obvious □

Remark 3.2.

(a) Owing to its more concise formulation, it is worth noticing that any of the conditions in Theorem 3.1 is equivalent to:
 (iv) Every $\mathfrak{p}$-equipartition of size 2 is a $\mathfrak{q}$-equipartition.
 In fact, it is clear that (ii) implies (iv). In the converse direction, one observes first that (iv) implies condition (i) in Theorem 3.1. To this end, let $E \in \Sigma$ be such that $\mathfrak{p}(E) = 1/2$ and take any $A \in \Sigma$ with $\mathfrak{p}(A) = \mathfrak{p}(E)$. As $\{A, A^c\}$ is a $\mathfrak{p}$-equipartition, it follows from (iv) that $\mathfrak{q}(A) = 1/2$. In particular, $\mathfrak{q}(A) = \mathfrak{q}(E)$. This yields (i). By Theorem 3.1, it then follows that (iv) implies (ii).

(b) In a different direction but in a somehow similar spirit, it is also worth observing that Theorem 3.1 implies that if $\mathfrak{p} \neq \mathfrak{q}$, then there are $\mathfrak{p}$-equipartitions of arbitrarily large size that are not $\mathfrak{q}$-equipartitions. In fact, by Theorem 3.1, $\mathfrak{p} \neq \mathfrak{q}$ if and only if there is a $\mathfrak{p}$-equipartition that is not a $\mathfrak{q}$-equipartition and from this, by using the convex rangedness of $\mathfrak{p}$, one readily obtains the above feature by successively splitting an event of the partition into smaller events of equal $\mathfrak{p}$-probability.

The second reformulation of the Marinacci-Svistula theorem can be seen as a functional counterpart to the previous one. It yields a new perspective that will prove useful in the sequel to extend the classical result on Fréchet-Hoeffeding bounds and, thence, for the applications to follow.

Theorem 3.3. *Any of the conditions in Theorem 3.1 is equivalent to:*

(v) There exists a $\mathfrak{p}$-nonconstant $X \in B_0(\Sigma)$ such that

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y) = \inf_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y).$$

Proof. (v) $\Longrightarrow$ (i): Let $x_1, \dots, x_k$ be the different values that X attains with strictly positive $\mathfrak{p}$-probability. Note that $k \geq 2$ because X is $\mathfrak{p}$-nonconstant. By definition, $\{X = x_i\} \in \Sigma$ for every $i = 1, \dots, k$. Moreover, we may assume without loss of generality that, for $i = 1, \dots, k-1$, $\mathfrak{p}(X = x_i) \leq \mathfrak{p}(X = x_{i+1})$. This entails that $\mathfrak{p}(X = x_1) \leq \frac{1}{2}$. Now, suppose an even $m \in \mathbb{N}$ satisfy $m > 1/\mathfrak{p}(X = x_1)$. Assume that $A, B \in \Sigma$ are pairwise disjoint and satisfy $\mathfrak{p}(A) = \mathfrak{p}(B) = 1/m$. By convex rangedness of $\mathfrak{p}$, we find disjoint $A', B' \in \Sigma$ such that $A', B' \subset (A \cup B)^c$ and

$$\mathfrak{p}(A') = \mathfrak{p}(X = x_1) - \tfrac{1}{m}, \quad \mathfrak{p}(B') = \mathfrak{p}(X = x_2) - \tfrac{1}{m}.$$

If $k = 2$, we may assume $\Omega = A \cup A' \cup B \cup B'$. Else, select pairwise disjoint $C_3, \dots, C_k \in \Sigma$ such that $C_i \subset (A \cup A' \cup B \cup B')^c$ and such that $\mathfrak{p}(C_i) = \mathfrak{p}(X = x_i)$ for $i = 3, \dots, k$. Now, define $X_1, X_2 \in B_0(\Sigma)$ by setting

$$X_1 = x_1 \mathbf{1}_{A \cup A'} + x_2 \mathbf{1}_{B \cup B'} + \sum_{i=3}^{k} x_i \mathbf{1}_{C_i}, \quad X_2 = x_1 \mathbf{1}_{B \cup A'} + x_2 \mathbf{1}_{A \cup B'} + \sum_{i=3}^{k} x_i \mathbf{1}_{C_i}.$$

Clearly, $X_1, X_2 \sim_{\mathfrak{p}} X$. Then, it follows from (v) that

$$0 = \mathbb{E}_{\mathfrak{q}}(X_1) - \mathbb{E}_{\mathfrak{q}}(X_2) = (x_1 - x_2)[\mathfrak{q}(A) - \mathfrak{q}(B)].$$

This can only hold if $\mathfrak{q}(A) = \mathfrak{q}(B)$. As a consequence, we infer that every $\mathfrak{p}$-equipartition of size m is also a $\mathfrak{q}$-equipartition. By combining Remark 3.2 and Lemma 2.2, we conclude that (i) holds.

(i) $\Longrightarrow$ (v): This is obvious because (i) implies that $\mathfrak{p} = \mathfrak{q}$. □

3.2. **The case of an algebra of events.** Trivially, any of the first two conditions in Theorem 3.1 as well as conditions (iv) in Remark 3.2 and (v) in Theorem 3.3, in that they imply $\mathfrak{p} = \mathfrak{q}$, imply also that $\mathfrak{p}$ and $\mathfrak{q}$ are comonotone: for all $A, B \in \Sigma$, $\mathfrak{p}(A) > \mathfrak{p}(B)$ implies $\mathfrak{q}(A) \geq \mathfrak{q}(B)$. Notably, as we will show momentarily, the reverse implication also holds whenever Σ is an algebra, i.e., it is closed under finite unions and intersections. Thus, in such a case, one may test the comonotonicity of $\mathfrak{p}$ and $\mathfrak{q}$ to determine whether or not the two charges are the same. This result extends [2], who obtained it for countably additive probabilities. The key to showing this additional equivalence is the Lemma 3.4 below, which is of independent interest. Incidentally, we will generalize this lemma in Section 8, where we will also prove a converse to it yielding a general representation for law-invariant, submodular games of bounded variation.

Lemma 3.4. *Let Σ be an algebra, $\mathfrak{p}$ and $\mathfrak{q}$ be finitely additive probabilities with $\mathfrak{p}$ convex ranged and let $\mathfrak{p} \neq \mathfrak{q}$. The set function $c\colon \Sigma \to [0,1]$ defined by*

$$c(A) := \sup\{\mathfrak{q}(B)\,;\ B \in \Sigma,\ \mathfrak{p}(B) = \mathfrak{p}(A)\}.$$

has the form $c = \gamma + v$ for a constant $\gamma \in [0,1]$ and a submodular, monotone, $\mathfrak{p}$-invariant game v. Moreover, $c(A) > \mathfrak{p}(A)$ for every $A \in \Sigma$ with $\mathfrak{p}(A) \in (0,1)$.

Proof. We set $\gamma := c(\varnothing)$ and $v := c - \gamma$. Clearly, $v(\varnothing) = 0$. Monotonicity of v follows once we establish monotonicity of c. To this effect, take $A, B \in \Sigma$ with $A \subset B$. Let $C \in \Sigma$ satisfy $\mathfrak{p}(C) = \mathfrak{p}(A)$. As $\mathfrak{p}$ has convex range, we find $D \in \Sigma$ such that $D \subset C^c$ and $\mathfrak{p}(D) = \mathfrak{p}(B) - \mathfrak{p}(A)$. Note that $C \cup D \in \Sigma$ and $\mathfrak{p}(C \cup D) = \mathfrak{p}(C) + \mathfrak{p}(D) = \mathfrak{p}(B)$. Hence, $\mathfrak{q}(C) \leq \mathfrak{q}(C \cup D) \leq c(B)$. Taking the supremum of $\mathfrak{q}(C)$ over all such C's yields $c(A) \leq c(B)$. This shows that v is a monotone game. To prove submodularity of v, it suffices to prove that, for arbitrary $A, B \in \Sigma$,

$$c(A \cup B) + c(A \cap B) \leq c(A) + c(B). \tag{3.1}$$

We may assume that $\mathfrak{p}(A) > 0$ and $\mathfrak{p}(B) > 0$, for otherwise (3.1) is clear. Let $C, D \in \Sigma$ satisfy $\mathfrak{p}(C) = \mathfrak{p}(A \cup B)$ and $\mathfrak{p}(D) = \mathfrak{p}(A \cap B)$. As

$$\mathfrak{p}(C \setminus D) \geq \mathfrak{p}(C) - \mathfrak{p}(D) = \mathfrak{p}(A \cup B) - \mathfrak{p}(A \cap B) \geq \mathfrak{p}(A \setminus B),$$

there is $E \in \Sigma$ such that $E \subset C \setminus D$ and $\mathfrak{p}(E) = \mathfrak{p}(A \setminus B)$. We then observe that $\mathfrak{p}(E \cup D) = \mathfrak{p}(E) + \mathfrak{p}(D) = \mathfrak{p}(A)$ and $\mathfrak{p}(C \setminus E) = \mathfrak{p}(C) - \mathfrak{p}(E) = \mathfrak{p}(B)$. As a result,

$$\mathfrak{q}(C) + \mathfrak{q}(D) = \mathfrak{q}(C \setminus E) + \mathfrak{q}(E \cup D) \leq c(A) + c(B).$$

Taking the supremum of $\mathfrak{q}(C) + \mathfrak{q}(D)$ over such C's and D's yields (3.1). It remains to show that, if $A \in \Sigma$ satisfies $\mathfrak{p}(A) \in (0,1)$, then $c(A) > \mathfrak{p}(A)$. Since $\mathfrak{p} \neq \mathfrak{q}$, as observed in Remark 3.2(b), there exists a strictly increasing sequence $(k_n) \subset \mathbb{N}$ such that, for every $n \in \mathbb{N}$, we find a $\mathfrak{p}$-equipartition of size k_n that is not a $\mathfrak{q}$-equipartition. Suppose that $\mathfrak{p}(A) = m/k_n$ for some $m, n \in \mathbb{N}$ with $m \leq k_n$. Select a $\mathfrak{p}$-equipartition $\{A_1, \ldots, A_{k_n}\}$ such that $\mathfrak{q}(A_1) \leq \cdots \leq \mathfrak{q}(A_{k_n})$ and $\mathfrak{q}(A_1) < \mathfrak{q}(A_{k_n})$, and set

$$B = \bigcup_{i=k_n-m+1}^{k_n} A_i.$$

Moreover, define $y_1 = \cdots = y_{k_n - m} = 0$ and $y_{k_n-m+1} = \cdots = y_{k_n} = 1$. Using the discrete Chebyshev's inequality, e.g., [17, p. 43], we obtain

$$c(A) \geq \mathfrak{q}(B) = \sum_{i=1}^{k_n} y_i \mathfrak{q}(A_i) > \tfrac{1}{k_n}\left(\sum_{i=1}^{k_n} y_i\right)\left(\sum_{i=1}^{k_n} \mathfrak{q}(A_i)\right) = \tfrac{m}{k_n} = \mathfrak{p}(A).$$

Next, we consider the general case where $A \in \Sigma$ with $\mathfrak{p}(A) \in (0,1)$. Note that for a unique function $h\colon [0,1] \to [0,1]$ with $h(m/k_n) > m/k_n$, $n \in \mathbb{N}$, $1 \leq m < k_n$, $c = h \circ \mathfrak{p}$. Let $n \in \mathbb{N}$ large enough such that we can find $m, \ell \in \mathbb{N}$ satisfying

$$0 < \tfrac{m}{k_n} < \mathfrak{p}(A) < \tfrac{\ell}{k_n}.$$

Moreover, let $\lambda \in (0,1)$ satisfy $\mathfrak{p}(A) = \frac{\lambda m+(1-\lambda)n}{k_n}$. Following the argument in the proof of [14, Proposition 4.75], one shows that the aforementioned function h is midpoint-concave. As h is nondecreasing, it is measurable. Sierpinski's Theorem [30] yields that h is concave, whence we infer

$$c(A) = h\big(\tfrac{\lambda m+(1-\lambda)n}{k_n}\big) \geq \lambda h\big(\tfrac{m}{k_n}\big) + (1-\lambda)h\big(\tfrac{\ell}{k_n}\big) > \tfrac{\lambda m+(1-\lambda)n}{k_n} = \mathfrak{p}(A).$$

This delivers the desired inequality and concludes the proof. □

Theorem 3.5. *Let Σ be an algebra. Any of the conditions in Theorem 3.1 is equivalent to:*

(vi) $\mathfrak{p}$ and $\mathfrak{q}$ are comonotone as functions from Σ to $[0,1]$: For all $A, B \in \Sigma$ with $\mathfrak{p}(A) > \mathfrak{p}(B)$ we have $\mathfrak{q}(A) \geq \mathfrak{q}(B)$.

