# Suffocating Fire Sales

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**Abstract**

Fire sales are among the major drivers of market instability in modern financial systems. Due to iterated distressed selling and the associated price impact, initial shocks to some institutions can be amplified dramatically through the network induced by portfolio overlaps. In this paper we develop models that allow us to investigate central characteristics that drive or hinder the propagation of distress. We investigate single systems as well as ensembles of systems that are alike, where similarity is measured in terms of the empirical distribution of all defining properties of a system. This approach ensures a great deal of robustness to statistical uncertainty and temporal fluctuations, and we give various applications. A natural characterization of systems resilient to fire sales emerges, and we provide explicit criteria that regulators can readily exploit in order to assess the stability of any system. Moreover, we propose risk management guidelines in form of minimal capital requirements, and we investigate the effect of portfolio diversification and portfolio overlap. We test our results by Monte Carlo simulations for exemplary configurations and we can quantify the trade-off between objectives for classical single firm risk management and those for systemic risk management.

## 1 Introduction

The inevitably complex structure of dependencies among institutions is one of the most defining characteristics of the modern financial environment. This complexity is visible on multiple levels, including for example the intricate distribution of corporate obligations and also the – sometimes significant – overlap in the asset holdings of the participating institutions. While this tremendous blend of dependencies and interactions provides the members with a great number of opportunities, it is also the source of an enormous threat: the interlocking and pervasive structures make the system in large parts or even as a whole fragile to possible initial local shock events. The problem of modeling, understanding, and managing this notion of *systemic risk* has been an important research topic, and it has become particularly prominent after the strike of the global financial crisis in the years 2007/08.

One of the first papers in the context of financial mathematics to address systematically the effect of – what they termed – cyclical dependencies among the market participants was the seminal work [16] by Eisenberg and Noe. They considered the channel of default contagion, that is, the successive propagation of balance-sheet insolvency across financial institutions, and showed, among other results, that under mild assumptions a unique clearing vector exists that

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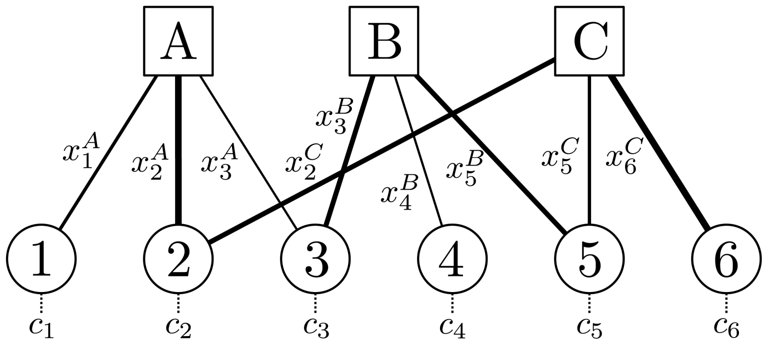



Figure 1: An illustration of a system with $n = 6$ financial institutions (circles) and $M = 3$ assets (squares). Edges represent investments of the institutions in the assets and their thickness indicates the investment volume $x_i^m$. Furthermore, capitals $c_i$ are attached to the institutions.

clears the obligations of all members. From today's viewpoint, however, it is well understood that there are various other important channels of contagion. For instance, in his book [22] Hurd lists *Asset Correlation*, *Default Contagion*, *Liquidity Contagion* and *Asset Fire Sales* as the four main channels of direct and indirect default propagation.

Here we focus on the fire sales channel, which is widely accepted as one of the main drivers of market instability. The underlying dynamics that we consider are defined abstractly as follows. Consider a system of institutions that are invested in certain assets, see also Figure 1, where each institution $i$ is equipped with some initial capital (equity) $c_i$ and holds some number $x_i^A$ of shares of an asset $A$. The portfolios of the institutions may therefore overlap, and the dependencies are quantified by the collection of the $x_i^A$'s. Then, as a reaction to some initial shock event, one or more of the institutions sell according to an individual strategy a non-negligible number of shares of their assets. These sales may cause a decline of the assets' share prices and all investors in the sold assets incur losses in their portfolio values. This may start another round of asset sales, possibly on an extended set of assets, again reducing share prices and so on. By this iterative process the initial stress is propagated through the system and can be amplified considerably. In particular, institutions who were spared from the initial shock can get into trouble if their portfolios overlap with those of distressed investors.

**Related Work** Various approaches have been developed to model and understand the impact of fire sales. The works [9, 27, 29] extend the classical Eisenberg-Noe setting from [16] for given financial networks so that various aspects of fire sales are incorporated. Caccioli et al. [8] use a branching process approximation to model fire sales, and they find that the system is stable when certain market parameters are below a critical value that they specify. Ibragimov et al. [23] study the benefits and disadvantages of overlapping portfolios not only from the perspective of single institutions, but also for the market as a whole. An important consequence of their model is that there may arise a divergence between private and social welfare depending on the various statistical features that they consider. Alike and related considerations are also made in Beale et al. [5], where a small number of assets is considered, and in Wagner [28], where using a microfounded model it is shown that the risk of joint liquidation motivates institutions to create heterogeneous portfolios. A related setting for reinsurance markets with overlapping insured objects is analyzed by Kley et al. in [25]. Cont and Wagalath analyze the impact of fire sales on asset price dynamics and correlation in continuous time in [12]. The same authors in [13], extending and building upon work of Khandani and Lo [24], describe the impact of the liquidation of large portfolios on the covariance structure of asset returns and provide a quantitative explanation for spikes in volatility and correlations observed.

Apart from the theoretical work there is a significant amount of empirical studies that consider the effect of fire sales, develop viable models and propose measures to capture quantitatively their effects. In Guo et al. [21] the topology of the network of common asset holdings is analyzed and in [6] Braverman and Minca propose a network representation to quantify the interrelations induced by common asset holdings. Cont an Schaanning develop a stress testing framework for fire sales in [10] and propose indices of centrality for institutions participating in a fire sales process in [11]. There are several works that develop exposure- and market-based measures to depict the effect of fire sales. For example, Girardi et al. [19] use the scalar product of two portfolios' weights, and Kritzman et al. [26] use the so-called absorption ratio, the degree of variation induced by the first principal components of asset returns, to measure dependence between different sources of risk in a portfolio. Additional measures are constructed from a statistical analysis of the equity returns of institutions, as in Acharya et al. [1], Brownlees and Engle [7] and Adrian and Brunnermeier [2].

