# A DYNAMIC LÉVY COPULA MODEL FOR THE SPARK SPREAD

THILO MEYER-BRANDIS AND MICHAEL MORGAN

Abstract. We present a model for the spark spread on energy markets which is implied by a two-dimensional model for the electricity and gas spot prices. The marginal price processes are supposed to follow sums of (not necessarily Gaussian) Ornstein-Uhlenbeck components and the main focus of this paper is on the two-dimensional dependence modeling via Lévy copulas. We will introduce a specific class of skewed Lévy copulas and estimate the complete model on data from UK markets. Further, due to the arithmetic structure of the model, we are able to employ Fourier transform techniques to derive semi-analytic expressions for option prices.

## 1. Introduction

The global economic growth highly depends on sustainable supply of energy which has caused a strong worldwide increase in demand of energy related assets over the last decades. At the same time, in many parts of the world electricity markets have been, or are in the process of being deregulated in order to establish a free float of prices in a competitive environment. This deregulation of electricity markets has led to the creation of a network of energy exchanges, where electricity is quoted almost as any other commodity. This new and highly complex environment has introduced a lot of price uncertainty that energy market participants have to cope with. Hence, the development of reliable and sustainable quantitative tools for valuation and management of energy risk should be a key consideration of decision makers from politics and industry.

In this paper we consider the specific problem to develop and estimate a dynamic model for the so called *spark spread*. The spark spread represents the difference between the price of electricity and the price of the gas required to generate this electricity. In other words, the spread is a proxy for the cost of igniting fuel and turning it into electricity, which indicates where the name "spark" comes from. Expressed mathematically that is

$$S(t) := \tilde{E}(t) - c\,\tilde{G}(t)\,,$$

where $\tilde{E}(t)$ is the spot price of electricity, and $\tilde{G}(t)$ the spot price of gas, both quoted in customary units. The factor $c$ is the (assumably) constant *heat rate*, which includes factors for matching the unit of $\tilde{G}$ to the unit of $\tilde{E}$ and moreover takes into account, that

*Date*: August 30, 2011.

*Key words and phrases.* Electricity markets, energy markets, spark spreads, electricity spot price model, gas spot price model, Lévy copulas, estimation of Ornstein-Uhlenbeck processes, estimation of Lévy copulas.

gas for most applications is less efficient than electricity or cannot be transformed into it without losses. However, the natural gas is a primary source of energy and environmental awareness is drawing more attention to gas-fired power plants since they run with very little pollution. The spark spread represents one of the primary cross commodity products in electricity markets.

When looking at the current literature, one basically finds two different approaches to model the spark spread:

(1) to model the spark spread directly,
(2) to specify a two-dimensional model for the underlying commodities and to infer the spark spread as the difference of the margins.

The first approach, which is a so called reduced form model of the spark spread, fully ignores the character of the marginal processes of electricity and gas. A model from this class is proposed in [4] where the spread price $S(t)$ is assumed to follow a non-gaussian OU process. The stress remains on tractability, since one can derive quite simple pricing formulas. However, the major disadvantage lies in loosing connection to the marginal processes, which results in less robustness, especially when the market conditions change for one margin it is not transparent in which way parameters are affected.

Contrary to the reduced form approach, models from the second class keep track of the dynamics and, in particular, the dependence of the marginal processes of the underlying commodities electricity and gas. Examples of models from this class can be found in [5], [15], and [3]. In [5] the marginal prices are assumed to follow exponentials of Gaussian Ornstein-Uhlenbeck (for short OU) processes. In order to gain analytic tractability the authors approximate the induced spark spread model as difference between two log-normal random variables with a normal distribution. However, one of the most prominent features of electricity prices (and also of gas prices on a smaller scale) are violent spikes, which are big upward jumps followed by a rapid decline to normal levels. Obviously, a Gaussian setting is not flexible enough to capture neither path nor distributional properties of this spiky behavior. An idea to improve this model was proposed in [15], where the approximation by a normal distribution for the difference in exponential OUs is critiqued and replaced by a Normal inverse Gaussian (for short NIG) distribution, so as to capture the observed heavy-tailedness. In [3] both electricity and gas price dynamics are modeled by discrete-time AR(1) processes with NIG-distributed noise terms. In this setting, a more thorough analysis of the dependence structure between the marginal processes is required to obtain the model of the spark spread in (2.1). The main reason to chose a time-discrete model is to facilitate the dependence modeling, and the authors propose to model the dependence structure by a copula

$$C_h(v, z) := vz + h(1 - |2v - 1|)(1 - (2z - 1)^2),$$

that connects the NIG distributions of the random innovations between the discrete points of time.

In this paper we will pick up the trail of [3] by following their explicit call for further development of continuous time modeling of the spark spread. We propose a two-dimensional dynamic model for the vector of electricity and gas prices where the marginals follow sums of, not necessarily Gaussian, OU processes (see (2.2) in Section 2). In the one-dimensional case, this model was successfully proposed and estimated for electricity spot prices in [1], [16], and [13]. In the present two-dimensional case the additional difficulty of modeling the multivariate dependence risk between electricity and gas prices arrives. While the specification of an appropriate covariance matrix is sufficient to connect the Gaussian OU components, we propose to employ Lévy copulas to model the dependence of the non-Gaussian OU components. For this purpose we introduce a new class of Lévy copulas that meet the requirement revealed by empirical spark spread data. Besides being able to capture both path and distributional properties of the marginal price processes as well as the multivariate dependence structure given by empirical data, the model exhibits great analytic tractability. In particular, due to its arithmetic structure we are able to develop semi-analytic pricing formulas for options written on the spark spread by employing Fourier transform techniques.

The remaining parts of the paper are structured as follows. In Section 2 we introduce our two-dimensional dynamic model for the underlying electricity and gas prices. In particular we present a class of Lévy copulas that we use to specify the dependence structure in our multivariate setting. In Section 3 we demonstrate how to fit the model to empirical data from the UK market. Finally, in Section 4 we employ Fourier transform techniques to develop call option prices written on the spark spread. Further we demonstrate how to numerically compute these option prices in the model fitted to UK data.

## 2. The model

The objective is to build a two-dimensional model for the process $(\tilde{E}(t), \tilde{G}(t))$ which implies a flexible and analytically tractable model for the spark spread

$$S(t) := \tilde{E}(t) - c\,\tilde{G}(t)\,, \tag{2.1}$$

that in particular provides an appropriate modeling of the dependence risk between electricity and gas prices. We start by specifying the dynamic model for the marginal electricity spot price process where we adopt the model proposed in [16] (see also [1], [13]) which has been shown to be analytically very tractable and to successfully reproduce stylized features of electricity spot prices such as

- seasonality on different time scales.
- stationarity of deseasonalized price series.
- multiscale autocorrelation structure.

- spike occurrence.
- non-Gaussianity, mainly caused by low-probability large-amplitude spikes.

According to this model we set

$$\tilde{E}(t) = \Lambda^e(t)\Big(Y_1^e(t) + Y_2^e(t)\Big)\,, \tag{2.2}$$

where $Y_1^e(t)$ and $Y_2^e(t)$ are OU processes of the form

$$dY_1^e(t) = -\lambda_1^e(\mu^e - Y_1^e(t))dt + \sigma^e dB^e(t)\,, \tag{2.3}$$

$$dY_2^e(t) = -\lambda_2^e Y_2^e(t)dt + dL^e(t)\,. \tag{2.4}$$

The OU component $Y_1^e(t)$ is driven by a Brownian motion $B^e(t)$ and is responsible for modeling the base signal of the electricity spot price. The OU component $Y_2^e(t)$ models the spike behavior of the spot price and is driven by a compound Poisson process

$$L^e(t) = \int_0^t \int_{\mathbb{R}} z\, N^e(dt, dz)\,,$$

where the associated Poisson jump measure $N^e(dt, dz)$ is characterized by its Lévy measure $\nu^e(dz) = \rho^e\, D_e(dz)$ with jump intensity $\rho^e > 0$ and jump distribution $D_e(dz)$. Note that since $Y_2^e$ is supposed to model spikes caused by upward jumps the jump distribution $D_e(dz)$ has positive support, i.e. $L^e(t)$ is a subordinator. Further, $\Lambda^e(t)$ is a deterministic seasonality function, $\lambda_1^e, \lambda_2^e > 0$ are constants determining the respective mean reversion rates of the OU components, while $\mu^e > 0$ notates the mean reversion level and $\sigma^e > 0$ the volatility of $Y_1^e(t)$ .

Next, we come to the marginal gas spot price process $\tilde{G}(t)$. Since gas prices exhibit similar qualitative behavior as electricity prices (the main difference being the smaller scale of spikes due to better storability of gas compared to electricity) we adopt the same model as in (2.2) for gas prices, differentiating all objects by a superscript $g$:

$$\tilde{G}(t) = \Lambda^g(t)\Big(Y_1^g(t) + Y_2^g(t)\Big)\,, \tag{2.5}$$

where $Y_1^g(t)$ and $Y_2^g(t)$ are OU processes of the form

$$dY_1^g(t) = -\lambda_1^g(\mu^g - Y_1^g(t))dt + \sigma^g dB^g(t)\,, \tag{2.6}$$

$$dY_2^g(t) = -\lambda_2^g Y_2^g(t)dt + dL^g(t)\,, \tag{2.7}$$

where $(B^e(t), B^g(t))$ is two-dimensional (correlated) Brownian motion and $(L^e(t), L^g(t))$ is a two-dimensional compound Poisson process.

In order to complete the specification of the two-dimensional model $(\tilde{E}(t), \tilde{G}(t))$ needed for the spark spread process (2.1), the remaining core problem is the specification of the dependence structure between the margins $\tilde{E}(t)$ and $\tilde{G}(t)$, i.e. the specification of the multivariate distributions of $(B^e(t), B^g(t))$ and $(L^e(t), L^g(t))$. As for the Gaussian distribution of $(B^e(t), B^g(t))$, it is sufficient to determine the correlation parameter $\delta$ between $B^e(1)$ and $B^g(1)$. The main focus of this paper, however, lies on the modeling of the multivariate Lévy process $(L^e(t), L^g(t))$, i.e. on the modeling of the dependence risk of electricity and gas spikes. This will be done by employing the concept of Lévy copulas as elaborated in the following subsection.