Proof. (vi) $\implies$ (i): We first prove that statement (vi) implies that $\mathfrak{q} \ll \mathfrak{p}$. Towards a contradiction, suppose we can find $N \in \Sigma$ such that $\mathfrak{p}(N) = 0$, but $\mathfrak{q}(N) > 0$. Let $m > 1/\mathfrak{q}(N)$ and select a $\mathfrak{p}$-equipartition π of Ω of size m. In particular, there is $B \in \pi$ such that $\mathfrak{q}(B) \leq \frac{1}{m}$. Now, $\frac{1}{m} = \mathfrak{p}(B) > \mathfrak{p}(N) = 0$, but $\mathfrak{q}(B) \leq \frac{1}{m} < \mathfrak{q}(N)$, contradicting (vi).
Now, assume that (i) does not hold, i.e., $\mathfrak{q} \ll \mathfrak{p}$, but also $\mathfrak{q} \neq \mathfrak{p}$. Let $C_1 \in \Sigma$ satisfy $0 < \mathfrak{p}(C_1) < 1$. By Lemma 3.4 and by convex rangedness of $\mathfrak{p}$, we find $C_2 \in \Sigma$ such that

$$\mathfrak{p}(C_1) < \mathfrak{p}(C_2) < \sup_{B\in\Sigma,\ \mathfrak{p}(B)=\mathfrak{p}(C_1)} \mathfrak{q}(B).$$

Use Lemma 3.4 once more to infer that

$$\inf_{A\in\Sigma,\ \mathfrak{p}(A)=\mathfrak{p}(C_2)} \mathfrak{q}(A) < \mathfrak{p}(C_2) < \sup_{B\in\Sigma,\ \mathfrak{p}(B)=\mathfrak{p}(C_1)} \mathfrak{q}(B).$$

Now, choose $A, B \in \Sigma$ such that $\mathfrak{p}(A) = \mathfrak{p}(C_2) > \mathfrak{p}(C_1) = \mathfrak{p}(B)$ but $\mathfrak{q}(A) < \mathfrak{p}(C_2) < \mathfrak{q}(B)$. This contradicts (vi) and yields the desired implication.
(i) $\implies$ (vi): This follows at once from Theorem 3.1. □

3.3. **The countably additive case.** We conclude this section by showing that, in the case of countably additive probabilities on a σ-algebra, the uniqueness result obtains via an extremely short proof. The proof is based on the observation from, e.g., [23], that, given an atomless probability measure $\mathbb{P}$ on the measurable space (Ω, Σ), for all measurable functions $X, Y : \Omega \to \mathbb{R}$ with $\mathbb{P}$-integrable product and all $\mathbb{P}$-quantile functions F_X^{-1} and F_Y^{-1} of X and Y

$$\{\mathbb{E}_{\mathbb{P}}[X'Y]\,;\ X' \sim_{\mathbb{P}} X\} = \left[\int_0^1 F_X^{-1}(1-s)F_Y^{-1}(s)ds, \int_0^1 F_X^{-1}(s)F_Y^{-1}(s)ds\right].$$

One can show that the interval is a singleton if and only if at least one between X and Y is $\mathbb{P}$-constant; see, e.g., [22]. Before stating the corollary below, let us recall that nonatomicity and convex rangedness are equivalent properties of a countably additive probability on a σ-algebra.

Corollary 3.6. *Let Σ be a σ-algebra and let $\mathbb{P}$ and $\mathbb{Q}$ be countably additive probabilities such that $\mathbb{P}$ is nonatomic. The following statements are equivalent:*

(i) There exists $E \in \Sigma$ such that $\mathbb{P}(E) \in (0,1)$ and

$$\mathbb{P}(A) = \mathbb{P}(E) \qquad \Longrightarrow \qquad \mathbb{Q}(A) = \mathbb{Q}(E)$$

(iii) $\mathbb{P} = \mathbb{Q}$.

Proof. If (i) holds, then $\mathbb{Q} \ll \mathbb{P}$ necessarily holds, whence nonatomicity of $\mathbb{Q}$ follows. Let $Y = \frac{d\mathbb{Q}}{d\mathbb{P}}$. Then $\{\mathbb{E}_{\mathbb{P}}[XY]\,;\ X \sim_{\mathbb{P}} \mathbf{1}_E\} = \{\mathbb{Q}(A)\,;\ A \in \Sigma,\ \mathbb{P}(A) = \mathbb{P}(E)\} = \{\mathbb{Q}(E)\}$. Since $\mathbf{1}_A$ is not $\mathbb{P}$-constant, Y must be $\mathbb{P}$-constant and $\mathbb{P} = \mathbb{Q}$ follows, showing (iii). □

4. Collapse of capacities

As we discussed in the Introduction, several results have been obtained in a variety of settings showing that – under a seemingly innocuous condition – certain law-invariant capacities (and the associated non-linear functionals) collapse to an additive (linear) form. The link between the uniqueness theorem and these results is precisely this seemingly innocuous condition: the existence of a non-trivial unambiguous event (see below). Yet, a number of extra assumptions are added in these derivations: notably, (1) that the core or the anticore of the capacity under consideration be nonempty ([4, 7, 15, 22, 24]); (2) that the capacities be continuous ([26]); and, in all applications, that (3) the capacities be distortions of a countably additive probability. With regard to the issue of extending these results, we should like to observe that while assumption (3) can be weakened (by using Svistula's extension) without changing the mechanics of the applications, assumptions (1) and (2) would continue to play a crucial role and would conceal, at least in part, those implications that stem solely from the uniqueness theorem.

In this section, we present a novel result that isolates such implications and, consequently, showcases what is needed to obtain a collapse-type result in any given setting. We then show that generalizations of the existing results obtain (almost) automatically in the non-convex/concave case from our theorem. In the first of the following three subsections, we recall and discuss some basic definitions and establish a collapse result for the core/anticore of law-invariant capacities. We also highlight, by means of examples, the role played by extra assumptions such as those mentioned above. In the second, we apply this result to obtain sufficient conditions for a law-invariant capacity to collapse to a finitely additive probability. In the final subsection, we re-examine the collapse of the core/anticore under a more permissive, yet natural, condition of law-invariance of the capacities under consideration.

4.1. Collapse of the core/anticore.

For a capacity v, an event $A \in \Sigma$ is *v-unambiguous* (or simply *unambiguous* when the reference to the capacity is clear from the context) if

$$v(A) + v(A^c) = 1.$$

The set of all v-unambiguous events is denoted by $\Sigma_{ua}(v)$.

We now isolate the implications of the existence of a non-trivial unambiguous event for the capacity or the functional under consideration. *Per se* this condition – that the core or the anticore of the capacity be at most a singleton (see Theorem 4.2, below) – does not imply the collapse, but it clarifies what extra conditions would be needed depending on the specific form of the capacity under consideration and makes it transparent why the collapse obtains in the cases considered in the existing literature.

Theorem 4.2 follows from a simple, but useful, lemma that extends to a finitely additive setting earlier observations of [4, Proposition 2.5], [6, Corollary 3.1] and [22, Lemma F.2]. It is worth noticing that, in the special case of capacities that are law-invariant with respect to a convex ranged charge, the condition stated in the lemma is the equivalent of the Bondareva-Shapley-Schmeidler condition for the nonemptiness of the core/anticore.

Lemma 4.1. *For a finitely additive probability $\mathfrak{p}$ with convex range and a $\mathfrak{p}$-invariant capacity v, the following statements hold:*

(i) $\mathrm{acore}(v) \neq \varnothing$ *if and only if* $\mathfrak{p} \in \mathrm{acore}(v)$.
(ii) $\mathrm{core}(v) \neq \varnothing$ *if and only if* $\mathfrak{p} \in \mathrm{core}(v)$.

Proof. We only prove (i); the proof of (ii) is essentially identical. Fix $\mathfrak{q} \in \mathrm{acore}(v)$. As $\mathfrak{p}$ has convex range and v is $\mathfrak{p}$-invariant, there is a unique nondecreasing function $h\colon [0,1] \to [0,1]$ such that $v = h \circ \mathfrak{p}$. It remains to show that, for every $p \in (0,1)$, we have $h(p) \geq p$. To this end, take natural numbers $m, n \in \mathbb{N}$ with $m \leq n$ and let $\pi \in \Pi$ be an arbitrary $\mathfrak{p}$-equipartition of size n, which exists because $\mathfrak{p}$ has convex range. Let $\mathcal{M}$ be the family of all subsets $\pi' \subset \pi$ with cardinality m. Then,

$$\binom{n}{m} h(\tfrac{m}{n}) = \sum_{\pi' \in \mathcal{M}} v\left(\bigcup_{B \in \pi'} B\right) \geq \sum_{\pi' \in \mathcal{M}} \sum_{B \in \pi'} \mathfrak{q}(B) = \sum_{B \in \pi} \sum_{\pi' \in \mathcal{M},\, B \in \pi'} \mathfrak{q}(B),$$

$$\binom{n-1}{m-1} = \binom{n-1}{m-1} \sum_{B \in \pi} \mathfrak{q}(B) = \sum_{B \in \pi} \sum_{\pi' \in \mathcal{M},\, B \in \pi'} \mathfrak{q}(B).$$

Hence, $h(\frac{m}{n}) \geq \frac{m}{n}$. At last, let $p \in (0,1)$ and take any $A \in \Sigma$ with $\mathfrak{p}(A) = p$. Then,

$$h(p) = v(A) \geq \sup_{B \in \Sigma,\, B \subset A,\, \mathfrak{p}(B) \in \mathbb{Q}} v(B) = \sup_{B \in \Sigma,\, B \subset A,\, \mathfrak{p}(B) \in \mathbb{Q}} h(\mathfrak{p}(B)) \geq \sup_{B \in \Sigma,\, B \subset A,\, \mathfrak{p}(B) \in \mathbb{Q}} \mathfrak{p}(B) = p.$$

This shows that $\mathfrak{p} \in \mathrm{acore}(v)$. ☐

Theorem 4.2. *Suppose $\mathfrak{p}$ is a finitely additive probability on Σ with convex range and let v be a $\mathfrak{p}$-invariant capacity. If there exists $A \in \Sigma_{ua}(v)$ such that $v(A) \in (0,1)$, then either* $\mathrm{core}(v) = \mathrm{acore}(v) = \varnothing$, *or* $\mathrm{core}(v) = \{\mathfrak{p}\}$, *or* $\mathrm{acore}(v) = \{\mathfrak{p}\}$.

Proof. If $\mathrm{acore}(v) \neq \varnothing$, then $\mathfrak{p} \in \mathrm{acore}(v)$ by Lemma 4.1. In particular, $\mathfrak{p}(A) \leq v(A)$ and $\mathfrak{p}(A^c) \leq v(A^c)$. In view of our assumptions on A, it follows that $\mathfrak{p}(A) = v(A) \in (0,1)$. Now, let $\mathfrak{q} \in \mathrm{acore}(v)$. We shall show that $\mathfrak{p} = \mathfrak{q}$ using Theorem 3.1. Take any $B \in \Sigma$ with $\mathfrak{p}(B) = \mathfrak{p}(A)$. Observe first that $v(B) = v(A)$ and $v(B^c) = v(A^c)$ by $\mathfrak{p}$-invariance of v. As a consequence, $\mathfrak{q}(B) \leq v(B) = v(A) = 1 - v(A^c) = 1 - v(B^c) \leq 1 - \mathfrak{q}(B^c) = \mathfrak{q}(B)$. This yields $\mathfrak{q}(B) = v(A) = \mathfrak{q}(A)$ and delivers that $\mathfrak{p} = \mathfrak{q}$. The reasoning is analogous in the case that $\mathrm{core}(v)$ is nonempty. ☐

Remark 4.3. Our definition of unambiguous event is taken from Marinacci [24, 26], and is well-suited for the type of applications we consider below. In the decision theory literature there are, nonetheless, several definitions of unambiguous event that greatly differ from one another and that, in fact, have been at the center of a somewhat heated debate ([3, 13, 16, 28, 33]). Notably, Ghirardato et al. [16] use a condition that is more stringent than ours. They start by representing a certain preference relation using a convex combination, with variable coefficients, of an upper and a lower expectation taken on the same set of charges. In this context, they term "unambiguous" those events for which all the charges in the aforementioned set – which is, in fact, the Clarke differential of the functional at 0 – agree. It should be noted, however, that most definitions, but typically not the ones in [13, 33], yield the same class of unambiguous events in the important case of exact capacities. Our interests here are not related, at least immediately, to the issues discussed within this literature, and we could have opted for a different terminology, but we decided to keep the diction *unambiguous* because of the direct reference to [24, 26].

4.2. **Collapse of capacities and implications.** As already said, the condition in Theorem 4.2 does not imply the collapse of the capacity to the underlying probability $\mathfrak{p}$, and we shall see momentarily examples highlighting this fact. But, before we do so, let us record that, in the exact case (see Section 2), Theorem 4.2 does imply the collapse at once.

Corollary 4.4. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v be a $\mathfrak{p}$-invariant exact capacity. If there exists $A \in \Sigma_{ua}(v)$ such that $v(A) \in (0,1)$, then $v = \mathfrak{p}$.*

Proof. It follows from exactness that $\operatorname{core}(v)$, resp. $\operatorname{acore}(v)$, is nonempty and that v is a minimum, resp. maximum, over $\operatorname{core}(v)$, resp. $\operatorname{acore}(v)$. But then Theorem 4.2 implies that $\operatorname{core}(v) = \{\mathfrak{p}\}$, resp. $\operatorname{acore}(v) = \{\mathfrak{p}\}$, showing that $v = \mathfrak{p}$. □

The following examples show that the conclusion $v = \mathfrak{p}$ might fail if either exactness or the implied (weaker) condition that either $\operatorname{acore}(v) \neq \varnothing$ or $\operatorname{core}(v) \neq \varnothing$ are not satisfied.