**Our Results & Perspective** In this paper we study various aspects of fire sales, in particular regarding its modeling and the associated risk management.

*The Deterministic Model.* As already mentioned, the underlying parameters of our model associate to each institution $i$ a capital $c_i$ and to each pair $(i, A)$ of an institution $i$ and an asset $A$ a number $x_i^A$ of shares of $A$ that $i$ holds. Apart from that, we equip each institution $i$ with an abstract strategy $\rho$ that dictates how many of the held shares are sold in case of losses, and we assume that there is a function $h$ that determines the impact of the ongoing sales to the prices of the shares. In contrast to previous works we only impose minimal assumptions at this point: the "strategy function" $\rho$ is right-continuous and non-decreasing – the more the price drops the more shares are pushed off – and the "price impact function" $h$ is continuous and non-decreasing – the more shares are sold the more the price drops. Thus, the basic setting that we consider is quite flexible and it enables us to model various different scenarios and strategies for the institutions. Given all these parameters, our first main result, given in Section 2, characterizes in terms of a fixed point equation the eventual loss caused by the initial stress (initial losses $\ell_i$ for institution $i$) for each institution in the system. Let us remark that we only consider dynamics provoked by price impact of sold shares, assuming that exogenous price changes are negligible over the short period of time in which the fire sales take place (unlike for example as in [12]).

*The Stochastic Model.* The results described so far enable us for any given finite set of institutions and assets equipped with all relevant parameters to determine for each institution the final loss incurred by fire sales. Thus, they are readily applicable to given systems. However, we are interested in studying a more fundamental question: what are the important underlying structures that propel the process of fire sales? In other words, which system characteristics favor the emergence of large fire sales cascades, and which ones prohibit them? Going even one step further, we want to understand how large cascades can be prevented, for example by determining minimal capital requirements for all institutions, so that the system as a whole is resilient to small initial shocks.

The main contribution of this paper is a qualitative and quantitative answer to these questions, where we proceed as follows. Instead of restricting our attention only to a single system, we consider an *ensemble* of systems that share common characteristics. This "structural resemblance" is measured in terms of the joint empirical distribution of all parameters that we consider: the capital of each institution, the number of shares of each asset that it holds and the initial losses that it suffers. Equivalently, it is the distribution of a vector $(\boldsymbol{X}, C, L)$, where $C$ is the capital, $L$ is the initial loss, and $\boldsymbol{X}$ the vector of the number of shares (there is one component for each asset) of an institution *drawn uniformly at random* among all institutions in the system. The route that we take is thus to depart from any particular system and to describe it solely by the distribution of the vector $(\boldsymbol{X}, C, L)$; hence we view any given system as a typical realization of a random experiment where the asset holdings, the capital and the initial loss of each institution are independent samples from $(\boldsymbol{X}, C, L)$. Similar to the deterministic model, we describe in Section 2 for this stochastic model the final state of the system after the fire sales cascade is completed.

This approach constitutes a key difference between our work and the previous literature on fire sales. By reducing any particular system to its bare bone characteristics, we obtain robustness and flexibility. For example, moderate uncertainties of the precise system configuration are absorbed. Moreover, our model is applicable to various configurations under the premise that these are typical samples of some given distribution – this puts us in a position to study future configurations or to develop system architectures that are resilient to fire sales. Finally, the model can readily be calibrated based on the empirical distribution of the asset holdings of the institutions in any given system.

The setting considered here is related to a prominent approach in a different community, namely the study of *random graphs/networks*. There, sequences of networks with alike statistical properties are studied, and the long term behavior is investigated. This approach was also used recently to study default contagion in the financial mathematics community, see the papers [3, 14, 15]. Some of our results are similar in nature, but the process that we consider has entirely different dynamics, and our parameter space is high-dimensional.

*Risk Management.* More than a mere description of the fire sales process, it is a central objective to identify instable system configurations and to devise measures to prevent disastrous fire sales cascades. A common approach taken in the literature and in the works already cited is to specify certain parameters, for example shock sizes and quantiles of the final damage, that are admissible for a system to pass a stress test.

The stochastic model studied here enables us to develop an attractive supplement and alternative to this approaches, as it puts us in the position to define in a natural and *parameter-free* way a notion of resilience (stability) of a given system. More specifically, in Section 3 we call a system non-resilient if an arbitrary small fraction of initially defaulted institutions leads to a cascade with the consequence that eventually a positive fraction of all institutions default. In other words, the system is non-resilient if *no matter* how small the initial shock is, there is a *positive lower bound* for the total fraction of defaulted institutions. Note that this definition makes no sense for any given finite system – the initial shock may be so small that nobody defaults – but here the effect of the stochastic model is greatly beneficial: we do not consider a system with a given number of institutions, but rather all systems where the empirical distribution of the asset holdings, capital and initial loss is close to the distribution $(\mathbf{X}, C, L)$. In this setting, (non-)resilience is a property solely of $(\mathbf{X}, C, L)$ that may become apparent only for large systems; however, we demonstrate in Section 4 that the behavior converges rather quickly and the results are a good approximation already for systems of moderate size.

Finally, as an application we are able to derive in Section 3 explicit capital requirements that ensure stability of the system, which are of fundamental interest to regulators. It turns out that for each institution this capital buffer is solely based on its own asset holdings, thus ensuring full transparency and fairness, which is an essential question to address when determining the systemic riskiness of the involved institutions. We want to stress again at this point that the particular feature of our model is that the requirements are independent of any chosen additional parameters. Moreover, we demonstrate that they can be combined with the classical single firm risk management policies defined by the *Basel Committee on Banking Supervision* in terms of value-at-risk [4] and we can expose their positive effects. In addition, in Section 4 our tools allow us to quantify the positive and negative effects of diversification, thus contributing to the latest discussions in the literature, see [5, 17, 23, 28] for example.