**Remark 2.1.** *We remark that in the case of compound Poisson processes we could also model the dependence structure by usual copulas applied to marginal jump distributions. However, this would require a cumbersome separation of the modeling and estimation procedure into dependent and independent components. The Lévy copula approach, on the other hand, conveniently integrates the complete dependence modeling and estimation into one concept. Also we remark that as soon as the compound Poisson processes are replaced by general Lévy processes with possibly infinite activity the concept of Lévy copulas becomes inevitable.*

2.1. **A class of Lévy copulas for the spark spread.** Since in our model spikes are driven by positive jumps, we restrict ourselves to the concept of Lévy copulas for two-dimensional Lévy measures with positive support for the dependence modeling of spike risk, i.e. for the specification of the dependence structure of the two-dimensional Lévy process $(L^e(t), L^g(t))$. We refer to Appendix A for a short review of the characterization of such Lévy measures with Lévy copulas.

A parametric class of Lévy copulas called Archimedean Lévy copulas, that is the analogue of the popular class of Archimedean copulas, is given by

$$F(x,y) = \phi^{-1}(\phi(x) + \phi(y)) \,, \tag{2.8}$$

where $\phi : [0,\infty] \to [0,\infty]$ is a strictly decreasing convex function such that $\phi(0) = \infty$ and $\phi(\infty) = 0$ (see Appendix A for some specific examples). This class, as all other classes we have found in literature, has the symmetry property $F(x,y) = F(y,x)$, for all $x, y \in [0,\infty]$. Our empirical spark spread data from the UK market, however, does not confirm this symmetry property (see FIGURE 5 in Section 3). We will therefore introduce the more flexible class of what we call skewed Archimedean Lévy copulas which is better suited to generate the dependency structure found in empirical spark spread data:

**Proposition 2.2** (Skewed Archimedean Lévy copulas)**.** *Let $\phi$ be as in* (2.8) *above. Further, for $i = 1, 2$ let $\psi_i : [0,\infty] \to [1,\infty]$ be decreasing functions satisfying $\psi_i(\infty) = 1$ then if*

$$F(x,y) = \phi^{-1}\left(\psi_1(y)\phi(x) + \psi_2(x)\phi(y)\right)$$

*is 2-increasing, it defines a 2-dimensional Lévy copula.*

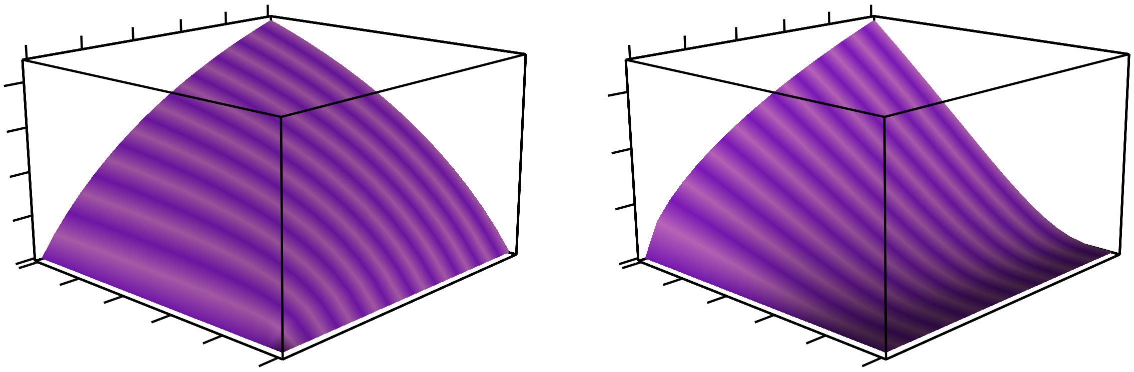
*Proof.* According to Definition A.7 it suffices to show that $F$ is grounded and has uniform margins, which in the setting of Proposition 2.2 is obviously fulfilled. □

We will nominate the two functions $\psi_1$ and $\psi_2$ as *skew factors.* It would be interesting to specify some conditions on the $\psi_i$, independent from $\phi$ at best, or at least in a simple way depending on $\phi$, that would guarantee the 2-increasingness of $F$. In this paper, however, we will leave this problem for future research and focus on a specific class of skewed Archimedean Lévy copulas where $\phi(x) = x^{-\theta}$ is the Clayton-Lévy generator (see Example (1) in Appendix A), and $\psi_2 \equiv 1$ and $\psi_1(x) = (\alpha v^{-\beta} + 1)$ introduce a single-sided skew:

**Corollary 2.3** (One-sided skewed Clayton-Lévy copula). *The function* $F : [0, \infty]^2 \to [0, \infty]$ *given by*

$$F(u, v) = \left((\alpha v^{-\beta} + 1)u^{-\theta} + v^{-\theta}\right)^{-\frac{1}{\theta}},$$

*for* $\alpha > 0,\ \theta > 0$ *and* $0 < \beta \leq \theta + 1$ *is a Lévy copula.*



Clayton-Levy copula:
$F(u, v) = \left(u^{-\theta} + v^{-\theta}\right)^{-\frac{1}{\theta}}$
$= F(v, u), \text{ for all } u, v$

One-sided skewed Clayton-Lévy copula:
$F(u, v) = \left((\alpha v^{-\beta} + 1)u^{-\theta} + v^{-\theta}\right)^{-\frac{1}{\theta}},$
$\neq F(v, u), \text{ for some } u, v$

FIGURE 1. Introducing the skewed Archimedean Lévy copulas: a more flexible class

*Proof.* By Proposition 2.2 we need to show that $F$ is 2-increasing. By Lemma A.4 it suffices to show $\frac{\partial^2 F}{\partial u \partial v} \geq 0$:

$$\frac{\partial}{\partial u} F(u, v) = (\alpha v^{-\beta} + 1)\left((\alpha v^{-\beta} + 1) + \frac{u^\theta}{v^\theta}\right)^{-(1+\frac{1}{\theta})} =: (\alpha v^{-\beta} + 1) h_{u,v}^{-(1+\frac{1}{\theta})},$$

and therefore

$$\begin{aligned}\frac{\partial^2}{\partial u \partial v} F(u, v) =& -\beta\alpha v^{-(\beta+1)} h_{u,v}^{-(1+\frac{1}{\theta})} + (\alpha v^{-\beta} + 1)\left[-\frac{\theta+1}{\theta}\left(-\beta\alpha v^{-(\beta+1)} - \theta\frac{u^\theta}{v^{\theta+1}}\right) h_{u,v}^{-(2+\frac{1}{\theta})}\right] \\ = -\beta\alpha v^{-(\beta+1)} & \left[h_{u,v} - (\alpha y^{-\beta} + 1)\frac{\theta+1}{\theta}\right] h_{u,v}^{-(2+\frac{1}{\theta})} + (\theta + 1)(\alpha v^{-\beta} + 1)\frac{u^\theta}{v^{\theta+1}} h_{u,v}^{-(2+\frac{1}{\theta})} \\ \geq & h_{u,v}^{-(2+\frac{1}{\theta})}(\theta + 1)\frac{u^\theta}{v^{\theta+1}} \geq 0\end{aligned}$$

$\square$

In FIGURE 1 a one-sided skewed Clayton-Lévy copula is compared to a symmetric Clayton-Lévy copula.

This concludes the entire specification of our model for the spark spread. In the next section we will pick a specific, well suited one-sided skewed Clayton-Lévy copula and provide a complete estimation of our model on data from the UK energy market.

## 3. A CASE STUDY ON UK DATA

In this section we will exemplarily fit the model presented in the preceding chapter to the data of a UK power and gas price series (see FIGURE 2). The data we use is quoted from February 6th, 2001 to December 31st, 2007 and was kindly provided by Icis Heren. In order to estimate and simulate the model we will proceed in six steps:

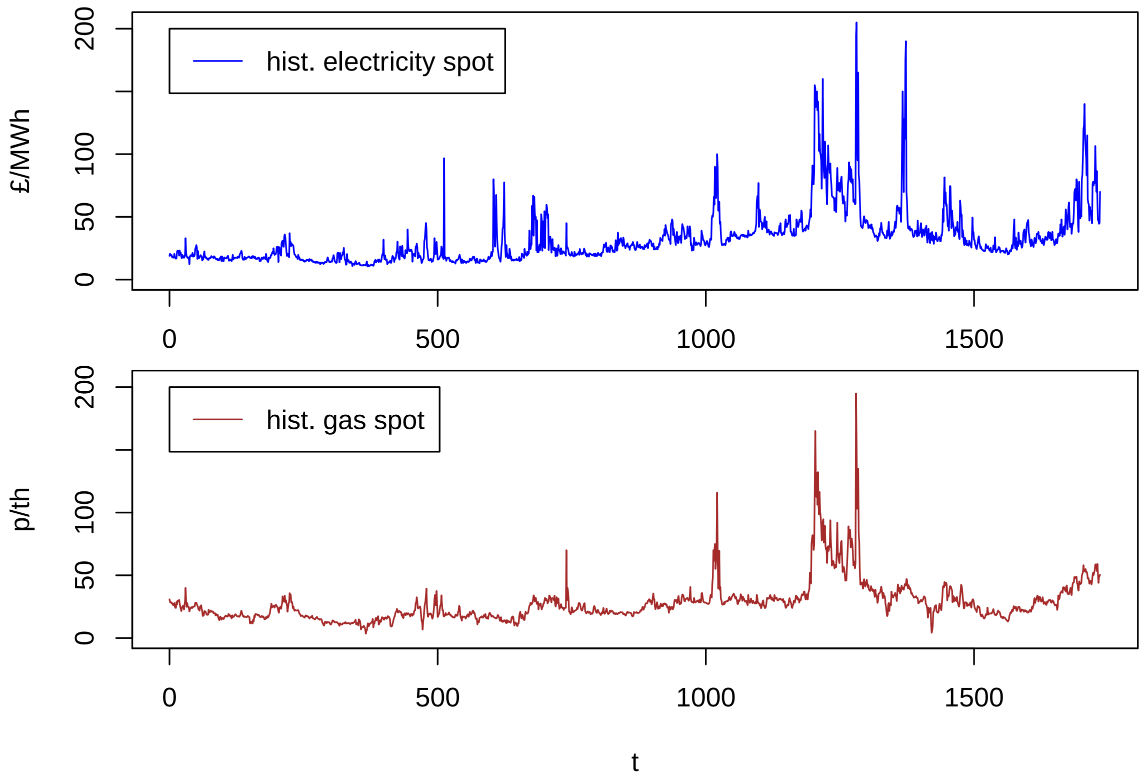



FIGURE 2. Daily quotes of UK power and gas spot prices spanning across 1736 workdays (based on data by ICIS Heren).