Example 4.5. Take $(\Omega, \Sigma, \mathfrak{p})$ to be $([0,1], \Lambda, \lambda)$ where Λ and λ are the usual Lebesgue σ-algebra and measure, respectively.
Consider the capacity $v = h \circ \lambda$ where

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly, v is λ-invariant and every event in Λ is unambiguous under v, but it is clear that $v \neq \lambda$. Note that $\operatorname{core}(v)$ and $\operatorname{acore}(v)$ are both empty.

Example 4.6. Consider the capacity $v = h \circ \lambda$ from [27, Example 4.3 p. 53] where

$$h(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{1}{2}, \\ \frac{1}{2} & \text{if } \frac{1}{2} < x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Clearly, v is λ-invariant and the interval $[0, 1/2]$ is an unambiguous event under v. Note also that $\mathrm{core}(v) = \{\lambda\}$. Yet, $v \neq \lambda$. In particular, v is not exact.

Exact capacities are necessarily either subadditive or superadditive. We now move away from these cases by considering capacities of the form

$$v = \alpha \sup_{\mathfrak{q} \in \mathcal{Q}} \mathfrak{q} + (1 - \alpha) \inf_{\mathfrak{q} \in \mathcal{Q}} \mathfrak{q}, \tag{4.1}$$

where $\alpha \in [0, 1]$ and $\mathcal{Q}$ is a set of finitely additive probabilities. These capacities are, generally speaking, neither subadditive nor superadditive. They have been studied in [18] under the assumption that $\mathcal{Q}$ consists of countably additive probabilities and collapse results for them have been obtained in [22] under the assumption that they are law-invariant with respect to an atomless countably additive reference probability. Here, we extend the collapse by allowing $\mathcal{Q}$ to be any set of probability charges and the reference probability $\mathfrak{p}$ to be finitely additive. In order to ease the notation, given a capacity v as in 4.1, we set

$$\underline{v} := \inf_{\mathfrak{q} \in \mathcal{Q}} \mathfrak{q}, \qquad \overline{v} := \sup_{\mathfrak{q} \in \mathcal{Q}} \mathfrak{q}.$$

The following lemma states that law-invariance of a capacity of the type 4.1 is equivalent to law-invariance of $\underline{v}$ and $\overline{v}$ as long as $\alpha \neq 1/2$. The peculiarities of the case $\alpha = 1/2$ are well known (see, for instance, [26]) and will be evident, in any case, from our proofs.

Lemma 4.7. *Let $\mathfrak{p}$ be a finitely additive probability. A capacity v of the form (4.1) for $\alpha \neq 1/2$ is $\mathfrak{p}$-invariant if and only if $\underline{v}$ and $\overline{v}$ are $\mathfrak{p}$-invariant.*

Proof. The "if" implication is obvious. Conversely, let v be $\mathfrak{p}$-invariant and take $A, B \in \Sigma$ such that $\mathfrak{p}(A) = \mathfrak{p}(B)$. Since $v(A) = v(B)$ and $v(A^c) = v(B^c)$ by $\mathfrak{p}$-invariance of v,

$$\overline{v}(A) = \tfrac{\alpha}{2\alpha-1} v(A) - \tfrac{1-\alpha}{2\alpha-1}(1 - v(A^c)) = \tfrac{\alpha}{2\alpha-1} v(B) - \tfrac{1-\alpha}{2\alpha-1}(1 - v(B^c)) = \overline{v}(B).$$

This shows that $\overline{v}$ is $\mathfrak{p}$-invariant. Noting that, for every $A \in \Sigma$, we have $\underline{v}(A) = 1 - \overline{v}(A^c)$, we infer that $\underline{v}$ is $\mathfrak{p}$-invariant as well. This establishes the "only if" implication. □

Capacities of the form 4.1, $\alpha \neq 1/2$, collapse in the presence of an unambiguous event:

Proposition 4.8. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v be as in (4.1) with $\alpha \neq 1/2$. If v is $\mathfrak{p}$-invariant and there exists $A \in \Sigma_{ua}(v)$ such that $v(A) \in (0, 1)$, then $v = \mathfrak{p}$.*

Proof. Since $A \in \Sigma_{ua}(v)$, we readily see that

$$\begin{aligned}1 = v(A) + v(A^c) &= \alpha(\overline{v}(A) + \overline{v}(A^c)) + (1-\alpha)(\underline{v}(A) + \underline{v}(A^c))\\ &= \alpha(\overline{v}(A) + \overline{v}(A^c)) + (1-\alpha)(1 - \overline{v}(A^c) + 1 - \overline{v}(A))\\ &= (2\alpha - 1)(\overline{v}(A) + \overline{v}(A^c)) + 2 - 2\alpha.\end{aligned}$$

As $\alpha \neq 1/2$, this yields $\overline{v}(A) + \overline{v}(A^c) = 1$, showing that A is unambiguous under $\overline{v}$. In particular, $v(A) \leq \overline{v}(A) = 1 - \overline{v}(A^c) \leq 1 - v(A^c) = v(A)$, whence $\overline{v}(A) = v(A) \in (0,1)$. Since $\overline{v}$ is exact and $\mathfrak{p}$-invariant by Lemma 4.7, we must have $\overline{v} = \mathfrak{p}$ by virtue of Corollary 4.4. This clearly implies $\underline{v} = \mathfrak{p}$ as well and delivers $v = \mathfrak{p}$ as desired. □

In the next result we deal with a *preference relation* $\succsim$ over $B(\Sigma)$ that is represented by

$$V(X) = \alpha \sup_{\mathfrak{q}\in\mathcal{Q}} \mathbb{E}_{\mathfrak{q}}[X] + (1-\alpha) \inf_{\mathfrak{q}\in\mathcal{Q}} \mathbb{E}_{\mathfrak{q}}[X], \tag{4.2}$$

where $\alpha \in [0,1]$ and $\mathcal{Q}$ is a set of finitely additive probabilities. We say that:

(a) $A \in \Sigma$ is *unambiguous* for $\succsim$ if $V(\mathbf{1}_A) + V(\mathbf{1}_{A^c}) = 1$.
(b) A preference relation $\succsim$ over $B(\Sigma)$ is *probabilistically sophisticated* if there exists a (unique) convex-ranged finitely additive probability $\mathfrak{p}$ such that, for all $A, B \in \Sigma$, one is $\succsim$-indifferent between $\mathbf{1}_A$ and $\mathbf{1}_B$ whenever $\mathfrak{p}(A) = \mathfrak{p}(B)$.
(c) A preference relation $\succsim$ is a *subjective expected utility preference* if it is represented by $\mathbb{E}_{\mathfrak{p}}$ for some finitely additive probability $\mathfrak{p}$ with convex range.

From Proposition 4.8, we obtain at once the following generalization of [26] where we remove altogether the assumption of monotone continuity.

Corollary 4.9. *Let $\succsim$ be a preference relation over $B(\Sigma)$ that is represented as in (4.2) for some $\alpha \neq \frac{1}{2}$. If there exists $A \in \Sigma$ such that A is unambiguous for $\succsim$ and $0 < V(\mathbf{1}_A) < 1$, then $\succsim$ is probabilistically sophisticated if and only if it is a subjective expected utility preference.*

Proof. Consider the capacity defined by $v(A) = V(\mathbf{1}_A)$ for $A \in \Sigma$. If $A \in \Sigma$ is unambiguous for $\succsim$ and $0 < V(\mathbf{1}_A) < 1$, then A is unambiguous under v and satisfies $v(A) \in (0,1)$. But then $\succsim$ is a subjective expected utility preference by Proposition 4.8. The converse is immediate. □

4.3. **Collapse of the core/anticore revisited.** In this subsection we show that the previous results remain true if we weaken the requirement of law-invariance and stipulate that a given capacity be law-invariant only with respect to those events that carry a sufficiently high degree of reliability. To capture this idea, we shall say that a capacity v is $\mathfrak{p}$-*invariant on unambiguous events* if it is constant on the equilikelihood classes under $\mathfrak{p}$ of unambiguous events, i.e., if for all $A \in \Sigma_{ua}(v)$ and $B \in \Sigma$

$$\mathfrak{p}(A) = \mathfrak{p}(B) \implies v(A) = v(B).$$

Clearly, $\mathfrak{p}$-invariance on unambiguous events is weaker than $\mathfrak{p}$-invariance, and the two properties coincide only when, for every event, there exists an unambiguous event with the same $\mathfrak{p}$-probability. In particular, the capacities fulfilling this weaker notion of law-invariance need not be distortions of probability measures. In connection with the notion of $\mathfrak{p}$-invariance on unambiguous events, we also introduce the notions of *unambiguous core* and *anticore* of v by

$$\mathrm{core}_{ua}(v) := \{\mathfrak{p} \in \mathcal{P}(\Sigma)\,;\ \forall A \in \Sigma_{ua}(v),\ \mathfrak{p}(A) \geq v(A)\},$$

$$\mathrm{acore}_{ua}(v) := \{\mathfrak{p} \in \mathcal{P}(\Sigma)\,;\ \forall A \in \Sigma_{ua}(v),\ \mathfrak{p}(A) \leq v(A)\}.$$

Clearly, $\mathrm{core}(v)$ is contained in $\mathrm{core}_{ua}(v)$ and the same holds for the anticore.
The following result shows that, albeit weaker than law-invariance, law-invariance on unambiguous events suffices to generate the same result on the core/anticore of a law-invariant capacity we obtained in Theorem 4.2.

Proposition 4.10. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v a capacity that is $\mathfrak{p}$-invariant on unambiguous events. If there is $A \in \Sigma_{ua}(v)$ such that $v(A) \in (0,1)$, then*

$$\mathrm{core}(v) \cup \mathrm{acore}(v) \subset \mathrm{core}_{ua}(v) \cup \mathrm{acore}_{ua}(v) \subset \{\mathfrak{p}\}.$$

Proof. We follow the argument used in the proof of Theorem 4.2. First, observe that $\mathfrak{p}(A) \in (0,1)$ by $\mathfrak{p}$-invariance on unambiguous events. Now, assume that $\mathrm{acore}_{ua}(v)$ is nonempty and take any $\mathfrak{q} \in \mathrm{acore}_{ua}(v)$. Take an arbitrary $B \in \Sigma$ with $\mathfrak{p}(B) = \mathfrak{p}(A)$. By assumption, $v(A) = v(B)$. Since A^c is also unambiguous, we have $v(A^c) = v(B^c)$ as well. As a consequence, B and B^c are both unambiguous and, hence, $\mathfrak{q}(B) \leq v(B) = v(A) = 1 - v(A^c) = 1 - v(B^c) \leq 1 - \mathfrak{q}(B^c) = \mathfrak{q}(B)$. This shows that $\mathfrak{q}(B) = v(A) = \mathfrak{q}(A)$. Thus, condition (i) in Theorem 3.1 is satisfied and we infer that $\mathfrak{q} = \mathfrak{p}$. The proof for the core is similar. □

Just like Theorem 4.2, the proposition does not imply the collapse of the capacity to the probability $\mathfrak{p}$ but only (quite trivially) the coincidence of v and $\mathfrak{p}$ on $\Sigma_{ua}(v)$ if $\mathfrak{p}$ is either in the unambiguous core or in the unambiguous anticore. Yet, just like in the case of $\mathfrak{p}$-invariance, the collapse follows seamlessly in the case of exact capacities or, more generally, capacities that can be represented as in (4.1). The simple proof is omitted.

Corollary 4.11. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v be as in (4.1) with $\alpha \neq 1/2$. If v is $\mathfrak{p}$-invariant on unambiguous events and there exists $A \in \Sigma_{ua}(v)$ such that $v(A) \in (0,1)$, then $v = \mathfrak{p}$.*

As an immediate consequence, the (linear) expectation under $\mathfrak{p}$ is the unique Choquet integral with respect to a capacity that fulfills the requirements in the corollary. Put differently, if a capacity is as in (4.1) with $\alpha \neq 1/2$ and is $\mathfrak{p}$-invariant on unambiguous events, then the corresponding Choquet integral collapses to the (linear) expectation under $\mathfrak{p}$ as soon as it is linear on the vector space spanned by a nontrivial indicator function. The corollary can therefore be viewed as an extension of the "collapse to the mean" results for Choquet integrals established in [7, 9, 22] where the *global* assumption of law-invariance is replaced by the weaker *local* assumption of law-invariance on unambiguous events. This is of especial interest when the collapse is applied to pricing rules in markets with frictions as done in [7, 9], as one would not expect a sound (market-consistent) pricing rule to be law-invariant on the entire payoff space but only on a portion of it (which typically consists of payoffs that are fully unreplicable in financial markets, as is often the case for losses arising from insurance risk or operational risk); see, e.g., the general discussion in [10].

5. Fréchet-Hoeffding bounds

From this section on, we depart from the existing literature by presenting a novel set of applications of the uniqueness theorems of Section 3. These applications stem from Proposition 3.3 that, as we observed, provides a new perspective on the uniqueness result.