**Outline** In Section 2, we present an analyze our model for fire sales. Then, in Section 3 we derive criteria for financial systems to be (non-)resilient with respect to initial shocks and apply them to obtain sufficient capital requirements. In Section 4, we discuss applications of the developed theory and provide simulations. All proofs can be found in Section 5.

## 2 A Model for Fire Sales

In this section we define our models of fire sales. We first describe the parameters and assumptions and then determine the final state of the system after the fire sales cascade has completed both in the deterministic as well as in the stochastic setting.

**Model parameters** We consider a financial system consisting of $n \in \mathbb{N}$ institutions which can invest in $M \in \mathbb{N}$ different (not perfectly liquid) assets or asset classes. That is, to each institution $i \in [n] := \{1, \ldots, n\}$ we assign a number $x_i^m \in \mathbb{R}_{+,0}$ of held shares of asset $m \in [M]$ (or any other index set of size $M$). See Figure 1 for an illustration. Further, we denote by $c_i \in \mathbb{R}_+$ the initial capital of institution $i$ (for example the equity for leveraged institutions or the portfolio value for institutions that exclusively invest in assets) and we assume that it incurs exogenous losses $\ell_i \in \mathbb{R}_{+,0}$ due to some shock event. In the case of a market crash for instance, it could be that $\ell_i = \sum_{1 \leq m \leq M} x_i^m \delta^m p^m$, where $p^m$ denotes the initial price of one share of asset $m \in [M]$ and $\delta^m \in (0, 1]$ is the relative price shock on the asset. Furthermore, let the empirical distribution function $F_n : \mathbb{R}_{+,0}^{M+2} \to [0, 1]$ of the institutions' parameters be denoted by

$$F_n(\boldsymbol{x}, c, \ell) = n^{-1} \sum_{i \in [n]} \mathbf{1}\{x_i^1 \leq x^1, \ldots, x_i^M \leq x^M, c_i \leq c, \ell_i \leq \ell\} \tag{2.1}$$

and let in the following $(\boldsymbol{X}_n, C_n, L_n)$ be a random vector with distribution $F_n$.

**Asset sales** We assume that due to the exogenous losses some of the institutions are forced to liquidate parts of their asset holdings in order to comply with regulatory or market-imposed constraints (e. g. leverage constraints), self-imposed risk preferences and policies to adjust the portfolio size, or to react to investor redemption. These sales are described by a non-decreasing function $\rho : \mathbb{R}_{+,0} \to [0, 1]$ such that each institution $i \in [n]$ incurring a loss of $\Lambda$ sells $x_i^m \rho(\Lambda/c_i)$ of its shares of asset $m$. The fraction $\Lambda/c_i$ describes the relative loss of institution $i$ measured against its initial equity. It is hence sensible to assume that

$$\rho(0) = 0, \quad \rho(u) \leq 1 \text{ and } \rho(u) = \rho(1) \text{ for all } u \geq 1.$$

If at default of an institution the whole portfolio is to be liquidated, then $\rho(1) = 1$. In general, however, the remaining assets at default may be frozen by the insolvency administrator and only be sold to the market on a longer time scale. In this case, $\rho(1) \in [0, 1)$. Our assumptions on $\rho$ are rather mild and allow for a flexible description of various scenarios. Some concrete examples for sales functions $\rho$ are as follows:

- The perhaps simplest non-trivial example is $\rho(u) = \mathbf{1}\{u \geq 1\}$. It describes complete liquidation of the portfolio at default (if the institution is leveraged) resp. dissolution.
- A more involved example can be derived from a leverage constraint that prohibits an institution from investing more money into risky assets than a certain multiple $\lambda_{\max} \geq 1$ of its capital/equity. In the one-asset case this means that $xp/c =: \lambda \leq \lambda_{\max}$, where $x$ denotes the number of shares held, $p$ is the price per share and $c$ denotes the institution's capital. Assume now that while the asset price $p$ stays constant, the institution suffers an exogenous shock $\ell$ and $c$ is reduced to $\tilde{c} = c - \ell$.

- If $\ell \leq (1-\lambda/\lambda_{\max})c$, then the leverage constraint $xp/\tilde{c} \leq \lambda_{\max}$ is satisfied and no reaction is required by the institution.
- However, if $\ell > (1-\lambda/\lambda_{\max})c$, then the institution must get rid of some shares; suppose that it sells $\delta x$ of them, for some $0 < \delta \leq 1$. In order for the leverage constraint $(1-\delta)xp/\tilde{c} \leq \lambda_{\max}$ to hold, it is easy to verify that $\delta \geq 1 - (1-\frac{\ell}{c})\frac{\lambda_{\max}}{\lambda}$.

The relative asset sales are hence given by $\rho(\ell/c)$, where $\rho(u) := (1-(1-u)\lambda_{\max}/\lambda)^+$ for $u \in [0,1]$; this amounts to linear sales (with respect to the losses) once the threshold $1-\lambda\lambda_{\max}^{-1}$ is reached.

- Taking an alternative route in the previous example, suppose that the loss of the institution stems only from a price change $p \to \tilde{p} < p$, which reduces the capital to $\tilde{c} = c - x(p-\tilde{p})$. If $\tilde{p} \geq p(1-\frac{1}{\lambda})/(1-\frac{1}{\lambda_{\max}})$, then no action is required to comply with the leverage constraint. In the remaining cases the institution must sell a fraction of $1-\lambda_{\max}+\lambda_{\max}\frac{p}{\tilde{p}}(1-\frac{1}{\lambda})$ of its assets, and we obtain $\rho(u) = (1-\lambda_{\max}(1-u)/(\lambda-u))^+$ for $u \in [0,1]$.
- Finally, it can be shown that for price changes combined with exogenous losses the sale function is bounded from above and below by the two previous cases. Leverage constraints hence imply a sale function which is 0 below a certain threshold and then grows linearly.