(1) Fitting and removing seasonality
(2) Spike filtering
(3) Base signal fitting
(4) Spike signal fitting
(5) Dependence of spike signals
(6) Simulation of the model

The first four steps apply mainly to the fitting of the marginal models (2.2) and (2.5) of the electricity and gas spot price series. We here follow the procedure given in [16] and thus do not give all the details. The main contribution of this paper lies in the modeling and estimation of the dependence risk performed in step five.

**(1) Fitting and removing seasonality.** For each of the marginal models (2.2) and (2.5) we are assuming the seasonal component to be $\Lambda(t) = e^{f(t)}$ with $f$ being a deterministic function of the form

$$f(t) = a + bt + c_1 \sin\left(\frac{2\pi}{252}t\right) + c_2 \cos\left(\frac{2\pi}{252}t\right) + d_1 \sin\left(\frac{4\pi}{252}t\right) + d_2 \cos\left(\frac{4\pi}{252}t\right), \tag{3.1}$$

The six parameters as shown in TABLE 1 are estimated by the method of least squares,

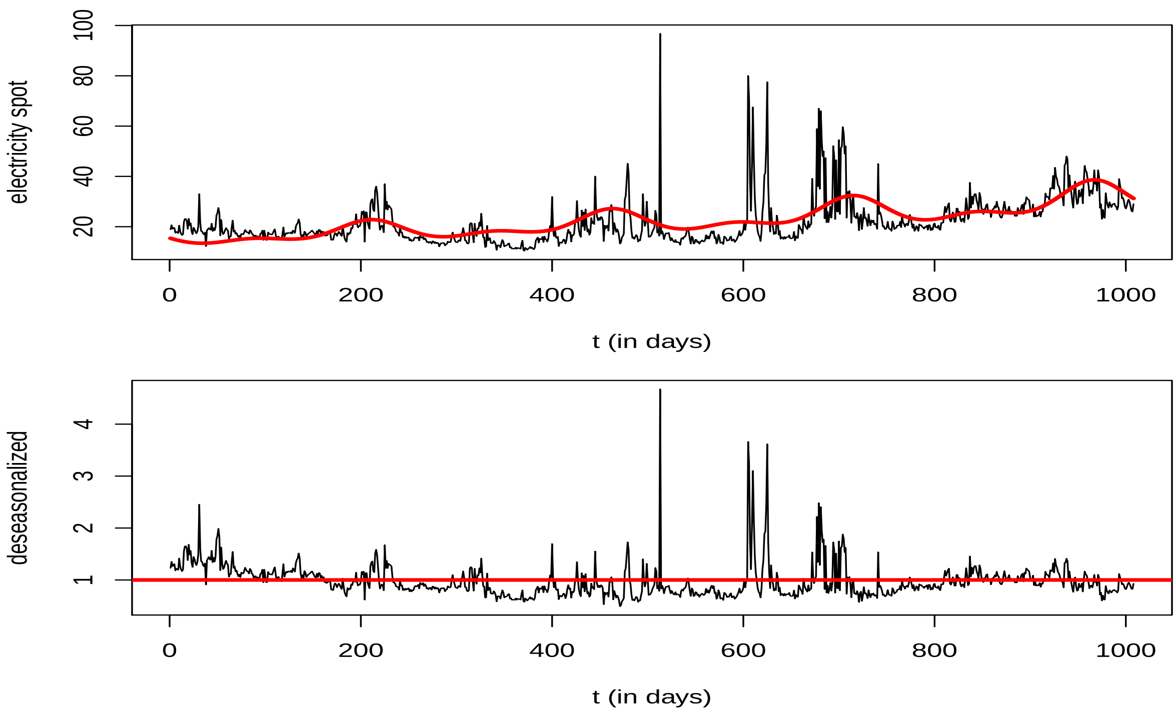



FIGURE 3. Top: The electricity spot prices and the seasonality function (fitted to the logarithms). Bottom: The residuals from dividing by the seasonality function (based on data by ICIS Heren).

fitting $f$ to the logarithms of each data series. FIGURE 3 illustrates the regression for the electricity margin.

| | $a$ | $b$ | $c_1$ | $c_2$ | $d_1$ | $d_2$ |
|---|---|---|---|---|---|---|
| UK power | 2.73 | $6.96 \cdot 10^{-4}$ | -0.147 | 0.0580 | -0.0876 | -0.0446 |
| UK gas | 2.79 | $4.85 \cdot 10^{-4}$ | -0.138 | 0.1918 | -0.0561 | -0.0124 |

TABLE 1. Fitted parameters of the seasonality function

**(2) Spike filtering** For separating the deseasonalized series into base signals - corresponding to $Y_1(t)$ - and spike signals - corresponding to $Y_2(t)$ - we use the adapted Potts filter presented in [16], which compared to the other candidate in [16] (hard-thresholding) performs better for clustered spikes as is the case for UK power and gas. The filter procedures are fed by the following parameters:

- $\lambda_2^e = 1.00$ and $\lambda_2^g = 0.84$, the mean reversion parameters describing the spikes: As proposed in [13], these are estimated directly from the largest relative decreases between two consecutive days. The largest decreases appear after a spike that is not followed by another spike and especially, where the spike is large enough to consider the current base signal negligible. For each margin we take the second largest (avoiding an exceptionally extreme case) relative decrease $\phi := Y(t)/Y(t-1)$ and we estimate $\lambda_2 := -\log \phi$.

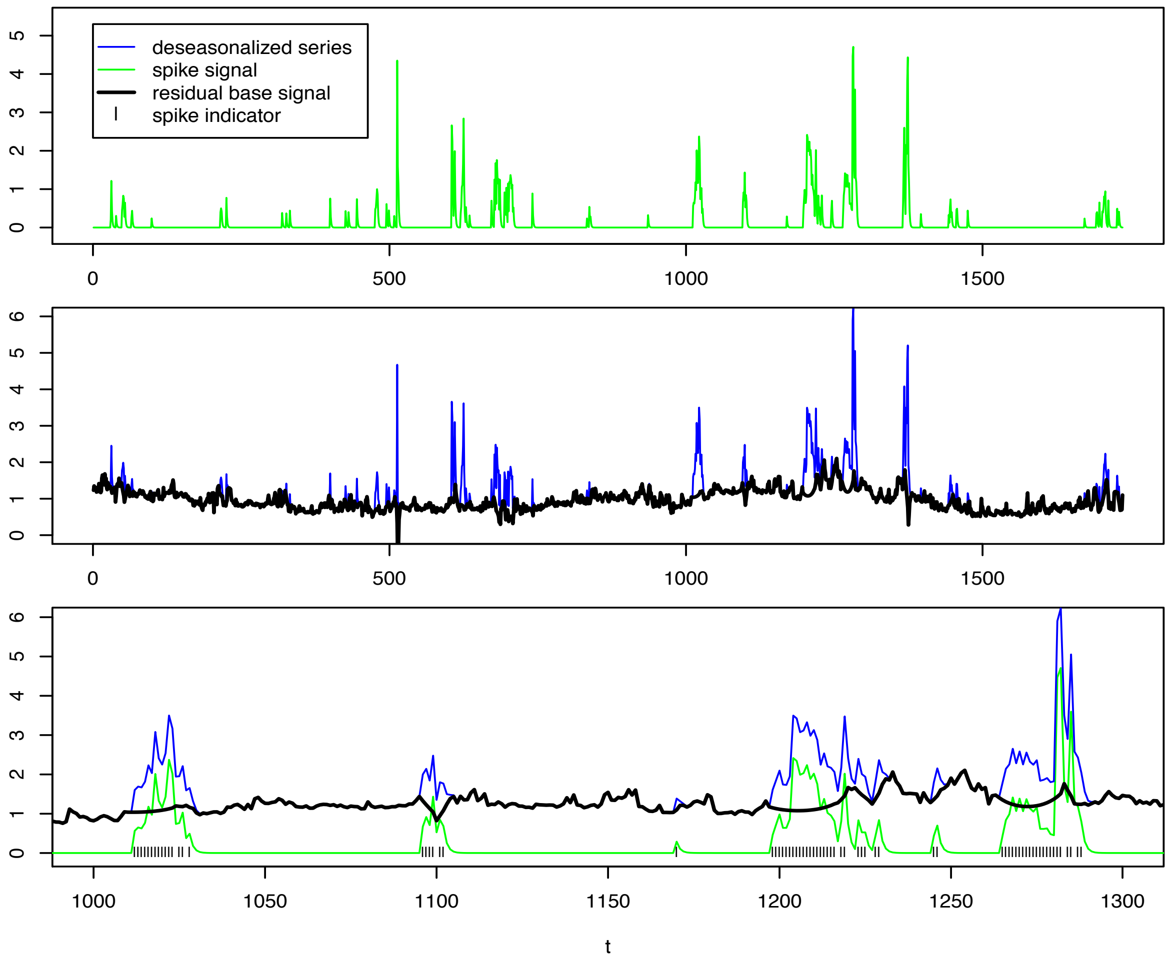



FIGURE 4. Performance of the adapted Potts filter on UK electricity. Top: Filtered spike signal. Middle: The deseasonalized series and the residual base signal after filtering. Bottom: the combination of both, at a higher resolution (based on data by ICIS Heren).