In Proposition 3.3 we showed that, given a finitely additive probability $\mathfrak{p}$ with convex range and any other finitely additive probability $\mathfrak{q} \neq \mathfrak{p}$, for every $\mathfrak{p}$-nonconstant function $X \in B_0(\Sigma)$

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y) > \inf_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y).$$

This result is reminiscent of the classic Fréchet-Hoeffding bounds (see, e.g., the formulation in [22]) yet it differs as the expectations are taken with respect to finitely additive probabilities. In the next theorem we sharpen it so to fully extend the classical result to a finitely additive setting. It is worth noting that, in this setting, we cannot resort to Radon-Nikodým derivatives and, hence, the standard arguments break down. Our proof shows how an appropriate application of Lemma 3.4 allows to circumvent this problem.

Theorem 5.1. *Let Σ be an algebra. Let $\mathfrak{p}$ be a finitely additive probability with convex range and take a finitely additive probability $\mathfrak{q} \neq \mathfrak{p}$. For every $\mathfrak{p}$-nonconstant $X \in B_0(\Sigma)$*

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y) > \mathbb{E}_{\mathfrak{p}}(X) > \inf_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}(Y).$$

In particular, for every $A \in \Sigma$ such that $\mathfrak{p}(A) \in (0,1)$,

$$\sup_{B \in \Sigma,\, \mathfrak{p}(B) = \mathfrak{p}(A)} \mathfrak{q}(B) > \mathfrak{p}(A) > \inf_{B \in \Sigma,\, \mathfrak{p}(B) = \mathfrak{p}(A)} \mathfrak{q}(B).$$

Proof. It suffices to show the left-hand side inequalities. To this effect, let X be $\mathfrak{p}$-nonconstant and let $x_1 < \cdots < x_k$ be the different values it takes with strictly positive $\mathfrak{p}$-probability. Suppose first that there is $N \in \Sigma$ such that $\mathfrak{p}(N) = 0$ but $\mathfrak{q}(N) > 0$. In this case, we can partition N^c into events $A_1, \ldots, A_k \in \Sigma$ such that, for $1 \leq i \leq k$, we have $\mathfrak{p}(A_i) = \mathfrak{p}(X = x_i)$. As for all $t > 0$ we have $\sum_{i=1}^k x_i \mathbf{1}_{A_i} + t\mathbf{1}_N \sim_{\mathfrak{p}} X$,

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}[Y] \geq \sup_{t > 0} \Big(\sum_{i=1}^k x_i \mathfrak{q}(A_i) + t\mathfrak{q}(N) \Big) = \infty > \mathbb{E}_{\mathfrak{p}}[X],$$

proving the desired inequality. Hence, we assume in the following that $\mathfrak{q} \ll \mathfrak{p}$ and establish the desired inequality by induction over k. Observe that $k \geq 2$ as X is $\mathfrak{p}$-nonconstant.

<u>$k = 2$.</u> Let $A = \{X = x_2\} \in \Sigma$. As $\mathfrak{p}(A) \in (0,1)$, it follows from Lemma 3.4 that

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_{\mathfrak{q}}[Y] = x_1 + (x_2 - x_1) \sup\{\mathfrak{q}(B)\,;\; B \in \Sigma,\; \mathfrak{p}(B) = \mathfrak{p}(A)\} > x_1 + (x_2 - x_1)\mathfrak{p}(A) = \mathbb{E}_{\mathfrak{p}}[X].$$

$\underline{k-1 \Rightarrow k.}$ Set $A = \{X \neq x_k\}$ and note that $x_k > \mathbb{E}_{\mathfrak{p}^A}(X)$. Using the case $k=2$, we find $B \in \Sigma$ with $\mathfrak{p}(B) = \mathfrak{p}(A)$ such that $Z = x_k \mathbf{1}_{B^c} + \mathbb{E}_{\mathfrak{p}^A}(X)\mathbf{1}_B$ satisfies

$$\mathbb{E}_{\mathfrak{q}}[Z] > \mathbb{E}_{\mathfrak{p}}[Z] = \mathfrak{p}(B^c)x_k + \mathfrak{p}(B)\mathbb{E}_{\mathfrak{p}^A}[X] = \mathfrak{p}(A^c)x_k + \mathfrak{p}(A)\mathbb{E}_{\mathfrak{p}^A}[X] = \mathbb{E}_{\mathfrak{p}}[X].$$

By convex rangedness of $\mathfrak{p}$, we find $Y \sim_{\mathfrak{p}} X$ such that $Y\mathbf{1}_{B^c} = x_k \mathbf{1}_{B^c}$. If $\mathfrak{q}(B) = 0$, then $\mathbb{E}_{\mathfrak{q}}[Y] = x_k > \mathbb{E}_{\mathfrak{p}}[X]$ and we are done. Hence, assume that $\mathfrak{q}(B) > 0$. Note that, in this case, Y takes $k-1$ different values with strictly positive $\mathfrak{p}^B$-probability, namely $x_1, \dots, x_{k-1}$. Moreover, $\mathbb{E}_{\mathfrak{p}^B}[Y] = \mathbb{E}_{\mathfrak{p}^A}[X]$ because $\mathfrak{p}(B) = \mathfrak{p}(A)$. Hence, by induction hypothesis, there exists $Y' \sim_{\mathfrak{p}^B} Y$ such that $\mathbb{E}_{\mathfrak{q}^B}[Y'] \geq \mathbb{E}_{\mathfrak{p}^B}[Y] = \mathbb{E}_{\mathfrak{p}^A}[X]$. But then $X' := x_k \mathbf{1}_{B^c} + Y'\mathbf{1}_B \sim_{\mathfrak{p}} X$ satisfies

$$\mathbb{E}_{\mathfrak{q}}[X'] = \mathfrak{q}(B^c)x_k + \mathfrak{q}(B)\mathbb{E}_{\mathfrak{q}^B}[Y'] \geq \mathfrak{q}(B^c)x_k + \mathfrak{q}(B)\mathbb{E}_{\mathfrak{p}^A}[X] = \mathbb{E}_{\mathfrak{q}}[Z] > \mathbb{E}_{\mathfrak{p}}[X].$$

This concludes the induction argument and delivers the first assertion. The second assertion already follows directly from Lemma 3.4. □

6. The Fatou property of law-invariant risk measures

The automatic Fatou property of law-invariant risk measures is a fundamental result in the theory of risk measures. The result has been obtained in [19, 31] under the assumption that the underlying probability be countably additive and atomless. We now derive a Fatou representation in the case of a finitely additive probability. Throughout this section we assume that Σ is a σ-algebra.

A *(monetary) risk measure* is a map $\rho : B(\Sigma) \to \mathbb{R}$ with the following properties:

(a) Monotonicity: $\rho(X) \leq \rho(Y)$ for all $X, Y \in B(\Sigma)$ with $X \geq Y$.
(b) Cash-additivity: $\rho(X+m) = \rho(X) + m$ for all $X \in B(\Sigma)$ and $m \in \mathbb{R}$.

As it is well known, any risk measure is (Lipschitz) continuous with respect to the supremum norm on $B(\Sigma)$. A risk measure ρ has the *Fatou property* if, for $(X_n) \subset B(\Sigma)$ and $X \in B(\Sigma)$

$$X_n \to X, \qquad \sup_{n\in\mathbb{N}} \|X_n\| < \infty \qquad \Longrightarrow \qquad \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

When ρ is law-invariant with respect to a countably additive probability $\mathbb{P}$, the assumption of pointwise convergence can be replaced by $\mathbb{P}$-almost sure convergence or convergence in $\mathbb{P}$-probability. Here, as we consider risk measures that are law-invariant with respect to finitely additive probabilities, we must replace convergence in probability with its *hazy convergence* generalization (cf. [8, Chapter 4.3]): A sequence $(X_n) \subset B(\Sigma)$ converges to $X \in B(\Sigma)$ hazily if, for every $\varepsilon > 0$, we have $\lim_{n\to\infty} \mathfrak{p}(|X_n - X| > \varepsilon) = 0$.

The extension of the automatic Fatou property (Theorem 6.2) is based on Lemma 6.1 below, which is a consequence of Theorem 5.1. For given $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}(\Sigma)$ and a finite measurable partition $\pi \in \Pi$, we introduce the finitely additive probability

$$\mathfrak{p}_{\pi,\mathfrak{q}} := \sum_{A\in\pi,\ \mathfrak{p}(A)>0} \mathfrak{q}(A)\mathfrak{p}^A.$$

In Lemma 6.1 (part (ii)), we show that the composition of $\mathfrak{q}$-expectations with conditional $\mathfrak{p}$-expectations given the information conveyed by π can be expressed as a straight expectation with respect to $\mathfrak{p}_{\pi,\mathfrak{q}}$. Along with part (iii), this provides a key step toward the proof of Theorem 6.2 as these expectations will appear naturally in the dual representation of $\mathfrak{p}$-invariant risk measures. Part (i) in Lemma 6.1 establishes the finitely additive counterpart of the *dilatation monotonicity* of a risk measure that is instrumental – just like dilation monotonicity in the countably additive case (see [19, 31]) – toward obtaining the automatic Fatou property.

Lemma 6.1. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and $\rho\colon B(\Sigma) \to \mathbb{R}$ be a $\mathfrak{p}$-invariant convex risk measure. For all $\pi \in \Pi$ and $\mathfrak{q} \in \mathcal{P}_{\mathfrak{p}}(\Sigma)$ the following statements hold:*

(i) $\rho\left(\mathbb{E}_{\mathfrak{p}}[X|\pi]\right) \leq \rho(X)$ *for every* $X \in B(\Sigma)$.
(ii) $\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] = \mathbb{E}_{\mathfrak{q}}[\mathbb{E}_{\mathfrak{p}}[X|\pi]]$ *for every* $X \in B(\Sigma)$.
(iii) $\rho^*(\mathfrak{p}_{\pi,\mathfrak{q}}) \leq \rho^*(\mathfrak{q})$.

Proof. It is easy to verify that any $\mu \in \mathrm{dom}(\rho^*)$ must satisfy $\mu(\Omega) = 1$ and $\mu(A) \geq 0$ for every $A \in \Sigma$ as well as $\mu \ll \mathfrak{p}$. Hence, for every $X \in B(\Sigma)$, we can rewrite (2.2) as

$$\rho(X) = \sup_{\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \left\{\mathbb{E}_{\mathfrak{q}}[X] - \rho^*(\mathfrak{q})\right\}. \tag{6.1}$$

Now, take $X \in B_0(\Sigma)$ and $A \in \Sigma$ such that $\mathfrak{p}(A) > 0$. For every $\mathfrak{q} \in \mathcal{P}_{\mathfrak{p}}(\Sigma)$ with $\mathfrak{q}(A) > 0$

$$\sup_{X'\sim_{\mathfrak{p}^A} X} \mathbb{E}_{\mathfrak{q}}[X'\mathbf{1}_A] = \mathfrak{q}(A) \sup_{X'\sim_{\mathfrak{p}^A} X} \mathbb{E}_{\mathfrak{q}^A}[X'] \geq \mathfrak{q}(A)\mathbb{E}_{\mathfrak{p}^A}[X] = \mathbb{E}_{\mathfrak{q}}[\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A] \tag{6.2}$$

by Theorem 5.1. Take an arbitrary $Y \in B_0(\Sigma)$ and observe that, for every $X' \sim_{\mathfrak{p}^A} X$, we have $X'\mathbf{1}_A + Y\mathbf{1}_{A^c} \sim_{\mathfrak{p}} X\mathbf{1}_A + Y\mathbf{1}_{A^c}$ as well. We then infer from (6.1) and (6.2) that

$$\begin{aligned}\rho(X\mathbf{1}_A + Y\mathbf{1}_{A^c}) &= \sup_{X'\sim_{\mathfrak{p}^A} X} \rho(X'\mathbf{1}_A + Y\mathbf{1}_{A^c}) = \sup_{\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \left\{ \sup_{X'\sim_{\mathfrak{p}^A} X} \mathbb{E}_{\mathfrak{q}}[X'\mathbf{1}_A + Y\mathbf{1}_{A^c}] - \rho^*(\mathfrak{q})\right\} \\ &\geq \sup_{\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \left\{\mathbb{E}_{\mathfrak{q}}[\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A + Y\mathbf{1}_{A^c}] - \rho^*(\mathfrak{q})\right\} = \rho(\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A + Y\mathbf{1}_{A^c}),\end{aligned}$$

where we used $\mathfrak{p}$-invariance in the first equality. Reiterating this argument yields

$$\rho(X) = \rho\Big(\sum_{A\in\pi,\ \mathfrak{p}(A)>0} X\mathbf{1}_A\Big) \geq \rho\Big(\sum_{A\in\pi,\ \mathfrak{p}(A)>0} \mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A\Big) = \rho\left(\mathbb{E}_{\mathfrak{p}}[X|\pi]\right). \tag{6.3}$$

Now, suppose that $X \in B(\Sigma)$ and select any sequence $(X_n) \subset B_0(\Sigma)$ satisfying $X_n \to X$, whence also $\mathbb{E}_{\mathfrak{p}}[X_n|\pi] \to \mathbb{E}_{\mathfrak{p}}[X|\pi]$ follows. Using (6.3) and continuity of ρ yields

$$\rho\left(\mathbb{E}_{\mathfrak{p}}[X|\pi]\right) = \lim_{n\to\infty} \rho\left(\mathbb{E}_{\mathfrak{p}}[X_n|\pi]\right) \leq \lim_{n\to\infty} \rho(X_n) = \rho(X).$$