The actual reasoning behind asset sales is in general more complex than the presented examples, and this is why we consider a general sale function $\rho$ in this article. A natural assumption is that $\rho$ is right-continuous. By replacing $\rho(u)$ with its right-continuous modification $\overline{\rho}(u) := \lim_{\epsilon\to 0+}\rho((1+\epsilon)u)$ throughout this article, our results become applicable also for arbitrary (not right-continuous) sale functions $\rho$. Finally, denote by $\mathring{\rho}(u) := \lim_{\epsilon\to 0+}\rho((1-\epsilon)u)$ the left-continuous modification of $\rho$.

Let us remark that more generally we may choose different sale functions $\rho^m$ for all assets $m \in [M]$. It is then possible to replace the scalar function $\rho(u)$ by the diagonal matrix $\mathrm{diag}(\rho^1(u),\ldots,\rho^M(u))$ in all the following considerations. Further, we may partition the set of institutions into different types (banks, insurance companies, hedge funds, ...) and choose different $\rho$ or $\rho^m$ for each type. Finally, our proofs in this article also work for other arguments than $\Lambda/c_i$ for $\rho$ (where $\Lambda$ are the losses), but for simplicity we stick to this particular form.

**Price impact** Since the assets are not perfectly liquid (the limit order book has finite depth), the sales of shares triggered by the exogenous shock cause prices to go down. This on the other hand causes losses for all the institutions invested in the assets due to mark-to-market accounting. We model the price loss of asset $m \in [M]$ by a continuous function $h^m : \mathbb{R}_{+,0}^M \to [0,1]$ which is non-decreasing in each coordinate. That is, if $\boldsymbol{y} = (y^1,\ldots,y^M) \in \mathbb{R}_{+,0}^M$ and $ny^m$ shares of asset $m$ have been sold in total, then we assume that the price of the asset $m$ drops by $h^m(\boldsymbol{y})$; hence each institution $i \in [n]$ suffers losses of $\boldsymbol{x}_i \cdot h(\boldsymbol{y})$, where $\boldsymbol{x}_i := (x_i^1,\ldots,x_i^M)$ and $h(\boldsymbol{y}) = (h^1(\boldsymbol{y}),\ldots,h^M(\boldsymbol{y}))$.

Two remarks are appropriate. First, note the relative parametrization with the number of institutions $n$, where we assumed that $ny^m$ (instead of $y^m$) shares of asset $m$ are sold. For fixed $n$ this is arbitrary; however, when we later consider the stochastic model (see Assumption 2.3), this parametrization will turn out to be rather convenient to state our results. Further note that $\boldsymbol{x}_i \cdot h(\boldsymbol{y})$ usually only describes an upper bound on institution $i$'s losses at the time that $\boldsymbol{y}$ shares were sold, since in general $i$ might already have sold parts of its shares at an earlier time and thus higher prices. Our model is in this sense conservative and also incorporates implementation losses (price changes for the particular trade itself) by selling institutions. Further, this will allow for explicit analytic results in the following. It is an interesting question for future research to extend the model so that it also accounts for intermediate sales.

**Fire sales** The fire sales process that we consider is described by the combination of the previous two ingredients. Triggered by some exogenous event the institutions start selling a portion of their assets hence driving down prices. Due to mark-to-market effects, however, this means that institutions experience further losses and are forced into further sales. This iterative process continues until the system stabilizes and no further sales, losses and price changes occur.

### 2.1 Fire Sales – The Deterministic Model

In this section we provide a complete description of the final state of the system after the fire sales process is completed. We are interested in the vector $\boldsymbol{\chi}_n$ of the number of finally sold shares divided by $n$ after the fire sales process and hence the final price impact $h^m(\boldsymbol{\chi}_n)$ on any asset $m \in [M]$. Further, for leveraged institutions such as banks or hedge funds, it makes sense to consider also the the size of the set of finally defaulted institutions $\mathcal{D}_n$. Given $\boldsymbol{\chi}_n$, we readily obtain that $\mathcal{D}_n := \{i \in [n] : \ell_i + \boldsymbol{x}_i \cdot h(\boldsymbol{\chi}_n) \geq c_i\}$ and hence

$$n^{-1}|\mathcal{D}_n| = n^{-1} \sum_{i\in[n]} \mathbf{1}\{\ell_i + \boldsymbol{x}_i \cdot h(\boldsymbol{\chi}_n) \geq c_i\} = \mathbb{P}\left(L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi}_n) \geq C_n\right). \tag{2.2}$$

In order to derive $\boldsymbol{\chi}_n$ we first consider the special case that the sale function $\rho$ is continuous. We consider the fire sales process in rounds, where in each round institutions react to the price changes from the previous round. Denote by $\boldsymbol{\sigma}_{(k)} = (\sigma^1_{(k)}, \ldots, \sigma^M_{(k)})$ the vector of cumulatively sold shares in round $k$. For $k = 1$ we readily obtain

$$\boldsymbol{\sigma}_{(1)} = \sum_{i\in[n]} \boldsymbol{x}_i \rho\left(\frac{\ell_i}{c_i}\right) = n\mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n}{C_n}\right)\right].$$

Similarly, in round $k \geq 2$

$$\boldsymbol{\sigma}_{(k)} = \sum_{i\in[n]} \boldsymbol{x}_i \rho\left(\frac{\ell_i + \boldsymbol{x}_i \cdot h(n^{-1}\boldsymbol{\sigma}_{(k-1)})}{c_i}\right) = n\mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(n^{-1}\boldsymbol{\sigma}_{(k-1)})}{C_n}\right)\right]. \tag{2.3}$$

Thus, by induction $(\boldsymbol{\sigma}_{(k)})_{k\in\mathbb{N}}$ is non-decreasing componentwise and bounded by $n\mathbb{E}[\boldsymbol{X}_n]$. The limit $n\boldsymbol{\chi}_n := \lim_{k\to\infty} \boldsymbol{\sigma}_{(k)}$ – the vector of finally sold shares – must hence exist.