- $\lambda_1^e = 0.17$ and $\lambda_1^g = 0.23$ as first estimates for the mean reversion rates of the Gaussian Ornstein-Uhlenbeck processes: We remark, that in step three they will be reestimated with more accuracy and also the choices here play a minor role for the performance of the filter, i.e. any $\lambda_1$ between 0.3 and 0.05 would perform comparably. However, here they are found by examination of the autocorrelation functions.
- The penalties for placing a spike are found by decreasing the values in small steps, until the filtering procedures reach the desired closeness to a Gaussian base signal. See [16] for details on this procedure.

The adapted Potts filter finally places $M = 169$ spikes for UK power. Its performance is illustrated in FIGURE 4. For UK gas it places $M = 74$ spikes. We remark that there seems to be evidence of clusters in spike occurrence. To capture cluster behavior one would have to extend the model to stochastic jump intensities, which, however, is beyond the scope of the paper.

**(3) Base signal fitting and base signal dependence** In our model the base signals $Y_1^e$ and $Y_2^g$ which correspond to the residuals after the respective spike removals are represented by one-factor Gaussian Ornstein-Uhlenbeck processes. These fail to capture the stochastic long-term movement of a trend line that the residual base signal exhibits: see FIGURE 4 (middle). A third Gaussian OU component as in [13] with a very slow mean reversion rate could be considered to more appropriately model the base signal. But since this paper focusses on the dependence modeling of spike risk, we restrict ourself to one-factor models for the base signals for simplicity of the exposition.

We estimate the mean level parameters $\mu^e, \mu^g$, the volatility parameters $\sigma^e, \sigma^g$, the mean reversion parameters $\lambda_1^{\mathrm{e}}, \lambda_1^{\mathrm{g}}$ and the correlation $\delta$ of the two dimensional Brownian motion by fitting the corresponding two dimensional AR(1)-process with a Maximum-Likelihood method to the historical base signals. The estimates are shown in TABLE 2.

| Set | $\mu$ | $\sigma$ | $\lambda_1$ |
|---|---|---|---|
| UK power | 0.95 | .11 | .066 |
| UK gas | 1.00 | .09 | .078 |

Correlation $\delta$: .23

TABLE 2. Estimates for the base signal

**(4) Spike signal fitting** We estimate the jump intensities $\rho^{\mathrm{e}}, \rho^{\mathrm{g}}$ of the two compound Poisson processes driving the spike signals by dividing the number of filtered spikes by the entire number of days under observation. For the jump size distributions $D_{\mathrm{e}}, D_{\mathrm{g}}$ we have found good fit in both cases with the heavy tailed shifted inverse Gaussian distributions (a random variable $X$ is said to follow a *shifted inverse Gaussian distribution* with *shift* $s$, mean $\mu > 0$ and shape parameter $\lambda > 0$ if $X - s \sim IG(\mu, \lambda)$).

For estimating the parameters, the method of Maximum-Likelihood is applied and the results are reproduced in TABLE 3.

| Set | $s^*$ | $\mu^*$ | $\lambda^*$ | $\rho$ |
|---|---|---|---|---|
| UK power (pos) | .15 | .60 | .56 | .10 |
| UK gas (pos) | .24 | .54 | .32 | .04 |

TABLE 3. Maximum-likelihood estimates for shifted inverse Gaussian distributions and the intensity of the respective compound Poisson process

**(5) Dependence of spike signals** As announced in Section 2 a Lévy copula shall link the two marginal processes $L^{\mathrm{e}}$ and $L^{\mathrm{g}}$. Our estimation procedure makes use of the fact that the dependence modeling via Lévy copulas is fully abstracted from the marginal processes, which means that the procedure does not rely on any choice we have made for the distributions of the spikes in step (4).

According to Definition A.6, let $U_1$, $U_2$ and $U$, respectively be the two marginal tail integrals and the joint tail integral of $(L^{\mathrm{e}}, L^{\mathrm{g}})$ Their empirical equivalences shall be $\bar{U}_1(x), \bar{U}_2(y), \bar{U}(x,y)$, returning the observed average number of jumps per day, which exceeded $x$ in their electricity component, respectively $y$ in their gas component, or respectively both. Denote the Lévy copula by $F$, that satisfies $F(U_1(x), U_2(y)) = U(x,y),\ x, y \in [0, \infty]$. In order to estimate a Lévy copula on our data we introduce the notion of the *empirical Lévy copula* $\bar{F}(u_1, u_2)$ as the function defined on the range of $\bar{U}_1$ and $\bar{U}_2$ satisfying $\bar{F}(\bar{U}_1(x), \bar{U}_2(y)) = \bar{U}(x,y)$.

In our case the empirical Lévy copula consists of 1632 points and a 3D-plot revealing its shape can be seen in FIGURE 5. We give a brief interpretation to a few features of the figure:

- All points of $\bar{F}$ are plotted, so the maximum value of $\bar{U}_2$ is smaller than the maximum value of$\bar{U}_1$, which simply reflects that historically (at least according to our filtering) there were more jumps in electricity.
- For fixed $u_1$, $\bar{F}(u_1, u_2)$ shows a concave behavior in $u_2$, growing rapidly for small values. The high values for small $u_2$ ensure that a large jump in gas is very likely accompanied by a jump in electricity.
- Contrarily, the behavior of $\bar{F}$ for fixed $u_2$ shows a slow growth. A large jump in electricity is not necessarily accompanied by a jump in the gas margin.

The last two items in the list represent the skewness of the Lévy copula, and show that we cannot use Lévy copulas where $F(x,y) = F(y,x)$. For choosing the best Lévy copula we tested 24 2-dimensional functions as candidates for skewed Archimedean Lévy copulas, temporarily assuming they are all 2-increasing. Precisely, we are using the four one-parameter Archimedean generators presented among the examples in Section A (i.e

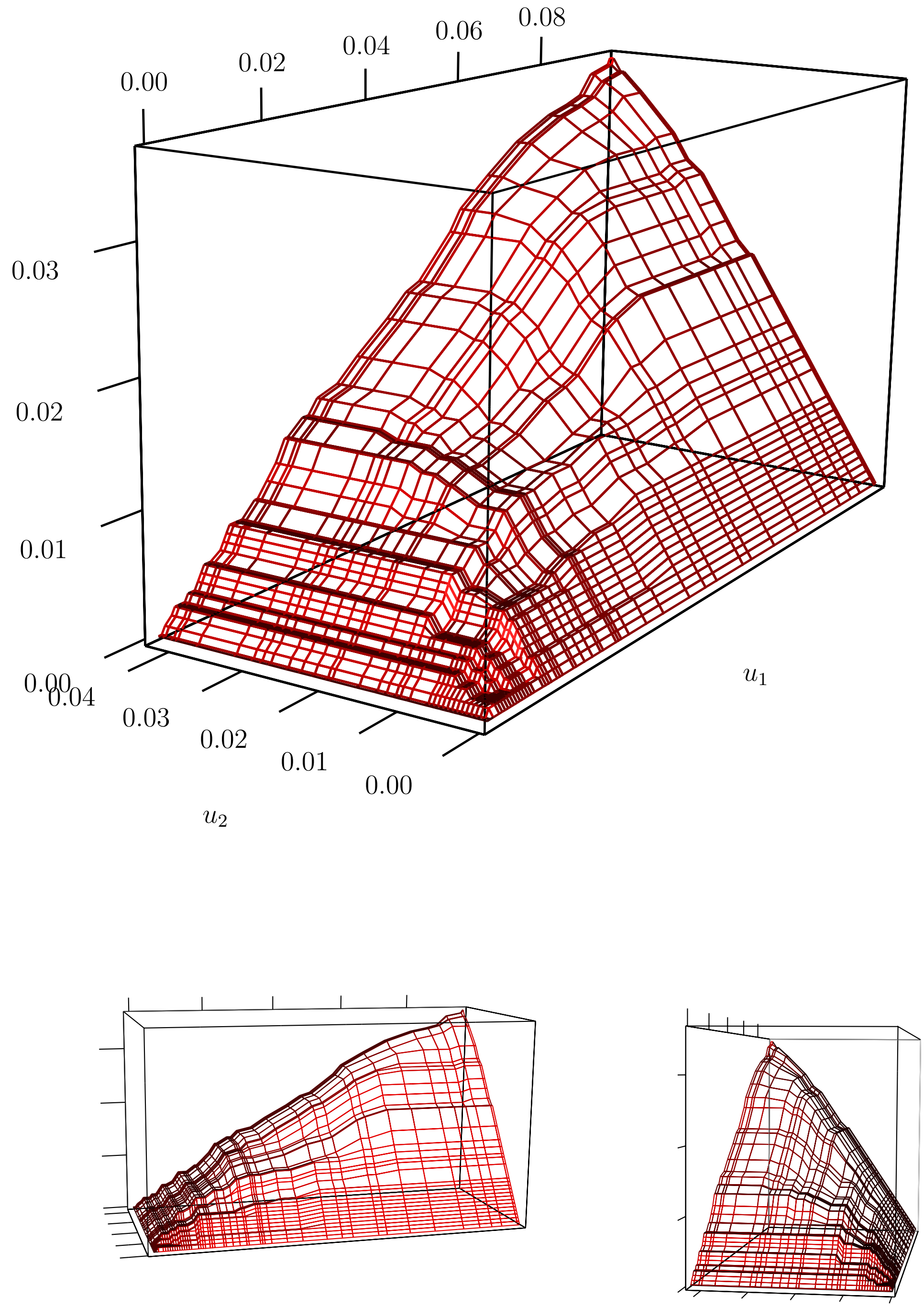



FIGURE 5. Empirical Lévy copula, and profile views (based on data by ICIS Heren).