This concludes the proof of (i). A direct calculation shows that (ii) holds. Finally,

$$\rho^*(\mathfrak{q}) \geq \sup_{X\in B(\Sigma)} \left\{\mathbb{E}_{\mathfrak{q}}[\mathbb{E}_{\mathfrak{p}}[X|\pi]] - \rho(\mathbb{E}_{\mathfrak{p}}[X|\pi])\right\} \geq \sup_{X\in B(\Sigma)} \left\{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] - \rho(X)\right\} = \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})$$

by combining (i) and (ii). This establishes (iii) and concludes the proof. ∎

We now give two dual representations of a $\mathfrak{p}$-invariant risk measure when $\mathfrak{p}$ is a probability charge with convex range. The first representation shows that the dual domain can always be restricted to the special finitely additive probabilities $\mathfrak{p}_{\pi,\mathfrak{q}}$'s. The key insight of the second representation is that the role played by the finitely additive probability $\mathfrak{q}$ in the expectation under $\mathfrak{p}_{\pi,\mathfrak{q}}$ is "discrete" in the sense that $\mathfrak{q}$ merely assigns weights on the (finitely many) events in the partition π, while the probabilistic "nucleus" of $\mathfrak{p}_{\pi,\mathfrak{q}}$ is entirely driven by $\mathfrak{p}$, reflecting the underlying $\mathfrak{p}$-invariance. In other words, for every $\pi \in \Pi$, denote by $\mathcal{W}_\pi$ the set of all functions $w \in [0,1]^\pi$ summing up to 1 over the events in π with strictly positive $\mathfrak{p}$-probability and consider the probability charge

$$\mathfrak{p}_{\pi,w} := \sum_{A\in\pi,\,\mathfrak{p}(A)>0} w(A)\mathfrak{p}^A.$$

The theorem below shows that a convex $\mathfrak{p}$-invariant risk measure can be represented as a supremum of penalized expectations with respect to probability charges of this type. This provides a new insight into the structure of $\mathfrak{p}$-invariance that does not seem to have been highlighted thus far already in the standard countably additive literature. As a byproduct of these new representations, we immediately obtain that, when the reference probability is countably additive, any law-invariant convex risk measure is automatically $\sigma(B(\Sigma),\mathbf{ca}(\Sigma))$-lower semicontinuous and, hence, possesses the Fatou property.

Theorem 6.2. *Let Σ be a σ-algebra and $\mathfrak{p}$ a finitely additive probability with convex range. Moreover, let $\rho\colon B(\Sigma)\to\mathbb{R}$ be a $\mathfrak{p}$-invariant convex risk measure. Then:*

(i) For every $X \in B(\Sigma)$

$$\rho(X) = \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_\mathfrak{p}(\Sigma)} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})\} = \sup_{\pi\in\Pi,\,w\in\mathcal{W}_\pi} \{\mathbb{E}_{\mathfrak{p}_{\pi,w}}[X] - \rho^*(\mathfrak{p}_{\pi,w})\}. \tag{6.4}$$

(ii) For all $X \in B(\Sigma)$,

$$\rho(X) = \sup_{\mathfrak{q}\in\mathcal{P}(\Sigma),\ \mathfrak{q}\lll\mathfrak{p}} \{\mathbb{E}_\mathfrak{q}[X] - \rho^*(\mathfrak{q})\}. \tag{6.5}$$

Moreover, ρ has the hazy Fatou property *in that, whenever $(X_n)\subset B(\Sigma)$ converges to $X\in B(\Sigma)$ hazily and satisfies $\sup_{n\in\mathbb{N}}\|X_n\|$,*

$$\rho(X) \le \liminf_{n\to\infty} \rho(X_n).$$

(iii) If $\mathfrak{p}$ is countably additive, then ρ is $\sigma(B(\Sigma),\mathbf{ca}(\Sigma))$-l.s.c. and has the Fatou property.

Proof. To prove (6.4), it clearly suffices to establish the left-hand side equality there. To this effect, let $X \in B_0(\Sigma)$ and $A = \{\omega\in\Omega\,;\ \mathfrak{p}(X = X(\omega)) > 0\} \in \Sigma$. Note that $X\mathbf{1}_A = \mathbb{E}_\mathfrak{p}[X|\pi_0]$ for a suitable $\pi_0 \in \Pi$. As $X\mathbf{1}_A \sim_\mathfrak{p} X$, we infer from $\mathfrak{p}$-invariance of ρ and (6.1) that

$$\rho(X) = \rho(\mathbb{E}_\mathfrak{p}[X|\pi_0]) \le \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_\mathfrak{p}(\Sigma)} \{\mathbb{E}_\mathfrak{q}[\mathbb{E}_\mathfrak{p}[X|\pi]] - \rho^*(\mathfrak{q})\} \le \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_\mathfrak{p}(\Sigma)} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})\},$$

where we used Lemma 6.1 in the last inequality. Another application of (6.1) yields the left-hand side equality in (6.4). Now, take $X \in B(\Sigma)$ and select any sequence $(X_n)\subset B_0(\Sigma)$ satisfying $X_n \le X_{n+1}$, $n\in\mathbb{N}$, and $X_n \to X$. Note that, for all $\pi\in\Pi$ and $\mathfrak{q}\in\mathcal{P}_\mathfrak{p}(\Sigma)$, we have $\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X_n|\pi] \to \mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X|\pi]$ and the sequence $(\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X_n|\pi])$ is increasing as well. Then, by continuity and monotonicity of ρ, we infer from (6.4) applied to $B_0(\Sigma)$ that

$$\rho(X) = \sup_{n\in\mathbb{N}} \rho(X_n) = \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \left\{ \sup_{n\in\mathbb{N}} \mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X_n] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}}) \right\} = \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})\}.$$

As a result, the left-hand side equality in (6.4) holds on $B(\Sigma)$.

Turning to (ii), let $\pi \in \Pi$ and $\mathfrak{q} \in \mathcal{P}_{\mathfrak{p}}(\Sigma)$ be arbitrary. We claim that $\mathfrak{p}_{\pi,\mathfrak{q}} \lll \mathfrak{p}$. Indeed, define $\alpha := \max\{\mathfrak{q}(A) \mid A \in \pi,\, \mathfrak{p}(A) > 0\}$ and $\beta := \min\{\mathfrak{p}(A) \mid A \in \Pi,\, \mathfrak{p}(A) > 0\}$. For $\varepsilon \in (0,1)$ arbitrary let $\delta := \frac{\beta}{\alpha}\varepsilon$. If $E \in \Sigma$ satisfies $\mathfrak{p}(E) \le \delta$, then

$$\mathfrak{p}_{\pi,\mathfrak{q}}(E) = \sum_{A\in\pi,\ \mathfrak{p}(A)>0} \frac{\mathfrak{q}(A)\mathfrak{p}(A\cap E)}{\mathfrak{p}(A)} \le \frac{\alpha}{\beta} \sum_{A\in\pi,\ \mathfrak{p}(A)>0} \mathfrak{p}(A\cap E) = \tfrac{\alpha}{\beta}\mathfrak{p}(E) \le \varepsilon,$$

finishing the proof that $\mathfrak{p}_{\pi,\mathfrak{q}} \lll \mathfrak{p}$. (8.5) is directly implied by (6.4). For the hazy Fatou property, suppose $(X_n) \subset B(\Sigma)$ converges to $X \in B(\Sigma)$ hazily and satisfies $\sup_{n\in\mathbb{N}} \|X_n\| < \infty$. Set $K := \sup_{n\in\mathbb{N}} \|X_n - X\| < \infty$, let $\pi \in \Pi$, $\mathfrak{q} \in \mathcal{P}_{\mathfrak{p}}(\Sigma)$, and $\varepsilon > 0$, and estimate

$$\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[|X_n - X|] \le \varepsilon + K\mathfrak{p}_{\pi,\mathfrak{q}}\left(|X_n - X| > \varepsilon\right) \le \varepsilon + K\mathfrak{p}(|X_n - X| > \varepsilon) \sum_{A\in\pi,\ \mathfrak{p}(A)>0} \frac{\mathfrak{q}(A)}{\mathfrak{p}(A)}.$$

Letting $n \to \infty$ now implies $\limsup_{n\to\infty} \mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[|X_n - X|] \le \varepsilon$. The arbitrariness of $\varepsilon > 0$ finally yields $\lim_{n\to\infty} \mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X_n] = \lim_{n\to\infty} \mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X]$. At last, use (6.4) to verify that

$$\begin{aligned}\rho(X) &= \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \liminf_{n\to\infty} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X_n] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})\} \\ &\le \liminf_{n\to\infty} \sup_{\pi\in\Pi,\,\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mathfrak{q}}}[X] - \rho^*(\mathfrak{p}_{\pi,\mathfrak{q}})\} = \liminf_{n\to\infty} \rho(X_n).\end{aligned}$$

As for (iii), if $\mathfrak{p}$ is countably additive, then it suffices to recognize that all $\mathfrak{p}_{\pi,\mathfrak{q}}$'s are also countably additive to infer from (6.4) that ρ is $\sigma(B(\Sigma), \mathbf{ca}(\Sigma))$-lower semicontinuous. The Fatou property follows from the hazy Fatou property in (ii). $\square$

To conclude the section, we observe that an inspection of the proofs above shows that the results remain true if Σ is assumed to be an algebra instead of a σ-algebra provided one replaces $B(\Sigma)$ with the smaller space $B_0(\Sigma)$.

7. Uniqueness of the reference probability

At the modeling level, an upside of law-invariant capacities is that they allow one to incorporate the evidence that is available in the form of estimated likelihoods of relevant events. The point becomes murky, nonetheless, when the chosen capacity is simultaneously a distortion of more than one probability. It becomes, thus, necessary to be able to single out such capacities. This issue is addressed in [21], where it is approached by means of Lyapunov's Convexity Theorem. As our focus is narrower than in [21], here we are able to present some sharper results that obtain by using the machinery we developed in the previous sections.

The first result stems from our Fréchet-Hoeffding bounds for finitely additive probabilities (Theorem 5.1). It yields a restrictive necessary condition on the distortion function for a capacity to be simultaneously law-invariant with respect to different finitely additive probabilities. In what follows, we denote by $h_{\mathfrak{p}}$ the distortion function associated to a $\mathfrak{p}$-invariant capacity v, i.e., a nondecreasing function $h_{\mathfrak{p}}\colon [0,1] \to [0,1]$ such that $v = h_{\mathfrak{p}} \circ \mathfrak{p}$.

Proposition 7.1. *Let Σ be an algebra and let $\mathfrak{p}$ and $\mathfrak{q}$ be finitely additive probabilities such that $\mathfrak{p}$ has convex range. If a capacity v is both $\mathfrak{p}$- and $\mathfrak{q}$-invariant, then $\mathfrak{p} \ll \mathfrak{q}$ and the following statements hold:*

(i) If $h_{\mathfrak{p}}(x) > 0$ for every $0 < x \leq 1$, then $\mathfrak{p} \approx \mathfrak{q}$.
(ii) If $\mathfrak{p} \approx \mathfrak{q}$ and $\mathfrak{p} \neq \mathfrak{q}$, then there exists $\alpha \in [0,1]$ such that

$$h_{\mathfrak{p}} = h_{\mathfrak{q}} = \alpha \mathbf{1}_{(0,1)} + \mathbf{1}_{\{1\}}.$$

Proof. It follows from [21] that $\mathfrak{p} \ll \mathfrak{q}$. To prove (i), assume that $h_{\mathfrak{p}}(x) > 0$ for every $0 < x \leq 1$ but we find $N \in \Sigma$ such that $\mathfrak{p}(N) = 0$ and $\mathfrak{q}(N) > 0$. Hence, $0 = h_{\mathfrak{p}}(0) = v(N) = h_{\mathfrak{q}}(\mathfrak{q}(N))$. Let $m \in \mathbb{N}$ satisfy $m > 1/\mathfrak{q}(N)$. Fix a $\mathfrak{p}$-equipartition $\pi \in \Pi$ of size m and note that

$$\min_{B \in \pi} \mathfrak{q}(B) \leq \tfrac{1}{m} < \mathfrak{q}(N).$$

Hence, we find a suitable $B \in \pi$ such that

$$0 < h_{\mathfrak{p}}(\tfrac{1}{m}) = v(B) \leq h_{\mathfrak{q}}(\tfrac{1}{m}) \leq h_{\mathfrak{q}}(\mathfrak{q}(N)) = 0.$$

This is a contradiction and we conclude that $\mathfrak{q} \approx \mathfrak{p}$ must hold. To establish (ii), suppose that $\mathfrak{p} \approx \mathfrak{q}$ and $\mathfrak{p} \neq \mathfrak{q}$. Set $\mathcal{R} = \{(\mathfrak{p}(A), \mathfrak{q}(A))\,;\ A \in \Sigma\}$. Moreover, fix $s \in (0,1)$ and define

$$\sigma = \sup\{x \in (0,1)\,;\ h_{\mathfrak{q}}(x) = h_{\mathfrak{q}}(s)\}.$$

As $h_{\mathfrak{q}}$ is nondecreasing, $\sigma \geq s$. Now, assume that $\sigma < 1$. By Theorem 5.1, we find sequences $(y_n^+) \subset (\sigma, 1)$ and $(y_n^-) \subset (0, \sigma)$ such that

$$y_n^+ \uparrow \sup_{B \in \Sigma,\ \mathfrak{p}(B) = \sigma} \mathfrak{q}(B), \qquad y_n^- \downarrow \inf_{B \in \Sigma,\ \mathfrak{p}(B) = \sigma} \mathfrak{q}(B)$$

while both (σ, y_n^+) and (σ, y_n^-) belong to $\mathcal{R}$ for each $n \in \mathbb{N}$. By construction and definition of σ, we have for every $n \in \mathbb{N}$

$$h_{\mathfrak{p}}(\sigma) = h_{\mathfrak{q}}(y_n^+) > h_{\mathfrak{q}}(s) \geq h_{\mathfrak{q}}(y_n^-) = h_{\mathfrak{p}}(\sigma).$$

Since this is impossible, we deduce that $\sigma = 1$. As s was arbitrary, there exists $\alpha \in [0,1]$ such that $h_{\mathfrak{q}}(x) = \alpha$ for every $x \in (0,1)$. It remains to observe that $h_{\mathfrak{p}} = h_{\mathfrak{q}}$ because $\mathfrak{p} \approx \mathfrak{q}$. ∎

As a next step, we investigate the impact of the existence of unambiguous events on the preceding result. We start with an intermediate result showing that, if a capacity is law-invariant on unambiguous events with respect to two non-equivalent finitely additive probabilities, one of which has convex range, then *every* event is actually unambiguous.