**Lemma 2.1.** *Consider the fire sales process with a continuous $\rho$. Then $\boldsymbol{\chi}_n = n^{-1} \lim_{k\to\infty} \boldsymbol{\sigma}_{(k)}$, the number of sold shares divided by $n$ at the end of the fire sales process, is the smallest (componentwise) solution of*

$$\mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi})}{C_n}\right)\right] - \boldsymbol{\chi} = \mathbf{0}. \tag{2.4}$$

*Proof.* By continuity of $\rho$ and the dominated convergence theorem

$$\begin{aligned}\boldsymbol{\chi}_n = n^{-1} \lim_{k\to\infty} \boldsymbol{\sigma}_{(k)} &= \lim_{k\to\infty} \mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(n^{-1}\boldsymbol{\sigma}_{(k-1)})}{C_n}\right)\right] \\ &= \mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(n^{-1}\lim_{k\to\infty}\boldsymbol{\sigma}_{(k-1)})}{C_n}\right)\right] = \mathbb{E}\left[\boldsymbol{X}_n \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(n^{-1}\boldsymbol{\chi}_n)}{C_n}\right)\right]\end{aligned}$$

and $\boldsymbol{\chi}_n$ is thus a solution of (2.4).

By the Knaster-Tarski theorem there must exist a least fixed point $\hat{\boldsymbol{\chi}}_n$. Clearly, $\boldsymbol{\sigma}_{(0)} := \mathbf{0} \leq$

$n\hat{\boldsymbol{\chi}}_n$. Hence assume inductively that $\boldsymbol{\sigma}_{(k)} \leq n\hat{\boldsymbol{\chi}}_n$ for $k \geq 1$. Then

$$\boldsymbol{\sigma}_{(k+1)} = \sum_{i\in[n]} \boldsymbol{x}_i \rho\left(\frac{\ell_i + \boldsymbol{x}_i \cdot h(n^{-1}\boldsymbol{\sigma}_{(k)})}{c_i}\right) \leq \sum_{i\in[n]} \boldsymbol{x}_i \rho\left(\frac{\ell_i + \boldsymbol{x}_i \cdot h(\hat{\boldsymbol{\chi}}_n)}{c_i}\right) = n\hat{\boldsymbol{\chi}}_n \tag{2.5}$$

by monotonicity of $\rho$, and hence $\boldsymbol{\chi}_n = n^{-1}\lim_{k\to\infty}\boldsymbol{\sigma}_{(k)} \leq \hat{\boldsymbol{\chi}}_n$. By definition of $\hat{\boldsymbol{\chi}}_n$ it thus holds that $\boldsymbol{\chi}_n = \hat{\boldsymbol{\chi}}_n$. □

It remains to study the case where the sale function $\rho$ is (only) right-continuous. The following simple example of a non-continuous $\rho$ shows that also in this case it may be possible to determine the final state of the system by the smallest solution of (2.4). Consider $\rho(u) = \mathbf{1}\{u \geq 1\}$, that is, institutions sell their portfolio as they go bankrupt. Then $\boldsymbol{\sigma}_{(k)} \neq \boldsymbol{\sigma}_{(k-1)}$ only if in round $k$ at least one institution defaults that was solvent in round $k-1$. Since there are only $n$ institutions, the fire sales process stops after at most $n-1$ rounds and the vector $\boldsymbol{\chi}_n$ of finally sold shares divided by $n$ solves (2.4). Again by (2.5) we then obtain $\boldsymbol{\chi}_n = \hat{\boldsymbol{\chi}}_n$ is the smallest solution of (2.4).

Finally, consider an arbitrary right-continuous sale function $\rho$. Again by the Knaster-Tarski theorem (2.4) has a smallest solution $\overline{\boldsymbol{\chi}}_n$ and by (2.5) it holds $n^{-1}\lim_{k\to\infty}\boldsymbol{\sigma}_{(k)} \leq \overline{\boldsymbol{\chi}}_n$. For left-continuous $\rho$ in fact we would derive equality but for right-continuous $\rho$ it is in general possible that $n^{-1}\lim_{k\to\infty}\boldsymbol{\sigma}_{(k)} \lneq \overline{\boldsymbol{\chi}}_n$. This is the case if $\lim_{k\to\infty}\boldsymbol{\sigma}_{(k)}$ sold shares would be enough to start a new round of fire sales but this quantity is actually never reached in finitely many rounds. Then the following holds; the proof is straight-forward by bounding $\rho$ from below with its left-continuous modification $\mathring{\rho}$.

**Proposition 2.2.** *Consider the fire sales process with a right-continuous sale function $\rho$ and the corresponding left-continuous modification $\mathring{\rho}$. Let $\overline{\boldsymbol{\chi}}_n \in \mathbb{R}^M_{+,0}$ denote the smallest solution of* (2.4)*. Moreover, let $\hat{\boldsymbol{\chi}}_n \in \mathbb{R}^M_{+,0}$ be the smallest solution of*

$$\mathbb{E}\left[\boldsymbol{X}_n \mathring{\rho}\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi})}{C_n}\right)\right] - \boldsymbol{\chi} = 0.$$

*Then the number of sold shares divided by $n$ at the end of the fire sales process satisfies*

$$\hat{\boldsymbol{\chi}}_n \leq n^{-1}\lim_{k\to\infty}\boldsymbol{\sigma}_{(k)} \leq \overline{\boldsymbol{\chi}}_n. \tag{2.6}$$

The equilibrium vector $n\overline{\boldsymbol{\chi}}_n$ (in the sense of (2.4)) is thus a conservative bound on the final number of sold shares $n\boldsymbol{\chi}_n = \lim_{k\to\infty}\boldsymbol{\sigma}_{(k)}$. However, as discussed above the convergence of the fire sales process to a non-equilibrium heavily relies on the assumption of *arbitrarily* small sale sizes towards the end of the process. For real systems this is obviously not realistic since the least possible number of shares sold by an institution is lower bounded by 1. For all practical purposes it will therefore hold that $\boldsymbol{\chi}_n = \overline{\boldsymbol{\chi}}_n$ and fire sales stop at an equilibrium state.