$\phi_C, \phi_G, \phi_{\bar{G}}, \phi_{\exp}$) and six skew factors, the neutral and five more with one parameter, as follows:

$$\begin{array}{ll} \psi_1(x) = 1, & \psi_4(x) = \frac{\pi}{2}\left(\arctan(\alpha x)\right)^{-1}, \\ \psi_2(x) = \frac{\alpha}{x} + 1, & \psi_5(x) = \tan\left(\frac{\pi}{2}(x^\alpha + 1)^{-1}\right) + 1, \\ \psi_3(x) = \frac{\alpha}{\sqrt{x}} + 1, & \psi_6(x) = (\log(\alpha x + 1)^{-1}) + 1 \end{array}$$

For each combination of generator and skew factor we construct the function:

$$F_{\phi,\psi}(x, y) = \phi^{-1}\left(\psi(y)\phi(x) + \phi(y)\right),$$

and fit its parameters to the empirical Lévy copula by least squares. Then we compare the sums of squared distances between the fitted and the empirical Lévy copula:

$$\Sigma_{\phi,\psi} = \sum_{(u_1,u_2,z)\in ELC} \left(F_{\phi,\psi}(u_1, u_2) - z\right)^2,$$

where $ELC$ denotes the graph of the empirical Lévy copula. All the values for $\Sigma_{\phi,\psi}$ are given in TABLE 4. The result of the measurement holds three clear statements:

- The two generators $\phi_{\exp}$ and $\phi_{\bar{G}}$ cannot compete with $\phi_C$ and $\phi_{\bar{G}}$, except in the non-skewed, case.
- The skew is essential and provides significantly better fit in all cases.
- Within the Clayton- and the Gumbel- generators there are only very small differences between the success of the different skew factors.

| $\phi\backslash\psi$ | $1$ | $\frac{\alpha}{x}+1$ | $\frac{\alpha}{\sqrt{x}}+1$ | $\psi_4$ | $\psi_5$ | $\psi_6$ |
|---|---|---|---|---|---|---|
| $\phi_C$ | 67.6 | 5.59 | 4.79 | 5.71 | 5.22 | 5.52 |
| $\phi_G$ | 69.2 | 5.35 | 4.92 | 5.45 | 5.18 | 5.30 |
| $\phi_{\bar{G}}$ | 103 | 42.1 | 48.4 | 41.4 | 44.2 | 42.5 |
| $\phi_{\exp}$ | 61.3 | 30.0 | 16.6 | 37.7 | 9.00 | 20.0 |

TABLE 4. The residual sum-of-squares after fitting the parameters for each combination, given as $\Sigma_{\phi,\psi} \cdot 10^4$

According to the TABLE 4 $\psi_3$ combined with the Clayton-Lévy generator (where 2-increasingness is already verified by Proposition 2.3) is our best option. The two fitted parameters are $\theta = 3.39$ and $\alpha = 2.56$, while the final Lévy copula $F$ chosen for the model in this paper takes the form:

$$F(u, v) = \left(\left(\frac{\alpha}{\sqrt{v}} + 1\right) u^{-\theta} + v^{-\theta}\right)^{-\frac{1}{\theta}}$$

In FIGURE 6 one can see how it mimics the empirical Lévy copula quite well and in particular the skewness mirrors the smaller slope in $u_1$.

**(6) Simulation of the model** We conclude this section with a short simulation study. A simulated path of the spark spread with the estimated parameters is presented in

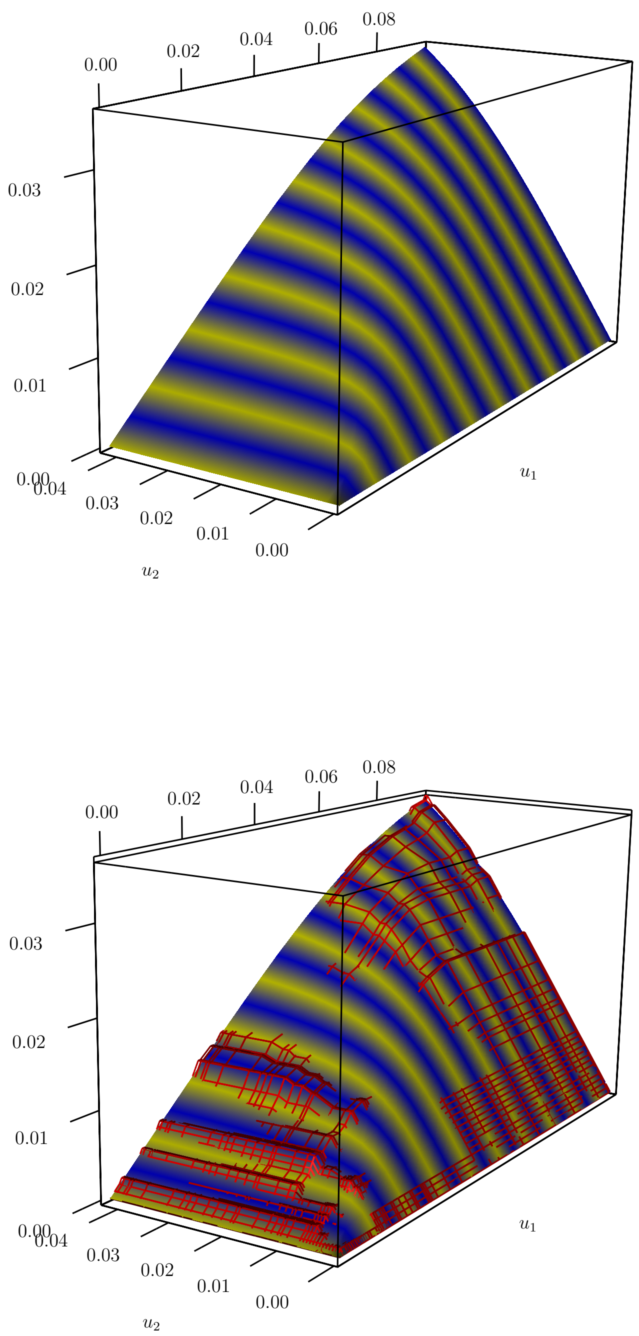



FIGURE 6. The estimated Lévy copula and its fit to the empirical Lévy copula (based on data by ICIS Heren).

FIGURE 7. The seemingly higher volatility in the simulated path is due to the fact that we restricted ourself to one OU component for the base signal for simplicity of the exposition (a more realistic modeling of the base signal with two OU components can be found in [13]). The focus of this paper is on the dependence modeling of the spike risk in electricity and gas prices. A detailed view on the spike dependence can be seen in FIGURE 8. In both cases, the historical and the simulated, we can see, how some spikes in the electricity spot are totally explained by a spike in the gas price, while others are not even noticed in the gas. Some are only partially accompanied. A large spike in the gas price which is not accompanied by a spike in the electricity is not found on any series.

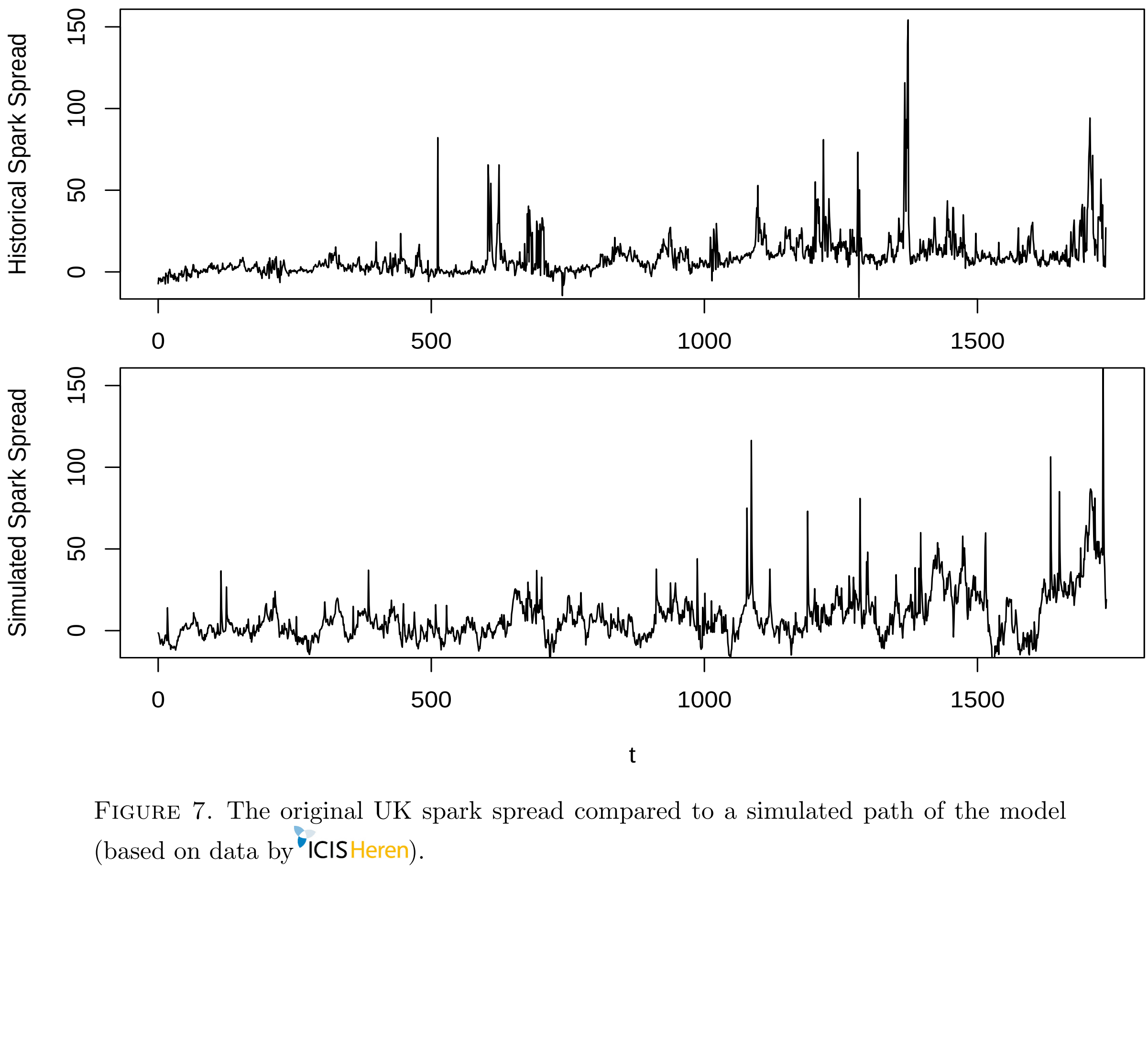



FIGURE 7. The original UK spark spread compared to a simulated path of the model (based on data by ICIS Heren).