Proposition 7.2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be finitely additive probabilities such that $\mathfrak{p}$ has convex range, $\mathfrak{q} \ll \mathfrak{p}$ and let v be a capacity that is $\mathfrak{p}$- and $\mathfrak{q}$-invariant on unambiguous events. Then, either $\mathfrak{p} \approx \mathfrak{q}$ or $\Sigma_{ua}(v) = \Sigma$.*

Proof. Suppose that $\mathfrak{p} \not\approx \mathfrak{q}$, so that we find $N \in \Sigma$ with $\mathfrak{p}(N) > 0 = \mathfrak{q}(N)$. We first prove the following claim:

$$\text{For all } A \in \Sigma_{ua}(v) \text{ and } B \in \Sigma \text{ with } 0 \le \mathfrak{p}(A) - \mathfrak{p}(B) \le \mathfrak{p}(N), \text{ we have } B \in \Sigma_{ua}(v). \tag{7.1}$$

To see this, abbreviate $x := \mathfrak{p}(B)$, $y := \mathfrak{p}(A)$, and note that, in view of the convex range of $\mathfrak{p}$, we can select $C, D \in \Sigma$ such that $C \subset N$, $D \subset N^c$, $\mathfrak{p}(C) = y - x$, and $\mathfrak{p}(D) = x$. As $\mathfrak{p}(A) = \mathfrak{p}(C \cup D)$ and $A \in \Sigma_{ua}(v)$, also $C \cup D$ is unambiguous. As $\mathfrak{q}(C \cup D) = \mathfrak{q}(D)$, we infer that D is unambiguous as well. At last, every $B \in \Sigma$ with $\mathfrak{p}(B) = \mathfrak{p}(D) = x$ must also be unambiguous.

Now, pick $A \in \Sigma_{ua}(v)$ so that $\mathfrak{p}(A) = 1$ and choose x so that $1 - x \le \mathfrak{p}(N)$. By (7.1), every event $B \in \Sigma$ whose probability satisfies $1 - \mathfrak{p}(N) \le \mathfrak{p}(B) \le 1$ is unambiguous.

Next, select an arbitrary event $E \in \Sigma$ with $\mathfrak{p}(E) = 1 - \mathfrak{p}(N)$ and note that E is unambiguous by the previous argument. Using the claim above once more in case $y = 1 - \mathfrak{p}(N)$, every event $B \in \Sigma$ with $1 - 2\mathfrak{p}(N) \le \mathfrak{p}(B) \le 1 - \mathfrak{p}(N)$ is unambiguous. If $\mathfrak{p}(N) \ge 1/2$, we have already proved that all events are unambiguous. Else, we iterate the argument to come to the same conclusion. □

The next proposition shows that the conditions for a capacity to be law-invariant with respect to different probabilities become much more stringent in the presence of nontrivial unambiguous events. Indeed, such a simultaneous law-invariance can only hold in the presence of the special distortion functions encountered in Example 4.5(b).

Proposition 7.3. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be finitely additive probabilities with convex range and let v be a capacity which is both $\mathfrak{p}$- and $\mathfrak{q}$-invariant. Then, $\mathfrak{p} \approx \mathfrak{q}$. In addition, if there is $E \in \Sigma_{ua}(v)$ with $v(E) \in (0,1)$, then $\mathfrak{p} = \mathfrak{q}$ or for every $A \in \Sigma$*

$$v(A) = \begin{cases} 0 & \mathfrak{p}(A) = 0, \\ \frac{1}{2} & 0 < \mathfrak{p}(A) < 1, \\ 1 & \mathfrak{p}(A) = 1. \end{cases}$$

Proof. As both $\mathfrak{p}$ and $\mathfrak{q}$ have convex range, Proposition 7.1 implies that $\mathfrak{p} \approx \mathfrak{q}$. Now, let $E \in \Sigma_{ua}(v)$ be as in the assertion. Since $v(E) \in (0,1)$, both $\mathfrak{p}(E)$ and $\mathfrak{p}(E^c)$ are in $(0,1)$. By Proposition 7.1 (ii), it follows that $v(E) = \alpha = v(E^c)$. Finally, E unambiguous implies $\alpha = 1/2$. □

We conclude the section with a few additional remarks. To begin, let us observe that Proposition 7.1 (ii) has a valid (immediate) converse:

> If $v = h_{\mathfrak{p}} \circ p$ with $h_{\mathfrak{p}}$ as in Proposition 7.1 (ii), then v is $\mathfrak{q}$-invariant for every finitely additive probability $\mathfrak{q}$ such that $\mathfrak{q} \approx \mathfrak{p}$.

At once, these capacities admit the representation

$$v(A) = \alpha \max_{\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \mathfrak{q}(A) + (1-\alpha) \min_{\mathfrak{q}\in\mathcal{P}_{\mathfrak{p}}(\Sigma)} \mathfrak{q}(A)$$

for some $\alpha \in [0,1]$. By a straightforward application of Lemma 4.1, one sees that $\operatorname{core}(v)$ ($\operatorname{acore}(v)$) is empty unless $\alpha = 0$ ($\alpha = 1$), in which case $\operatorname{core}(v) = \mathcal{P}_{\mathfrak{p}}(\Sigma)$ ($\operatorname{acore}(v) = \mathcal{P}_{\mathfrak{p}}(\Sigma)$), the set of probability charges that are absolutely continuous with respect to $\mathfrak{p}$.
Another immediate implication of Proposition 7.1 is that the law-invariant capacities whose reference probability is non-unique are never continuous as they must fail continuity at either $\varnothing$ or Ω. Equivalently, if a capacity that is law-invariant with respect to a convex-ranged charge is continuous at $\varnothing$ or Ω, then the reference probability is automatically unique. Notice, however, that the converse of the latter statement is false: the capacity $v = \mathfrak{p}$ is not continuous at $\varnothing$ unless $\mathfrak{p} \in \mathbf{ca}(\Sigma)$.

8. Extensions to signed charges

In this final section, we generalize the Marinacci-Svistula theorem by allowing the charge $\mathfrak{q}$ of Theorem 3.1 of Section 3 to be any element of $\mathbf{ba}(\Sigma)$; in particular, a signed charge. We will then quickly present a number of applications, which mirror those in Sections 3 to 6. Throughout the section, we assume that Σ is an algebra; thus, $\mathbf{ba}(\Sigma)$ is a vector lattice ([1, Chapter 10.10]).

8.1. A generalization of the Marinacci-Svistula theorem.

Our extension of the Marinacci-Svistula theorem is based the two lemmata below. Let $\mu \in \mathbf{ba}(\Sigma)$ and $\varepsilon > 0$. Let us recall that a partition $\{H^+, H^-\} \in \Pi$ is an *ε-Hahn decomposition* for μ if, for every $A \in \Sigma$, we have

$$A \subset H^+ \implies \mu(A) \geq -\varepsilon, \qquad A \subset H^- \implies \mu(A) \leq \varepsilon.$$

By [8, Theorem 2.6.2], each element of $\mathbf{ba}(\Sigma)$ admits an ε-Hahn decomposition if Σ is an algebra, no matter the value ε. The first lemma is immediately verified.

Lemma 8.1. *For $\mu \in \mathbf{ba}(\Sigma)$ and $\varepsilon > 0$ small enough, and an ε-Hahn decomposition $(H^+, H^-) \in \Pi$ for μ, we have:*

(i) For every $A \in \Sigma$, $\mu^+(A) \leq \varepsilon$ if $A \subset H^-$ and $\mu^-(A) \leq \varepsilon$ if $A \subset H^+$.
(ii) $\mu^+(H^+) \geq \mu^+(\Omega) - \varepsilon$ and $\mu^-(H^-) \geq \mu^-(\Omega) - \varepsilon$.
(iii) $\mu(H^+) \geq \mu^+(\Omega) - 2\varepsilon$.

The next lemma extends Lemma 3.4 and is key to our generalization.

Lemma 8.2. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and suppose $\mu \in \mathbf{ba}(\Sigma)$ satisfies $\mu \neq \mu(\Omega)\mathfrak{p}$. Then the set function $c\colon \Sigma \to [0,1]$ defined by*

$$c(A) := \sup\{\mu(B)\,;\ B \in \Sigma,\ \mathfrak{p}(B) = \mathfrak{p}(A)\}$$

is submodular in the sense of (2.1) and, for every $A \in \Sigma$, it satisfies

$$\mathfrak{p}(A) \in (0,1) \quad \implies \quad c(A) > \mu(\Omega)\mathfrak{p}(A). \tag{8.1}$$

Moreover, c is nondecreasing if $\mu \in \mathbf{ba}_+(\Sigma)$.

Proof. First, assume that $\mu \in \mathbf{ba}_+(\Sigma)$. In this case μ is, up to a normalization by $\mu(\Omega)$, a finitely additive probability and, hence, the statement follows at once from Lemma 3.4. Next, take a general $\mu \in \mathbf{ba}(\Sigma)$. We can prove submodularity as in the proof of Lemma 3.4. If $\mu \in -\mathbf{ba}_+(\Omega)$, then for every $A \in (0,1)$ with $\mathfrak{p}(A) \in (0,1)$,

$$c(A) = \sup\{\mu(\Omega) - \mu(B^c)\,;\ B \in \Sigma,\ \mathfrak{p}(B) = \mathfrak{p}(A)\} > \mu(\Omega) - \mu(\Omega)\mathfrak{p}(A^c) = \mu(\Omega)\mathfrak{p}(A)$$

again by Lemma 3.4. Hence, we shall assume that both μ and $-\mu$ lie outside of $\mathbf{ba}_+(\Sigma)$. We are now going to show that for any $r \in (0,1)$, we can find an $A \in \Sigma$ such that $\mathfrak{p}(A) = r$ and $\mu(A) > r\mu(\Omega)$, which yields (8.1). To this end, let us select $\varepsilon := \frac{1}{3}\min\{(1-r)\mu^+(\Omega), r\mu^-(\Omega)\}$ and let $\{H^+, H^-\} \in \Pi$ be an ε-Hahn decomposition for μ. We distinguish three cases. First, assume that $r = \mathfrak{p}(H^+)$ and set $A = H^+$. In this case, $\mathfrak{p}(A) = r$ and

$$\mu(A) = \mu(H^+) \geq \mu^+(\Omega) - 2\varepsilon > r\mu^+(\Omega) > r\mu(\Omega).$$

Second, assume that $r < \mathfrak{p}(H^+)$ and take $A \in \Sigma$ satisfying $A \subset H^+$ and $\mathfrak{p}^{H^+}(A) = r/\mathfrak{p}(H^+)$ as well as $\mu^+(A) \geq \mathfrak{p}^{H^+}(A)\mu^+(H^+)$, which is possible by Lemma 3.4 applied to $\mathfrak{p}^{H^+}$ defined on the algebra $\{A \cap H^+\,;\ A \in \Sigma\}$. Then, $\mathfrak{p}(A) = r$ and

$$\mu(A) \geq \mathfrak{p}^{H^+}(A)\mu^+(H^+) - \varepsilon \geq r(\mu^+(\Omega) - \varepsilon) - \varepsilon > r\mu^+(\Omega) - 2\varepsilon > r\mu(\Omega).$$

Third, assume that $r > \mathfrak{p}(H^+)$. The partition $\{H^-, H^+\}$ is also an ε-Hahn decomposition for the signed charge $\nu := -\mu$, but with the roles of H^+ and H^- inverted. As $1 - r < \mathfrak{p}(H^-)$ in the present case, we are therefore back in the second case for ν and find an event $B \in \Sigma$ with $\mathfrak{p}(B) = 1 - r$ and $-\mu(B) = \nu(B) > \nu(\Omega)(1-r) = -\mu(\Omega) + r\mu(\Omega)$. Consequently, $A := B^c$ satisfies $\mathfrak{p}(A) = r$ and $r\mu(\Omega) < \mu(\Omega) - \mu(B) = \mu(A)$. $\square$

Our extension of the Marinacci-Svistula theorem follows seamlessly from Lemma 8.2. For convenience, we have collected in one statement all the equivalent conditions encountered in Section 3.