## 2.2 Fire Sales – The Stochastic Model

The previous section describes fire sales in any specific (finite) system. Our aim is, however, to understand qualitatively how and which characteristics of a system promote or hinder the spread of fire sales. As portrayed in detail in the introduction, in the following we thus consider an ensemble of systems that are similar in the sense that they all share some (observed) statistical characteristics. This similarity is measured in terms of the most natural parameters, namely the joint empirical distribution function (2.1) of the asset holdings, the capital/equity, and the initial losses. In particular, we assume that we have a collection of systems with a varying number $n$ of institutions with the property that the sequence $(F_n)_{n\in N}$ stabilizes, i. e. has a limit. Additionally, we assume convergence of the average asset holdings to a finite value; this is a standard assumption avoiding condensation of the distribution of the asset holdings. Our assumptions are collected in the following definition.

**Assumption 2.3.** *Let $M \in \mathbb{N}$. For each $n \in \mathbb{N}$ consider a system with $n$ institutions and $M$ assets specified by the sequences $\boldsymbol{x}(n) = (\boldsymbol{x}_i(n))_{1\le i\le n}$ of asset holdings, $\boldsymbol{c}(n) = (c_i(n))_{1\le i\le n}$ of capitals and $\boldsymbol{\ell}(n) = (\ell_i(n))_{1\le i\le n}$ of exogenous losses. Let $F_n$ be the empirical distribution function of these parameters for $n \in \mathbb{N}$ (as in (2.1)) and let*

$$(\boldsymbol{X}_n, C_n, L_n) = \big((X_n^1, \dots, X_n^M), C_n, L_n)\big) \sim F_n.$$

*Then assume the following.*

*(a)* **Convergence in distribution:** *There is a distribution function $F$ such that as $n \to \infty$, $F_n(\boldsymbol{x}, y, z) \to F(\boldsymbol{x}, y, z)$ at all continuity points of $F$.*

*(b)* **Convergence of means:** *Let $(\boldsymbol{X}, C, L) = ((X^1, \dots, X^M), C, L) \sim F$. Then as $n \to \infty$,*

$$\mathbb{E}[X_n^m] \to \mathbb{E}[X^m] < \infty, \quad m \in [M].$$

An ensemble of systems satisfying Assumption 2.3 will be called in the sequel an $(\boldsymbol{X}, C)$-*system with initial shock $L$*. A particular and probably the most relevant scenario is as follows. Suppose that the distribution $F$ is specified, for example by considering a real system. Then, for each $n \in \mathbb{N}$ we construct a system by assigning to each institution $i \in [n]$ independently asset holdings, capital and losses distributed like $F$. Then, by the strong law of large numbers, with probability 1, the sequence of systems we obtain satisfies Assumption 2.3.

As in the *deterministic model* our aim is to describe in this broader setting the final state of the system. Before we do so, let us give some definitions that are handy in the forthcoming description. First, recall that for $n \in \mathbb{N}$ the eventual number of sold shares is characterized by the smallest solution to (2.4), and the number of defaulted institutions is given by (2.2). Let therefore $f^m, g : \mathbb{R}_{+,0}^M \to \mathbb{R}$, $m \in [M]$ be analogously defined for the "limiting object" by

$$\begin{aligned} f^m(\boldsymbol{\chi}) &:= \mathbb{E}\left[X^m \rho\left(\frac{L + \boldsymbol{X}\cdot h(\boldsymbol{\chi})}{C}\right)\right] - \chi^m, \quad m \in [M], \\ g(\boldsymbol{\chi}) &:= \mathbb{P}(L + \boldsymbol{X}\cdot h(\boldsymbol{\chi}) \ge C), \end{aligned} \tag{2.7}$$

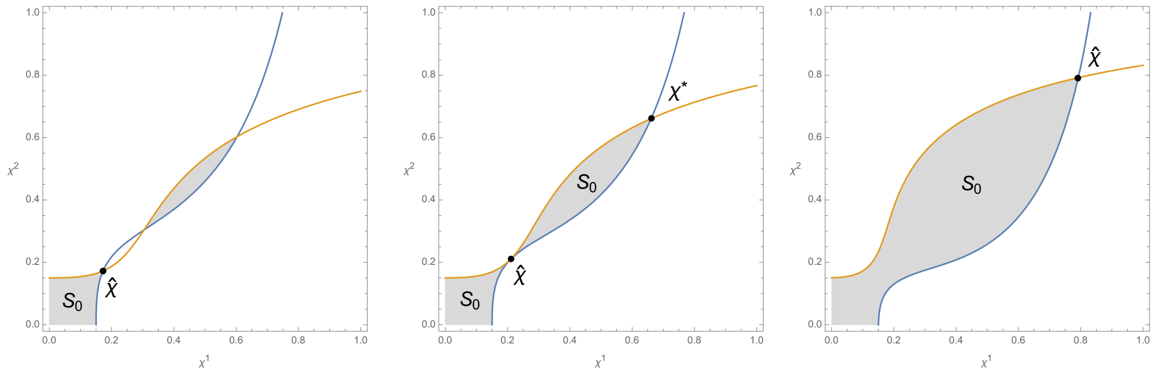
which are clearly upper semi-continuous, and let (cf. Proposition 2.2)

$$\mathring{f}^m(\boldsymbol{\chi}) := \mathbb{E}\left[X^m \mathring{\rho}\left(\frac{L + \boldsymbol{X}\cdot h(\boldsymbol{\chi})}{C}\right)\right] - \chi^m, m \in [M], \text{ and } \mathring{g}(\boldsymbol{\chi}) := \mathbb{P}(L + \boldsymbol{X}\cdot h(\boldsymbol{\chi}) > C) \tag{2.8}$$

be their lower semi-continuous modifications. Further, define the sets

$$\mathring{S} := \bigcap_{m\in[M]} \left\{\boldsymbol{\chi} \in \mathbb{R}_{+,0}^M : \mathring{f}^m(\boldsymbol{\chi}) \ge 0\right\} \quad \text{and} \quad S := \bigcap_{m\in[M]} \{\boldsymbol{\chi} \in \mathbb{R}_{+,0}^M : f^m(\boldsymbol{\chi}) \ge 0\}$$

and denote by $\mathring{S}_0$ resp. $S_0$ the largest connected subsets of $\mathring{S}$ and $S$ containing $\mathbf{0}$ (clearly $f^m(\mathbf{0}) \ge \mathring{f}^m(\mathbf{0}) \ge 0$ for all $m \in [M]$). Note that $S$ and $S_0$ are closed sets by upper semi-continuity of $f^m$, $m \in [M]$.