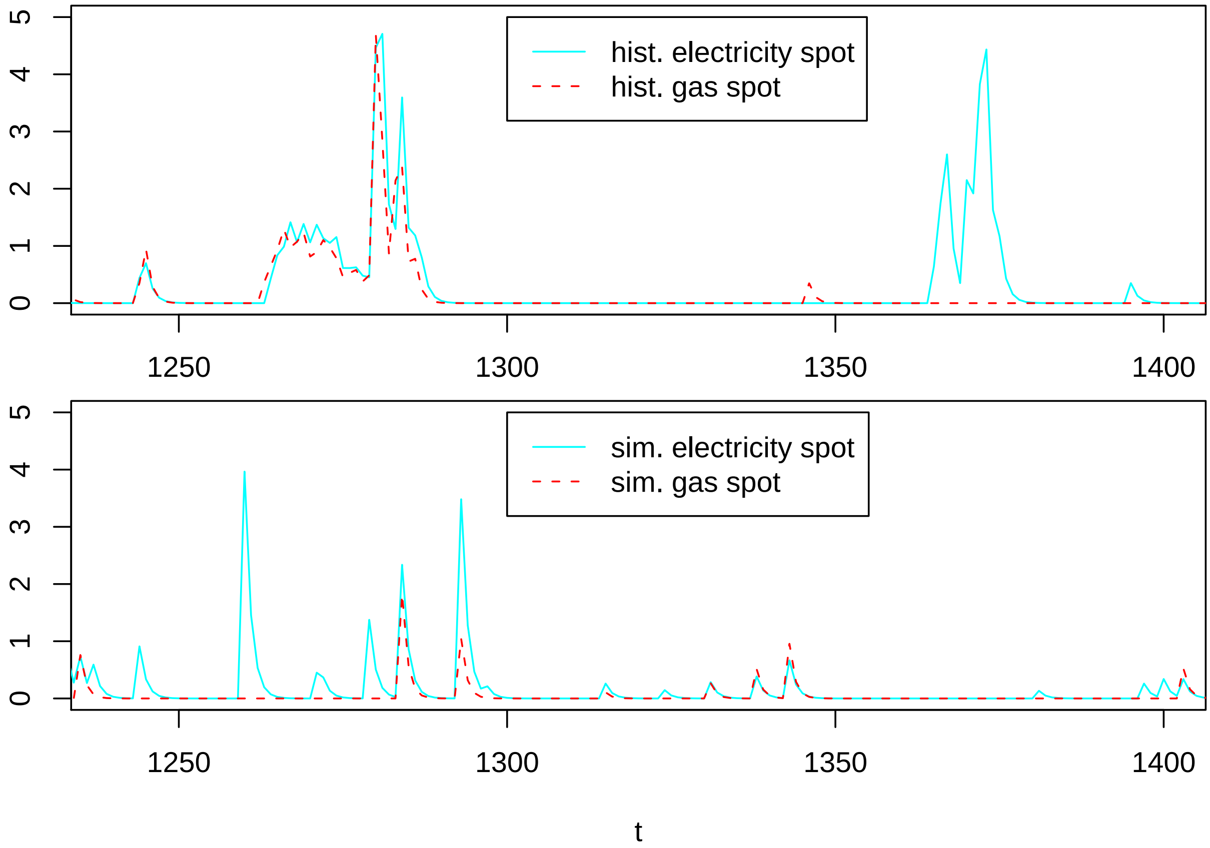



FIGURE 8. Detailed views on the isolated spike signals and the spike dependence in the historical and in a simulated setup (based on data by ICIS Heren).

## 4. Pricing of options written on the Spark Spread

In this section we demonstrate the applicability of our model in pricing options on the spark spread. We will focus on computing arbitrage free prices of call options written on the spark spread with maturity $T$ and strike $K$:

$$C_{T,K} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}(S(T)-K)^+\right],$$

where $(x)^+ := \max(x,0)$, $r(s)$ is the risk free rate, and $\mathbb{Q}$ is a risk neutral pricing measure. In the following we assume for the sake of simplicity that $r = 0$ and that $\mathbb{Q}$ is model structure preserving, i.e. without loss of generality we set $\mathbb{Q} = \mathbb{P}$. Then the call option price reads

$$C_{T,K} = \mathbb{E}\left[(S(T)-K)^+\right]. \tag{4.1}$$

To compute the price in (4.1) we will employ Fourier transform pricing techniques as presented in [6, Section 3.1]). Define the *dampened payoff function* $f : \mathbb{R} \to [0,\infty)$ by

$$f(x) := e^{-\alpha x}(x-K)^+, \ \alpha > 0, x \in \mathbb{R},$$

which is needed in the sequel, since $(\cdot - K)^+$ is not integrable on the right. Note that $f$ is an integrable function for any $\alpha > 0$. It is easy to show that the Fourier transform of $f$, denoted by $\hat{f}$, is also integrable and given by

$$\hat{f}(u) = \int_{\mathbb{R}} e^{ixu} f(x)dx = \frac{1}{(iu-\alpha)^2} e^{(iu-\alpha)K}. \tag{4.2}$$

So the Fourier inversion theorem (see for example [14, Section 8.2]) says that

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ixu}\hat{f}(u)du.$$

We can thus derive the following semi-analytic expression for the call price

$$\begin{aligned} C_{T,K} &= \mathbb{E}\left[(S(T)-K)^+\right] = \mathbb{E}\left[e^{\alpha S(T)} f(S(T))\right] \\ &= \frac{1}{2\pi}\mathbb{E}\left[e^{\alpha S(T)}\int_{\mathbb{R}} e^{-iuS(T)}\hat{f}(u)du\right] \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}\left[e^{(\alpha-iu)S(T)}\right]\hat{f}(u)du, \end{aligned} \tag{4.3}$$

provided the extended Laplace transform $\mathbb{E}\left[e^{(\alpha-iu)S(T)}\right]$ exists for the chosen $\alpha > 0$. Now, due to the arithmetic structure of the marginal price models, the resulting model for the spark spread $S(t)$ is also of arithmetic form and analytically very tractable. In particular, we can analytically compute the extended Laplace transform. Indeed, by explicitly solving for the involved OU components we can write

$$S(T) = \phi^0 + \int_0^T \phi_s^1 dB_s + \int_0^T \phi_s^2 dL_s, \tag{4.4}$$

where the two dimensional Brownian motion $B = (B^e, B^g)$ has correlation parameter $\delta$ and the two-dimensional compound Poisson process $L = (L^e, L^g)$ is characterized by its Lévy measure $\nu(dx)$ (with support on $\mathbb{R}_+ \times \mathbb{R}_+$), and where we define

$$\begin{aligned}
\phi^0 :=& \Big[\Lambda^{\mathrm{e}}(T)\left(\mu^{\mathrm{e}} + e^{-\lambda_1^{\mathrm{e}} T}(Y_1^{\mathrm{e}}(0) - \mu^{\mathrm{e}})\right) - c\Lambda^{\mathrm{g}}(T)\left(\mu^{\mathrm{g}} + e^{-\lambda_1^{\mathrm{g}} T}(Y_1^{\mathrm{g}}(0) - \mu^{\mathrm{g}})\right)\Big],\\
\phi_s^1 :=& \begin{pmatrix} \Lambda^{\mathrm{e}}(T)\sigma^e e^{-\lambda_1^{\mathrm{e}}(T-s)} \\ -c\Lambda^{\mathrm{g}}(T)\sigma^g e^{-\lambda_1^{\mathrm{g}}(T-s)} \end{pmatrix},\\
\phi_s^2 :=& \begin{pmatrix} \Lambda^{\mathrm{e}}(T) e^{-\lambda_2^{\mathrm{e}}(T-s)} \\ -c\Lambda^{\mathrm{g}}(T) e^{-\lambda_2^{\mathrm{g}}(T-s)} \end{pmatrix}.
\end{aligned}$$

By employing the Lévy-Khinchin representation for Lévy processes we can use this explicit representation to derive as in [12, Proposition 1.9] the extended Laplace transform of $S(T)$ if it exists:

(4.5)

$$\begin{aligned}
&\mathbb{E}\left[e^{(\alpha - iu)S(T)}\right]\\
&= \exp\left((\alpha - iu)\phi^0\right)\mathbb{E}\left[\exp\left(\int_0^T (\alpha - iu)\phi_s^1 dB_s\right)\right]\mathbb{E}\left[\exp\left(\int_0^T (\alpha - iu)\phi_s^2 dL_s\right)\right]\\
&= \exp\left((\alpha - iu)\phi^0\right)\exp\left(\int_0^T \frac{1}{2}(\alpha - iu)^2\langle \phi_s^1, A\phi_s^1\rangle ds\right)\exp\left(\int_0^T \int_{\mathbb{R}^2}\left(e^{\langle(\alpha - iu)\phi_s^2, x\rangle} - 1\right)\nu(dx)ds\right)
\end{aligned}$$

where $\alpha, u \in \mathbb{R}$ and

$$A = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}$$

is the correlation matrix of the Brownian motion. In order to specify for which values of $\alpha > 0$ the extended Laplace transform exists, we only need to consider the Poisson process $L^{\mathrm{e}}$, since $L^{\mathrm{g}}$ points into the negative direction and therefore can be ignored, while the Brownian and deterministic components clearly have finite expectation. Further, the non-decreasing paths of the compound Poisson process allows us to ignore the mean reversion:

$$\mathbb{E}\left[\exp\left(\alpha\int_0^T \Lambda^{\mathrm{e}}(T)e^{-\lambda_2^{\mathrm{e}}(T-s)}dL^{\mathrm{e}}(s)\right)\right] \le \mathbb{E}\left[\exp\left(\alpha\Lambda^{\mathrm{e}}(T)L^{\mathrm{e}}(T)\right)\right].$$