Theorem 8.3. *Let $\mathfrak{p}$ be a finitely additive probability with convex range. For every $\mu \in \mathbf{ba}(\Sigma)$ the following statements are equivalent:*

(i) There exists $E \in \Sigma$ such that $\mathfrak{p}(E) \in (0,1)$ and

$$\mathfrak{p}(A) = \mathfrak{p}(E) \qquad \Longrightarrow \qquad \mu(A) = \mu(E).$$

(ii) There exists a $\mathfrak{p}$-nonconstant $X \in B_0(\Sigma)$ such that

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_\mu[Y] = \inf_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_\mu[Y].$$

(iii) Every $\mathfrak{p}$-equipartition is a μ-equipartition.

(iv) Every $\mathfrak{p}$-equipartition of size 2 is a μ-equipartition.

(v) $\mu = \mu(\Omega)\mathfrak{p}$.

Proof. To show that (i) implies (v), let $E \in \Sigma$ satisfy $\mathfrak{p}(E) \in (0,1)$ and assume that $\mu \neq \mu(\Omega)\mathfrak{p}$. By applying Lemma 8.2 to $\pm\mu$, we find $A, B \in \Sigma$ such that $\mathfrak{p}(A) = \mathfrak{p}(B) = \mathfrak{p}(E)$ and

$\mu(A) > \mu(\Omega)\mathfrak{p}(E) > \mu(B)$. This delivers the desired implication. The proofs of the remaining implications are identical to those of the corresponding statements in Section 3 with Lemma 8.2 replacing Lemma 3.4. □

8.2. **Collapse of games.** The results obtained for capacities in Section 4 extend to general games. The key step is in the following generalization of Lemma 4.1:

Proposition 8.4. *For a convex-ranged probability charge $\mathfrak{p}$ and a $\mathfrak{p}$-invariant game v, the following statements hold:*

(i) $\operatorname{acore}(v) \neq \varnothing$ *if and only if* $v(\Omega)\mathfrak{p} \in \operatorname{acore}(v)$.
(ii) $\operatorname{core}(v) \neq \varnothing$ *if and only if* $v(\Omega)\mathfrak{p} \in \operatorname{core}(v)$.

Proof. We shall only prove statement (i). Take an arbitrary $A \in \Sigma$. If $\mathfrak{p}(A) = 0$, then $v(A) = 0 = v(\Omega)\mathfrak{p}(A)$ by $\mathfrak{p}$-invariance of v. Similarly, if $\mathfrak{p}(A) = 1$, then $v(A) = v(\Omega) = v(\Omega)\mathfrak{p}(A)$. Now, let $\mathfrak{p}(A) \in (0,1)$ and take any $\mu \in \operatorname{acore}(v)$. It follows from Lemma 8.2 that there exists $B \in \Sigma$ with $\mathfrak{p}(B) = \mathfrak{p}(A)$ such that

$$v(A) = v(B) \geq \mu(B) \geq \mu(\Omega)\mathfrak{p}(B) = v(\Omega)\mathfrak{p}(A),$$

again by $\mathfrak{p}$-invariance. This yields $v(\Omega)\mathfrak{p} \in \operatorname{acore}(v)$. □

Conditions for the collapse of a game to an additive form obtain by following the same patterns as in Section 4 and by simply replacing Lemma 4.1 with Proposition 8.4. In particular, we can give the following extended version of Theorem 4.2:

Theorem 8.5. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v be a $\mathfrak{p}$-invariant game. If there exists $A \in \Sigma$ such that $v(A) \notin \{0, v(\Omega)\}$ and $v(A) + v(A^c) = v(\Omega)$, then* $\operatorname{core}(v) = \{v(\Omega)\mathfrak{p}\}$, $\operatorname{acore}(v) = \{v(\Omega)\mathfrak{p}\}$, *or both core and anticore of v are empty.*

8.3. **Fréchet-Hoeffding bounds.** The following extends our version of the Fréchet-Hoeffding bounds (Theorem 5.1) to the world of signed charges.

Theorem 8.6. *Let $\mathfrak{p}$ a finitely additive probability with convex range and $\mu \in \mathbf{ba}(\Sigma)$ such that $\mu \neq \mu(\Omega)\mathfrak{p}$. Then, for every $\mathfrak{p}$-nonconstant $X \in B_0(\Sigma)$,*

$$\sup_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_\mu[Y] > \mu(\Omega)\mathbb{E}_{\mathfrak{p}}[X] > \inf_{Y \sim_{\mathfrak{p}} X} \mathbb{E}_\mu[Y].$$

In particular, for every $A \in \Sigma$ such that $\mathfrak{p}(A) \in (0,1)$

$$\sup_{B \in \Sigma,\, \mathfrak{p}(B) = \mathfrak{p}(A)} \mu(B) > \mu(\Omega)\mathfrak{p}(A) > \inf_{B \in \Sigma,\, \mathfrak{p}(B) = \mathfrak{p}(A)} \mu(B).$$

Proof. Let $x_1 < \ldots < x_k$ be the different values that X takes with strictly positive $\mathfrak{p}$-probability. Suppose first that $|\mu| \not\ll \mathfrak{p}$, i.e., there is $N \in \Sigma$ such that $\mathfrak{p}(N) = 0$ but $|\mu|(N) \neq 0$. In this case, partition N^c into events $A_1, \ldots, A_k \in \Sigma$ such that $\mathfrak{p}(A_i) = \mathfrak{p}(X = x_i)$ for every $1 \leq i \leq k$. For all $t > 0$ and $E \in \Sigma$ with $E \subset N$, we have $\sum_{i=1}^k x_i \mathbf{1}_{A_i} + t\mathbf{1}_E - t\mathbf{1}_{N \setminus E} \sim_{\mathfrak{p}} X$ and thus

$$\sup_{Y\sim_{\mathfrak{p}}X}\mathbb{E}_\mu[Y]\geq\Big\{\sum_{i=1}^k x_i\mu(A_i)+t(\mu(E)-\mu(N\setminus E)\,;\ t>0,\,E\in\Sigma,\,E\subset N\Big\}$$
$$=\sup\Big\{\sum_{i=1}^k x_i\mu(A_i)+t|\mu|(N)\,;\ t>0\Big\}=\infty>\mathbb{E}_{\mathfrak{p}}[X].$$

As a result, we may assume that $\mu\ll\mathfrak{p}$. From this point on, one can argue as in the proof of Theorem 5.1 by replacing Lemma 3.4 with Lemma 8.2. $\square$

8.4. **Fatou property.** In this subsection we extend to general convex functionals on $B(\Sigma)$ the automatic Fatou property established in Section 6 for convex risk measures. The appropriate version of the Fréchet-Hoeffding bounds, which is recorded in the previous subsection, plays a key role also in this case. In what follows, for a given $\mathfrak{p}\in\mathcal{P}(\Sigma)$, we define for all $\pi\in\Pi$ and $\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)$ the finitely additive set function

$$\mathfrak{p}_{\pi,\mu}:=\sum_{A\in\pi,\,\mathfrak{p}(A)>0}\mu(A)\mathfrak{p}^A.$$

Lemma 8.7. *Let Σ be a σ-algebra and $\mathfrak{p}$ a finitely additive probability with convex range. Moreover, let $\varphi\colon B(\Sigma)\to\mathbb{R}$ be $\mathfrak{p}$-invariant, lower semicontinuous, and convex. For all $\pi\in\Pi$ and $\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)$ the following statements hold:*

(i) $\varphi\left(\mathbb{E}_{\mathfrak{p}}[X|\pi]\right)\leq\varphi(X)$ *for every* $X\in B(\Sigma)$.
(ii) $\mathbb{E}_{\mathfrak{p}_{\pi,\mu}}[X]=\mathbb{E}_\mu[\mathbb{E}_{\mathfrak{p}}[X|\pi]]$ *for every* $X\in B(\Sigma)$.
(iii) $\varphi^*(\mathfrak{p}_{\pi,\mu})\leq\varphi^*(\mu)$.

Proof. It is not difficult to show that any $\mu\in\mathrm{dom}(\varphi^*)$ must satisfy $\mu\ll\mathfrak{p}$. Hence, for every $X\in B(\Sigma)$ we can rewrite the dual representation in (2.2) as

$$\varphi(X)=\sup_{\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)}\left\{\mathbb{E}_\mu[X]-\varphi^*(\mu)\right\}. \tag{8.2}$$

Now, take $X\in B_0(\Sigma)$ and $A\in\Sigma$ such that $\mathfrak{p}(A)>0$. An application of Theorem 8.6 to the sample space A equipped with the algebra $\Sigma^A:=\{A\cap B\,;\ B\in\Sigma\}$ and with the finitely additive probability with convex range given by the restriction of $\mathfrak{p}^A$ to Σ^A shows that, for every $\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)$,

$$\sup_{X'\sim_{\mathfrak{p}^A}X}\mathbb{E}_\mu[X'\mathbf{1}_A]\geq\mu(A)\mathbb{E}_{\mathfrak{p}^A}[X]=\mathbb{E}_\mu[\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A]. \tag{8.3}$$

Now, take an arbitrary $Y\in B_0(\Sigma)$ and observe that, for every $X'\sim_{\mathfrak{p}^A}X$, we have $X'\mathbf{1}_A+Y\mathbf{1}_{A^c}\sim_{\mathfrak{p}}X\mathbf{1}_A+Y\mathbf{1}_{A^c}$ as well. Using $\mathfrak{p}$-invariance, we infer from (8.2) and (8.3) that

$$\varphi(X\mathbf{1}_A+Y\mathbf{1}_{A^c})=\sup_{X'\sim_{\mathfrak{p}^A}X}\varphi(X'\mathbf{1}_A+Y\mathbf{1}_{A^c})=\sup_{\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)}\Big\{\sup_{X'\sim_{\mathfrak{p}^A}X}\mathbb{E}_\mu[X'\mathbf{1}_A+Y\mathbf{1}_{A^c}]-\varphi^*(\mu)\Big\}$$
$$\geq\sup_{\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)}\left\{\mathbb{E}_\mu[\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A+Y\mathbf{1}_{A^c}]-\varphi^*(\mu)\right\}=\varphi(\mathbb{E}_{\mathfrak{p}^A}[X]\mathbf{1}_A+Y\mathbf{1}_{A^c}).$$

Repeating the argument yields (i) on $B_0(\Sigma)$. From this point on, we can argue as in the proof of Lemma 6.1 by exploiting the fact that, being convex and lower semicontinuous and being finitely valued, φ is continuous on $B(\Sigma)$ by [12, Corollary 2.5]. $\square$

The next result extends Theorem 6.2 beyond the class of risk measures. To obtain a parallel statement, for every $\pi \in \Pi$ and $w \in \mathbb{R}^\pi$ we consider the finitely additive set function

$$\mathfrak{p}_{\pi,w} := \sum_{A\in\pi,\, \mathfrak{p}(A)>0} w(A)\mathfrak{p}^A.$$

Theorem 8.8. *Let Σ be a σ-algebra and $\mathfrak{p}$ a finitely additive probability with convex range. Moreover, let $\varphi\colon B(\Sigma) \to \mathbb{R}$ be $\mathfrak{p}$-invariant, convex, and lower semicontinuous.*

(i) For every $X \in B(\Sigma)$

$$\rho(X) = \sup_{\pi\in\Pi,\, \mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)} \{\mathbb{E}_{\mathfrak{p}_{\pi,\mu}}[X] - \rho^*(\mathfrak{p}_{\pi,\mu})\} = \sup_{\pi\in\Pi,\, w\in\mathbb{R}^\pi} \{\mathbb{E}_{\mathfrak{p}_{\pi,w}}[X] - \rho^*(\mathfrak{p}_{\pi,w})\}. \tag{8.4}$$

(ii) For all $X \in B(\Sigma)$,

$$\rho(X) = \sup_{\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma),\ \mu\lll\mathfrak{p}} \{\mathbb{E}_\mu[X] - \rho^*(\mu)\}. \tag{8.5}$$

Moreover, ρ has the hazy Fatou property.

(iii) If $\mathfrak{p}$ is countably additive, then ρ is $\sigma(B(\Sigma), \mathbf{ca}(\Sigma))$-l.s.c. and has the Fatou property.