(a) (b) (c)

Figure 2: Plot of the root sets of the functions $f^1(\chi^1, \chi^2)$ (blue) and $f^2(\chi^1, \chi^2)$ (orange) for three different example systems. In gray the set $S$ is depicted.

Let us immediately give an illustrative explanation of all the quantities above. Leave aside for the moment the $\mathring{\cdot}$ modifications and assume that during the fire sales process assets are sold continuously in time $\tau$, that is, let $n\boldsymbol{\chi}(\tau)$ at any given time $\tau$ be the current vector of sold shares. Then by definition of the process, $n\mathbb{E}[X^m \rho((L + \boldsymbol{X} \cdot h(\boldsymbol{\chi}(\tau)))/C)]$ is the number of shares of asset $m$ the system as a whole needs to sell. Moreover, as $\boldsymbol{\chi}^m(\tau)$ is the current number of sold shares, at any point in time $\tau$ it must hold $f^m(\boldsymbol{\chi}(\tau)) \geq 0$. Since the process starts with zero sold shares ($\boldsymbol{\chi}(0) = \boldsymbol{0}$) it is therefore intuitive that $\boldsymbol{\chi}(\tau)$ will never leave the set $S_0$. At the end of the fire sales process, however, the final number of sold shares must equal the required number of sold shares for each asset $m \in [M]$ (cf. (2.4) for the *deterministic model*). So the final state will be described by a joint root of the functions $f^m$, $m \in [M]$, that lies in $S_0$. The $\mathring{\cdot}$ modifications come into play because in certain pathological cases it can be important if the limiting system (as $n \to \infty$) is approached from below or from above.

To formalize this intuition about joint roots in $S_0$, consider the following lemma. Let

$$\boldsymbol{\chi}^* \in \mathbb{R}_{+,0}^M \quad \text{with} \quad (\chi^*)^m := \sup_{\boldsymbol{\chi} \in S_0} \chi^m.$$

**Lemma 2.4.** *There exists a smallest joint root $\hat{\boldsymbol{\chi}}$ of all functions $\mathring{f}^m(\boldsymbol{\chi})$, $m \in [M]$ with $\hat{\boldsymbol{\chi}} \in \mathring{S}_0$. Further, $\boldsymbol{\chi}^*$ as defined above is a joint root of the functions $f^m$, $m \in [M]$, and $\boldsymbol{\chi}^* \in S_0$.*

At the end of the section we will give a couple of examples illustrating the situation in concrete settings. Using the quantities $\hat{\boldsymbol{\chi}}$ and $\boldsymbol{\chi}^*$ as well as the functions $\mathring{g}$ and $g$, we can then describe the final state of the system after the fire sales process asymptotically as $n \to \infty$.

**Theorem 2.5.** *Consider a system satisfying Assumption 2.3. Then for the final default fraction $n^{-1}|\mathcal{D}_n|$ and $\chi_n^m$, the number of finally sold shares of asset $m \in [M]$ divided by $n$*

$$\mathring{g}(\hat{\boldsymbol{\chi}}) + o(1) \leq n^{-1}|\mathcal{D}_n| \leq g(\boldsymbol{\chi}^*) + o(1), \qquad \hat{\chi}^m + o(1) \leq \chi_n^m \leq (\chi^*)^m + o(1).$$

*In particular, for the final price impact $h^m(\boldsymbol{\chi}_n)$ on asset $m \in [M]$*

$$h^m(\hat{\boldsymbol{\chi}}) + o(1) \leq h^m(\boldsymbol{\chi}_n) \leq h^m(\boldsymbol{\chi}^*) + o(1).$$

Only in rather pathological cases, when $\hat{\boldsymbol{\chi}}$ is a point of discontinuity for some $f^m$ (cf. Figure 3) or it is instable (compare Figure 2(b) to Figures 2(a) and 2(c)), it happens that $\hat{\boldsymbol{\chi}} \neq \boldsymbol{\chi}^*$.

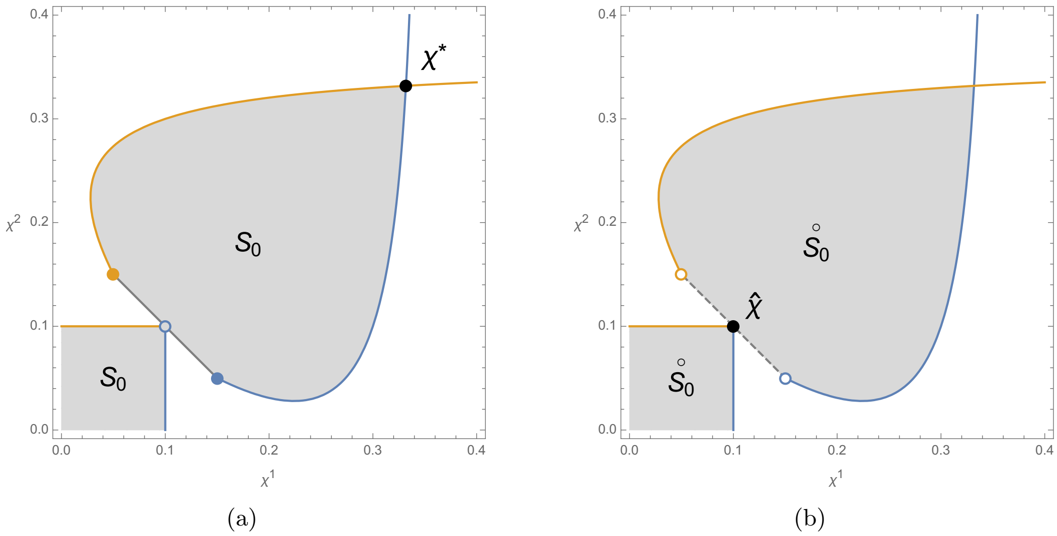



Figure 3: Plot of the root sets of the functions $f^1(\chi^1, \chi^2)$ (blue), $f^2(\chi^1, \chi^2)$ (orange) in (a) and $\mathring{f}^1(\chi^1, \chi^2)$ (blue), $\mathring{f}^2(\chi^1, \chi^2)$ (orange) in (b) respectively. In gray the sets $S_0$ and $\mathring{S}_0$ respectively, where the solid line in (a) belongs to $S_0$ whereas the dashed line in (b) does not.