The exponential moments of Lévy processes $\mathbb{E}\left[e^{uL_t}\right]$ are finite if and only if their Lévy measure satisfies $\int_{|x|\geq 1} e^{ux}\nu(dx) < \infty$ (see [17, Theorem 25.17]). So expressed by the jump size distribution $D_\mathrm{e}$, in our case shifted inverse Gaussian with parameters $(\mu, \lambda)$, we are looking for an $\alpha > 0$ such that

$$\int_{\mathbb{R}_+} e^{(\alpha\Lambda^\mathrm{e}(T))x} D_\mathrm{e}(dx) < \infty\,.$$

For the regular inverse Gaussian as a special case of the generalized inverse Gaussian distribution, according to [8] this holds as long as $\alpha\Lambda^\mathrm{e}(T) < \frac{\lambda}{2\mu^2}$. The shift, however, does not change the asymptotic behavior and therefore the same applies here. Therefore $\alpha$ must be chosen in dependence on $T$ as:

$$0 < \alpha < \lambda \left(2\mu^2\Lambda^\mathrm{e}(T)\right)^{-1}\,.$$

By plugging (4.5) into (4.3), we thus obtain

**Proposition 4.1.** *Let the electricity jump distribution $D_\mathrm{e}(dx)$ be shifted inverse Gaussian with parameters $\mu$ and $\lambda$, and let $0 < \alpha < \lambda\left(2\mu^2\Lambda^\mathrm{e}(T)\right)^{-1}$ be a given constant. Then the price of a call option written on the spark spread with maturity $T$ and strike $K$ is given by*

$$C_{T,K} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\alpha-iu)\phi^0} \exp\left(\int_0^T \frac{1}{2}(\alpha-iu)^2\langle\phi_s^1, A\phi_s^1\rangle ds\right) \cdot \exp\left(\int_0^T \int_{\mathbb{R}^2}\left(e^{\langle(\alpha-iu)\phi_s^2,x\rangle}-1\right)\nu(dx)ds\right)\hat{f}(u)du, \tag{4.6}$$

To compute prices given in (4.6), we observe that the integral in the first parentheses can be calculated analytically as

$$\begin{aligned}\int_0^T \frac{1}{2}(\alpha-iu)^2\langle\phi_s^1, A\phi_s^1\rangle ds = \frac{1}{2}(\alpha-iu)^2\Bigg[&\frac{1}{2\lambda_1^\mathrm{e}}a_{11}(\Lambda^\mathrm{e}(T))^2\left(1-e^{-2\lambda_1^\mathrm{e}T}\right)\\ &-\frac{2}{\lambda_1^\mathrm{e}+\lambda_1^\mathrm{g}}a_{21}c\Lambda^\mathrm{e}(T)\Lambda^\mathrm{g}(T)\left(1-e^{-\left((\lambda_1^\mathrm{e}+\lambda_1^\mathrm{g})\right)T}\right)\\ &+\frac{1}{2\lambda_1^\mathrm{g}}a_{22}c^2(\Lambda^\mathrm{g}(T))^2\left(1-e^{-2\lambda_1^\mathrm{g}T}\right)\Bigg].\end{aligned}$$

The double integral in the second parentheses will require numerical evaluation, where we shortly sketch how to compute the inner integral in terms of the Lévy copula and the tail integrals of the marginal compound Poisson processes. We denote the integrand by

$$g_{u,s}(x_1,x_2) := e^{\langle(\alpha-iu)\phi^2(s),x\rangle}-1$$

and fix a grid $R \times R$ with $R = \{r_0, \ldots, r_k\} \subset \mathbb{R}$, $0 = r_0 < r_1 < \cdots < r_k$. Then

$$\begin{aligned}
&\int_{[0,\infty[^2} g_{u,s}(x_1, x_2)\nu(dx_1 \times dx_2) \\
&\approx \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} g_{u,s}(r_i, r_j)\nu\left([r_i, r_{i+1}[\times[r_j, r_{j+1}[\right) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} g_{u,s}(r_i, r_j) \cdot [U(r_i, r_j) - U(r_i, r_{j+1}) - U(r_{i+1}, r_j) + U(r_{i+1}, r_{j+1})] \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} g_{u,s}(r_i, r_j) \cdot \Big[F(U_1(r_i), U_2(r_j)) - F(U_1(r_i), U_2(r_{j+1})) \\
&\qquad\qquad - F(U_1(r_{i+1}), U_2(r_j)) + F(U_1(r_{i+1}), U_2(r_{j+1}))\Big],
\end{aligned}$$

where the tail integrals are given by

$$\begin{aligned}
U_1(x) &= \rho^{\mathrm{e}}(1 - D_{\mathrm{e}}(x))\,, \\
U_2(x) &= \rho^{\mathrm{g}}(1 - D_{\mathrm{g}}(x))\,.
\end{aligned}$$

We conclude this section by computing some call prices for our estimated model according to the formula given in Proposition 4.1. Parallel to [2], in Table 5, we consider call prices with a maturity of $T = 20$ days for five strike prices $K \in \{0, 5, 10, 15, 20\}$ and four periods of time equally spaced around the yearly cycle beginning on the same day the sample begins: $t_0 \in \{0, 63, 126, 189\}$. Those periods are referred to as Period 1,2,3 and 4. We also applied the same routine for two non-skewed Clayton Lévy copulas with $\theta_1 = 0.1$ and $\theta_2 = 10$, representing continuous approximations for respectively the independence and the complete dependence Lévy copula (more extreme values for $\theta_1, \theta_2$ do not change the prices noticeably). Additionally we fitted an alternative skewed Clayton Lévy copula with skew factor $\psi(x) = \frac{\alpha}{x} + 1$ in order to study the model's robustness across different choices for the Lévy copula.

The first finding is that the seasonal effect and trend plays the major role for the option prices. This is a common and natural observation in energy markets.Another conspicuous observation is that the relative effect of the assumed dependence becomes stronger when moving towards out-of-the-money options. This results in a rather absolute effect on the prices for in-the-money options.

Choosing any of the skew Lévy copulas keeps the prices nicely between those of an independent and fully dependent assumption, except for the two cases of $K \in \{5, 10\}$ in Period 1, where the fitted skew copulas leave a higher value for the option. The choice of the skew factor only has a very small effect. This is the ideal case in terms of robustness. However, in general we find that the stronger the dependence assumption, the higher the value of the call option. It would be more intuitive to think that a strong dependence

| Strike | Period 1 | Period 2 | Period 3 | Period 4 | Levy copula |
|---|---|---|---|---|---|
| -10 | 6.0887 | 10.4511 | 10.3304 | 8.6884 | Complete dep. |
| -10 | 5.8444 | 10.1682 | 10.0413 | 8.2714 | Alternative LC |
| -10 | 5.8392 | 10.1629 | 10.0359 | 8.2632 | Skew LC |
| -10 | 5.7914 | 10.0144 | 9.8876 | 8.0804 | Independence |
| -5 | 1.1094 | 5.4637 | 5.3315 | 3.6909 | Complete dep. |
| -5 | 0.9024 | 5.1821 | 5.0432 | 3.2851 | Alternative LC |
| -5 | 0.8913 | 5.1764 | 5.0374 | 3.2747 | Skew LC |
| -5 | 0.8158 | 5.0229 | 4.8869 | 3.0837 | Independence |
| 0 | 0.0350 | 0.6360 | 0.5745 | 0.3793 | Complete dep. |
| 0 | 0.0272 | 0.3856 | 0.3250 | 0.1332 | Alternative LC |
| 0 | 0.0184 | 0.3773 | 0.3162 | 0.1164 | Skew LC |
| 0 | 0.0002 | 0.2320 | 0.1744 | 0.0043 | Independence |
| 5 | 0.0031 | 0.0884 | 0.0883 | 0.1193 | Complete dep. |
| 5 | 0.0112 | 0.0389 | 0.0393 | 0.0561 | Alternative LC |
| 5 | 0.0070 | 0.0299 | 0.0299 | 0.0415 | Skew LC |
| 5 | 0.0002 | 0.0004 | 0.0003 | 0.0005 | Independence |
| 10 | 0.0003 | 0.0208 | 0.0210 | 0.0389 | Complete dep. |
| 10 | 0.0056 | 0.0183 | 0.0187 | 0.0322 | Alternative LC |
| 10 | 0.0035 | 0.0128 | 0.0130 | 0.0220 | Skew LC |
| 10 | 0.0001 | 0.0010 | 0.0007 | 0.0003 | Independence |

Table 5. The prices of call options of maturity $T = 20$ derived from the model with complete dependence, fitted dependence and independence of spikes

reduced the value of the spark spread options, i.e. by cancelling out situations where the electricity spot jumps without the gas spot, thus producing less situations where the option is worthless, but as one can see here the opposite is true. This phenomenon was also found in the studies of [3].

## Appendix A. Lévy copulas

We will shortly review the main definitions and results needed for the concept of two-dimensional Lévy copulas for Lévy measures with positive support. For further informations on Lévy copulas we refer to [7] and [10]. We first need a multidimensional extension of the notion of increasing functions.