Proof. It suffices to show the left-hand side equality in (8.4). To this effect, take $X \in B(\Sigma)$ and $\mu \in \mathrm{dom}(\varphi^*)$. For every $n \in \mathbb{N}$ set $\pi_n := \{\{X \in [k2^{-n}, (k+1)2^{-n})\};\ k \in \mathbb{Z}\} \in \Pi$ and observe that

$$|\mathbb{E}_\mu[X] - \mathbb{E}_\mu[\mathbb{E}_{\mathfrak{p}}[X|\pi_n]]| \le \sum_{A\in\pi_n,\, \mathfrak{p}(A)>0} \mathbb{E}_{|\mu|}[|\mathbb{E}_{\mathfrak{p}^A}[X] - X|\mathbf{1}_A] \le 2^{-n}|\mu|(\Omega),$$

where we used that $\mu \ll \mathfrak{p}$. As a result,

$$\mathbb{E}_\mu[X] - \varphi^*(\mu) \le \inf_{n\in\mathbb{N}} \{\varphi(\mathbb{E}_{\mathfrak{p}}[X|\pi_n]) + 2^{-n}|\mu|(\Omega)\} \le \sup_{\pi\in\Pi} \varphi(\mathbb{E}_{\mathfrak{p}}[X|\pi]).$$

The desired equality follows at once from (8.2) and from Lemma 8.7. Statements (ii) and (iii) are verified similarly to the corresponding statements in Theorem 6.2. $\square$

A direct inspection of the proofs reveals that the results of this section hold true under alternative assumptions. Firstly, they remain valid if Σ is assumed to be an algebra instead of a σ-algebra provided one replaces $B(\Sigma)$ with the smaller space $B_0(\Sigma)$. Secondly, they hold true if φ, rather than only taking finite values, takes values in $(-\infty, \infty]$, is proper, lower semicontinuous, and, for every $X \in \mathrm{dom}(\varphi)$, admits a sequence of simple functions (Y_n^X) such that $\varphi(X) = \lim_{n\to\infty} \varphi(Y_n^X)$.

8.5. **Representation of law-invariant games.** We conclude the section, and with it the paper, by providing a representation theorem for law-invariant, submodular games of bounded variation. A (cooperative) game v:

(1) is *bounded* if $\sup_{A\in\Sigma} v(A) < \infty$.

(2) has *bounded variation* ([5]) if

$$\sup_{n\in\mathbb{N}}\sup\left\{\sum_{i=1}^{n}|v(A_i)-v(A_{i-1})|\,;\ \varnothing=A_0\subset A_1\subset\cdots\subset A_n=\Omega,\ A_1,\dots,A_n\in\Sigma\right\}<\infty.$$

An example of a law-invariant, submodular game of bounded variation was encountered in Lemma 8.2 and was given for $\mu\in\mathbf{ba}_{\mathfrak{p}}(\Sigma)$ by

$$v(A)=\sup\{\mu(B)\,;\ B\in\Sigma,\ \mathfrak{p}(B)=\mathfrak{p}(A)\},\quad A\in\Sigma.$$

Indeed, as $\mu\ll\mathfrak{p}$, $v(\varnothing)=0$ and v is a game. As $\mu\in\mathbf{ba}(\Sigma)$, v is bounded. Clearly, v is $\mathfrak{p}$-invariant and it is submodular by Lemma 8.2. Being bounded and submodular, v has bounded variation by [27, Theorem 4.7].

Functionals of the previous type have been a key tool throughout our analysis. Our representation result focuses on them once more by showing that *every* law-invariant, submodular game of bounded variation can always be represented as the above v.

Theorem 8.9. *Let $\mathfrak{p}$ be a finitely additive probability with convex range. A game v is a $\mathfrak{p}$-invariant, submodular game of bounded variation iff there is $\mu\in\mathbf{ba}(\Sigma)$ such that $\mu\ll\mathfrak{p}$ and, for every $A\in\Sigma$,*

$$v(A)=\sup\{\mu(B)\,;\ B\in\Sigma,\ \mathfrak{p}(B)=\mathfrak{p}(A)\}.\tag{8.6}$$

On our way to this representation, we begin with the following

Lemma 8.10. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and v a submodular, $\mathfrak{p}$-invariant game of bounded variation. Then there is a unique concave function $h\colon[0,1]\to\mathbb{R}$ such that $v=h\circ\mathfrak{p}$.*

The property stated in Lemma 8.10 is well-known, at least in the case of capacities. For completeness, we provide a proof.

Proof. By $\mathfrak{p}$-invariance there is a unique function $h\colon[0,1]\to\mathbb{R}$ satisfying $h(0)=0=h(1)-v(\Omega)$ and $v=h\circ\mathfrak{p}$. In particular, h is of bounded variation and therefore Lebesgue measurable. Using the same arguments as in the proof of Lemma 3.4, we see that h is continuous and concave on $(0,1)$. It remains to show concavity on $[0,1]$. First, we show concavity at 0. Let $x\in(0,1)$ and $\lambda=m/n$ for integers $m,n\in\mathbb{N}$ with $m<n$. Take $A\in\Sigma$ with $\mathfrak{p}(A)=x$ and partition A into events $A_1,\dots,A_n\in\Sigma$ with the same $\mathfrak{p}$-probability. Applying submodularity repeatedly, we obtain $h(x)/n=v(\bigcup_{i=1}^n A_i)/n\le h(x/n)$, whence we infer

$$h(\tfrac{m}{n}x)=v\Big(\bigcup_{i=1}^{m}A_i\Big)\ge\sum_{i=1}^{m}v(A_i)=mh\big(\tfrac{x}{n}\big)=\tfrac{m}{n}h(x).$$

We can now use continuity on $(0,1)$ to show for arbitrary $\lambda\in(0,1)$ that

$$h(\lambda x)=\lim_{n\to\infty}h\big(\tfrac{\lfloor n\lambda\rfloor}{n}x\big)\ge\tfrac{\lfloor n\lambda\rfloor}{n}h(x)=\lambda h(x).$$

Next, we establish concavity at 1. As h is midpoint concave, for every $n \in \mathbb{N}$ with $n \geq 2$

$$h\big(\tfrac{2^n-1}{2^n}\big) = h\big(\tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot \tfrac{2^{n-1}-1}{2^{n-1}}\big) \geq \tfrac{1}{2}h(1) + \tfrac{1}{2}h\big(\tfrac{2^{n-1}-1}{2^{n-1}}\big).$$

This entails that $\ell := \limsup_{n\to\infty} h\big(\frac{2^n-1}{2^n}\big) \geq h(1)$. Now let $x, \lambda \in (0,1)$. By concavity and continuity of h on $(0,1)$, we have

$$h(\lambda x + 1 - \lambda) = \limsup_{n\to\infty} h\big(\lambda x + (1-\lambda)\tfrac{2^n-1}{2^n}\big) \geq \lambda h(x) + (1-\lambda)\ell \geq \lambda h(x) + (1-\lambda)h(1).$$

This concludes the proof. $\square$

Given $\mu \in \mathbf{ba}(\Sigma)$ that is absolutely continuous with respect to a convex-ranged probability charge $\mathfrak{p}$, let us denote by v_μ the corresponding game defined by (8.6) and let $h_\mu \colon [0,1] \to \mathbb{R}$ be the unique function such that $v_\mu = h_\mu \circ \mathfrak{p}$.

Lemma 8.11. *Let $\mathfrak{p}$ be a finitely additive probability with convex range and $h\colon [0,1] \to \mathbb{R}$ a concave function. Suppose that $A_1, \dots, A_n \in \Sigma$ form a $\mathfrak{p}$-equipartition and let $\mu = \sum_{i=1}^n \big[h\big(\frac{i}{n}\big) - h\big(\frac{i-1}{n}\big)\big]\mathfrak{p}^{A_i} \in \mathbf{ba}(\Sigma)$. Then, for every $m \in \mathbb{N}$ with $m \leq n$,*

$$h_\mu(\tfrac{m}{n}) = \mu\Big(\bigcup_{i=1}^m A_i\Big).$$

Proof. Set $w_i := h\big(\frac{i}{n}\big) - h\big(\frac{i-1}{n}\big)$ for every $1 \leq i \leq n$ and define the function $\chi\colon \mathbb{R}^n \to \mathbb{R}$ by $\chi(y) := \sum_{i=1}^n w_i y_i$. For $A \in \Sigma$ with $\mathfrak{p}(A) = m/n$ and $\mathbf{y} := \big(\mathfrak{p}(A \cap A_1), \dots, \mathfrak{p}(A \cap A_n)\big)$, $\mu(A) = \chi(\mathbf{y})$. This immediately implies that for the set Δ of vectors $\mathbf{y} = (y_1, \dots, y_n) \in [0, \frac{1}{n}]^n$ satisfying $\sum_{i=1}^n y_i = \frac{m}{n}$,

$$h_\mu(\tfrac{m}{n}) = \sup_{\mathbf{y}\in\Delta} \chi(\mathbf{y}).$$

As χ is continuous and Δ is a simplex, we find an extreme point $\mathbf{e} = (e_1, \dots, e_n) \in \Delta$ satisfying $h_\mu(m/n) = \chi(\mathbf{e})$. The extreme points of Δ are characterized by the fact that $\mathcal{I} := \{1 \leq i \leq n\,;\ e_i = \frac{1}{n}\}$ has cardinality m. By concavity of h, $w_1 \geq w_2 \geq \cdots \geq w_n$, whence we obtain

$$h_\mu(\tfrac{m}{n}) = \max_{|\mathcal{I}|=m} \sum_{i\in\mathcal{I}} \tfrac{w_i}{n} = \sum_{i=1}^m \frac{w_i}{n} = \mu\big(\bigcup_{i=1}^m A_i\big).$$

$\square$

We are now in a position to prove Theorem 8.9:

Proof of Theorem 8.9. Let v be a game that is $\mathfrak{p}$-invariant, submodular and of bounded variation. Let h be the concave function associated with v by Lemma 8.10. For every $n \in \mathbb{N}$, we can find a $\mathfrak{p}$-equipartition $\{A_1^n, \dots, A_{2^n}^n\} \subset \Sigma$ such that, for each $1 \leq i \leq 2^n$, we have $A_i^n = A_{2i}^{n+1} \cup A_{2i+1}^{n+1}$. For $n \in \mathbb{N}$, set

$$\mu_n := \sum_{i=1}^{2^n} \big[h\big(\tfrac{i}{n}\big) - h\big(\tfrac{i-1}{n}\big)\big]\mathfrak{p}^{A_i^n} \in \mathbf{ba}(\Sigma).$$

Let v_{μ_n} be the corresponding games as defined by (8.6) and let h_{μ_n} be the corresponding distortion. Observe that for all $n \in \mathbb{N}$, $1 \le i \le 2^n$, and $x \in [(i-1)/2^n, i/2^n]$, we have

$$h_{\mu_n}(x) = h(\tfrac{i-1}{2^n}) + \left[h(\tfrac{i}{2^n}) - h(\tfrac{i-1}{2^n})\right]2^n(x - \tfrac{i-1}{2^n}).$$

As additionally $h_{\mu_n}(0) = h(0) = 0$ and $h_{\mu_n}(1) = h(1) = v(\Omega)$, we conclude $h_{\mu_n}(x) \uparrow h(x)$ for every $x \in [0,1]$. Next, denoting by $TV(\mu_n)$ the total variation of μ_n, observe that

$$\sup_{n\in\mathbb{N}} TV(\mu_n) \le \sup_{n\in\mathbb{N}} \sup\left\{\sum_{i=1}^{n} |h(x_i) - h(x_{i-1})|\,;\ 0 = x_0 < x_1 < \dots < x_n = 1\right\} < \infty.$$

Hence, by the $\sigma(\mathbf{ba}(\Sigma), B(\Sigma))$-compactness of the unit ball in $\mathbf{ba}(\Sigma)$, we can find a net $(n_\alpha) \subset \mathbb{N}$ such that (μ_{n_α}) converges to a suitable $\mu \in \mathbf{ba}(\Sigma)$ in the topology $\sigma(\mathbf{ba}(\Sigma), B(\Sigma))$. Now, fix any index α_0 and set $n_0 := n_{\alpha_0}$. For all $\ell \in \{1, \dots, 2^{n_0}\}$ and $n \in \mathbb{N}$ with $n \ge n_0$ define $A := \bigcup_{i=1}^{\ell 2^{n-n_0}} A_i^n$. Then, $h_{\mu_n}(\ell 2^{-n_0}) = \mu_n(A)$ by Lemma 8.11. As a result,

$$h_\mu(\ell 2^{-n_0}) \ge \mu(A) = \lim \mu_{n_\alpha}(A) = \sup h_{\mu_{n_\alpha}}(\ell 2^{-n_0}) = h(\ell 2^{-n_0}).$$

Now observe that each h_{μ_n} is a function whose variation is bounded by the variation of h. Thus also h is of bounded variation and hence measurable. Arguing several times as above, both h_μ and h are continuous on $(0,1)$, and the previous estimate implies $h_\mu \ge h$ on $(0,1)$. By construction, $\mu \ll \mathfrak{p}$ and therefore $h_\mu(0) = 0 = h(0)$ and $h_\mu(1) = \mu(\Omega) = h(1)$. Hence, $h_\mu \ge h$ on the entire $[0,1]$. Conversely, however, we observe for all $E \in \Sigma$ that

$$\mu(E) = \lim \mu_{n_\alpha}(E) \le \sup h_{\mu_{n_\alpha}}(\mathfrak{p}(E)) = h(\mathfrak{p}(E)).$$

Taking the supremum over all $F \in \Sigma$ with $\mathfrak{p}(F) = \mathfrak{p}(E)$ on the left-hand side yields $h_\mu \le h$, and the proof is complete.

The converse direction is the observation immediately preceding assertion of Theorem 8.9 $\square$

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