**Definition A.1** ($F$-volume). *Let $F : S \to \bar{\mathbb{R}}$ for some subset $S \subset \bar{\mathbb{R}}^d$. For $u^1 = (u^1_1, \ldots, u^1_d)$, $u^2 = (u^2_1, \ldots, u^2_d) \in S$ with $u^1 \leq u^2$ and $[u^1, u^2] \subset S$, the* $F$-volume *is* *defined by*

$$V_F([u^1,u^2]) := \sum_{j_1=1}^{2}\cdots\sum_{j_d=1}^{2}(-1)^{j_1+\cdots+j_d}F(u_1^{j_1},\ldots,u_d^{j_d})$$

**Definition A.2** ($d$-increasing)**.** *A function $F : S \to \bar{\mathbb{R}}$ for some subset $S \subset \bar{\mathbb{R}}^d$ is called* $d$-increasing *if $V_F([u,v]) \geq 0$ for all $u,v \in S$ with $u \leq v$ and $[u,v] \subset S$.*

**Definition A.3** (Grounded functions)**.** *A function $F : [0,\infty]^d \to [0,\infty]$ is* grounded *if $F(x) = 0,\ x = (x_1,\ldots,x_d)$ as soon as any of $x_1,\ldots,x_d$ equals 0.*

Note, that in general the notion of $d$-increasing functions does not coincide with functions that increase in each margin. However, in the following lemma some connection is made in the 2-dimensional and positive case.

**Lemma A.4.** *Let $F : [0,\infty]^2 \to [0,\infty]$ be grounded and in $C^{1,1}$ then $F$ is 2-increasing, if $\frac{\partial^2}{\partial u \partial v}F(u,v) > 0, \forall (u,v) \in [0,\infty]^2$*

Next, we need the definition of the tail integral of a positive Lévy measure.

**Definition A.5** (Tail integral)**.** *A d-dimensional* tail integral *is a function $U : [0,\infty]^d \to [0,\infty]$ such that*

1. *$(-1)^d U$ is a d-increasing function.*
2. *$U$ is equal to zero if one of its arguments is equal to $\infty$.*
3. *$U$ is finite everywhere except at zero and $U(0,\ldots,0) = \infty$.*

*The margins $U_k,\ k = 1,\ldots,d$ of a tail integral $U$ are defined by*

$$U_k(x) = U(0,\ldots,0, \underset{\substack{\uparrow \\ k\text{-th posisition}}}{x}, 0,\ldots,0).$$

The name "tail integral" makes more sense, as soon as the following link between Lévy measures and tail integrals is made:

**Definition A.6** (Tail integral of a positive Lévy measure)**.** *The tail integral $U$ of a Lévy measure $\nu$ on $[0,\infty)^d$ for $x = (x_1,\ldots,x_d)$ is defined by*

- *$U(x) = 0$, if $x_k = \infty$ for any $k = 1,\ldots,d$;*
- *$U(x) = \nu([x_1,\infty) \times \cdots \times [x_d,\infty))$ for $x \in [0,\infty)^d \backslash \{0\}$;*
- *$U(0,\ldots,0) = \infty$.*

We are now ready to introduce Lévy copulas for two-dimensional Lévy measures with positive support. We remark that the general framework would need some small modification.

**Definition A.7** (Lévy copula)**.** *A two-dimensional* Lévy copula *for Lévy processes with positive Lévy jumps is a function $F : [0,\infty]^2 \to [0,\infty]$, which*

- *is grounded,*

- *2-increasing and*
- *has uniform margins, that is* $F(x, \infty) = F(\infty, x) = x, \ \forall x \in [0, \infty]$.

In analogy to Sklar's Theorem for copulas, the following theorem forms the main result in the theory of Lévy copulas.

**Theorem A.8** (Sklar's Theorem for Lévy copulas)**.** *Let* $(X_t, Y_t)$ *be a two-dimensional Lévy process with positive jumps having tail integral* $U$ *and marginal tail integrals* $U_1$ *and* $U_2$*. There exists a positive Lévy copula* $F$ *such that:*

$$U(x_1, x_2) = F(U_1(x_1), U_2(x_2)), \ \forall x_1, \ x_2 \in [0, \infty] \tag{A.1}$$

*It is unique on* Ran $U_1 \times$ Ran $U_2$*, the product of the ranges of one-dimensional tail integrals.*
*Conversely, if* $F$ *is a positive Lévy copula and* $(X_t)$*,* $(Y_t)$ *are two one-dimensional Lévy processes with tail integrals* $U_1$*,*$U_2$ *then there exists a two-dimensional Lévy process such that its tail integral is given by (A.1).*

Finally, we recall the major representatives of Lévy copulas applied in practice.

**Proposition A.9** (Archimedean Lévy copula)**.** *Let* $\phi : [0, \infty] \to [0, \infty]$ *be a strictly decreasing convex function such that* $\phi(0) = \infty$ *and* $\phi(\infty) = 0$*. Then*

$$F(x, y) = \phi^{-1}(\phi(x) + \phi(y))$$

*defines a two-dimensional Lévy copula.*

In analogy to ordinary copulas, the Lévy copulas constructed this way, are called Archimedean Lévy copulas and $\phi$ is called the generator.

**Examples** A few examples of Archimedean Lévy copulas, each with respect to a parameter $\theta > 0$, following the naming in [11] wherever given, are:

(1) The *Clayton-Lévy copula* generated by $\phi_{\mathrm{C}}(u) = u^{-\theta}$:

$$F_\theta(x, y) = \left(x^{-\theta} + y^{-\theta}\right)^{-1/\theta},$$

In this case a greater parameter $\theta$ means higher dependence of jumps. This includes $F_\perp$ for $\theta \to 0$ and $F_{\uparrow\uparrow}$ for $\theta \to \infty$.

(2) The *Gumbel-Lévy copula* generated by $\phi_{\mathrm{G}}(u) = [\log(u+1)]^{-\theta}$

$$F_\theta(x, y) = \exp\left\{\left[(\log(x+1))^{-\theta} + (\log(y+1))^{-\theta}\right]^{-1/\theta}\right\} - 1.$$

(3) The *complementary Gumbel-Lévy copula* generated by the inverted generator: $\phi_{\bar{\mathrm{G}}}(u) = \exp(u^{-\theta}) - 1$

$$F_\theta(x, y) = \left\{\log\left[\exp(x^{-\theta}) + \exp(y^{-\theta}) - 1\right]\right\}^{-1/\theta}.$$

(4) Generated by $\phi_{\exp}(u) = \frac{1}{e^{\theta u}-1}$

$$F_\theta(x,y) = \frac{1}{\theta}\log\left[\left(\frac{1}{e^{\theta x}-1}+\frac{1}{e^{\theta y}-1}\right)^{-1}+1\right].$$

**Remark A.10.** *Even though unnamed in literature, the last example should be included in any relevant list. It could play an important role as an alternative to the Clayton-Lévy, since, in the above list, these are the only two cases where the inverse of the partial derivative can be given in closed form. The relevance of this feature comes with the simulation algorithm of Lévy copulas.*

## References


[1] Benth, F.E. ; Kallsen, J. ; Meyer-Brandis, T. (2007): A Non-Gaussian Ornstein-Uhlenbeck Process of Electricity Spot Price Modeling and Derivatives Pricing. *Applied Mathematical Finance 14, No. 2, pp. 153–169.*

[2] Benth, F.E. ; Kettler, P.C. (2006): Dynamic copula models for the spark spread. Preprint #14, Pure Mathematics, Dept. of Math., Univ. of Oslo.

[3] Benth, F.E. ; Kettler, P.C. (2011): Dynamic copula models for the spark spread. *Quantitative Finance, 11(3), pp. 407–421.*

[4] Benth, F. E. and Šaltytÿe-Benth, J. (2006): Analytical approximation for the price dynamics of spark spread options. *Stud. Nonlinear Dynam. Econometrics 10(3), Article 8 (electronic).*

[5] Carmona, R., and Durrleman, V. (2003): Pricing and hedging spread options, *SIAM Reviews, 45, 627–685.*

[6] Carr, P. and Madan, D.(1999): Option valuation using the fast Fourier transform. *Journal of Computational Finance 2, No. 4, pp. 61–73.*

[7] Cont R. and Tankov P. (2004): *Financial Modelling With Jump Processes.* CHAPMAN & HALL/CRC

[8] . Embrechts, P. (1983): A property of the generalized inverse Gaussian distribution with some applications. *Journal of Applied Probability 20, No. 3, pp. 537-544, ISSN 0021-9002.*

[9] Gillespie, D.T. (1996): Exact numerical simulation of the Ornstein-Uhlenbeck process and its integral. *Physical review E 54 , No. 2, pp. 2084-2091, ISSN 1550-2376.*

[10] Kallsen J. and Tankov, P. (2006): Characterization of dependence of multidimensional Lévy processes using Lévy copulas. *Journal of Multivariate Analysis 97, No. 7, pp. 1551–1572.*

[11] Kettler, P.C. (2006): Lévy-copula-driven financial processes. Preprint #23,Pure Mathematics, Dept. of Math., Univ. of Oslo.

[12] Kluge, W. (2005): *Time-inhomogeneous Lévy processes in interest rate and credit risk models.* Ph. D. thesis, Fakultät für Mathematik und Physik, Albert-Ludwigs-Universität Freiburg im Breisgau.

[13] Klüppelberg, C.; Meyer-Brandis, T.; Schmidt, A.(2010): Electricity spot price modelling with a view towards extreme spike risk, *Quantitative Finance, 10, No. 9, pp. 963–974.*

[14] Königsberger, K. (2004): *Analysis 2.* Springer, ISBN 3540203893.

[15] Lima, B. (2005): *Spark spread options.* Master thesis, Department of Mathematics, University of Oslo.

[16] Meyer-Brandis T., and Tankov P. (2008): Multi-factor jump-diffusion models of electricity prices. *IJTAF, Vol. 11 (5), pp. 503–528.*

[17] Sato, K.I. (1999): *Lévy processes and infinitely divisible distributions*, Cambridge Univ Pr, ISBN 0521553024.

(Thilo Meyer-Brandis), Department of Mathematics, University of Munich, D-80333, Munich, Germany
*E-mail address*: meyerbr@math.uio.no

(Michael Morgan), Rottenbucher Str. 35, D-81377, Munich, Germany
*E-mail address*: michael.morgan@gmx.net