Similarly it usually holds that $\mathring{g}(\hat{\boldsymbol{\chi}}) = g(\boldsymbol{\chi}^*)$. Then Theorem 2.5 determines the limits of $\boldsymbol{\chi}_n$ and $n^{-1}|\mathcal{D}_n|$ as $n \to \infty$. We conclude this section with two illustrative examples.

**Example 1** It is often (e. g. if $\rho$ is continuous or $\boldsymbol{X}$ is absolutely continuous) the case that $f^m(\boldsymbol{\chi}) = \mathring{f}^m(\boldsymbol{\chi})$ and hence $S = \mathring{S}$ as well as $S_0 = \mathring{S}_0$. See Figure 2 for an illustration of three different two-dimensional examples. We chose $h^m(\boldsymbol{\chi}) = \chi^m$, $m = 1, 2$, $\rho(y) = \mathbf{1}\{y \geq 1\}$, $X^1 = X^2 \sim \text{Exp}(1)$, $C = c$, $\mathbb{P}(L = c) = 0.15$ and $\mathbb{P}(L = 0) = 0.85$, where $c = 1.9$ in (a), $c \approx 1.82$ in (b) and $c = 1.5$ in (c). Whereas in (a) and (c) it holds $\hat{\boldsymbol{\chi}} = \boldsymbol{\chi}^*$, in (b) the points are distinct.

**Example 2** Since it is of advantage for the proofs in the rest of this article to gain intuition about the general case as well, in Figure 3 we further provide an example of a system where $\mathring{f}^m \neq f^m$ and $\mathring{S}_0 \subsetneq S_0$. Again we chose $h^m(\boldsymbol{\chi}) = \chi^m$, $m = 1, 2$, $\rho(y) = \mathbf{1}\{y \geq 1\}$ and $X^1 = X^2$. Further, $C = 0.1$, $\mathbb{P}(X^1 = 1, L = 0.1) = 0.1$, $\mathbb{P}(X^1 = 0.5, L = 0) = 0.1$ and with the remaining probability of 0.8 it holds that $L = 0$ and $X^1$ is uniformly distributed on the interval $[0, 0.5]$.

# 3 Resilient and Non-Resilient Systems

In the previous section we derived results that allow us to determine the final default fraction in $(\mathbf{X}, C)$-systems caused by fire sales and sparked by some exogenous shock $L$. In this section we go one step further and investigate whether a given system in an *initially* unshocked state is likely to be resilient to small shocks or susceptible to fire sales.

Note that all information about an initial shock comes from the random variable $L$, whereas the system itself is specified by $(\boldsymbol{X}, C)$. So we can easily consider shocks of different magnitude $L$ on the same a priori unshocked system. In the following, if we use the notation $g$, $f^m$, $\mathring{g}$, $\mathring{f}^m$, $\hat{\boldsymbol{\chi}}$ and $\boldsymbol{\chi}^*$, we mean the quantities from the previous section (see Equations (2.7) and (2.8)) for the $(\boldsymbol{X}, C)$-system, that is, with initial shock $L \equiv 0$.

## 3.1 Resilience

From a regulator's perspective a desirable property of an $(\boldsymbol{X}, C)$-system is the ability to absorb small local shocks $L$ without larger parts of the system being harmed. In our model we can even choose $L$ arbitrarily small. The following way of defining resilience is hence natural: we let the shock $L$ become small in the sense that $\mathbb{E}[L/C] \to 0$, and we call the system *resilient* if the asymptotic final default fraction $n^{-1}|\mathcal{D}_{n,L}|$ caused by $L$ also tends to 0.

**Definition 3.1** (Resilience)**.** An $(\boldsymbol{X}, C)$-system is said to be *resilient* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $L$ with $\mathbb{E}[L/C] < \delta$ it holds $\limsup_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| \leq \epsilon$.

We chose our definition of resilience (and non-resilience in Definition 3.4 below) in terms of the final default fraction and thus in line with the default contagion literature [14, 15]. Alternatively depending on the quantity of interest, it can also be sensible to define resilience via the final number of sold shares $n\boldsymbol{\chi}_{n,L}$ (and hence the final price impacts $h^m(\boldsymbol{\chi}_{n,L})$ which also affect the wider economy). Theorem 3.2 determines upper bounds for both $n^{-1}|\mathcal{D}_{n,L}|$ and $\boldsymbol{\chi}_{n,L}$ in the limit $\mathbb{E}[L/C] \to 0$.

**Theorem 3.2.** *For each $\epsilon > 0$ there exists $\delta > 0$ such that for all $L$ with $\mathbb{E}[L/C] < \delta$ it holds for the final fraction of defaulted institutions $n^{-1}|\mathcal{D}_{n,L}|$ and the number $n\chi^m_{n,L}$ of finally sold shares of each asset $m \in [M]$ in the shocked system that*

$$\limsup_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| \leq g(\boldsymbol{\chi}^*) + \epsilon \quad \textit{and} \quad \limsup_{n\to\infty} \chi^m_{n,L} \leq (\chi^*)^m + \epsilon, \quad m \in [M].$$

We immediately obtain the following handy resilience criterion.

**Corollary 3.3** (Resilience Criterion)**.** *If $g(\boldsymbol{\chi}^*) = 0$, then the $(\boldsymbol{X}, C)$-system is resilient.*

Note that $g(\mathbf{0}) = 0$ and hence the system is resilient if $\boldsymbol{\chi}^* = \mathbf{0}$ (i. e. $S_0 = \{\mathbf{0}\}$). It is possible, however, that $g(\boldsymbol{\chi}^*) = 0$ while $\boldsymbol{\chi}^* \neq \mathbf{0}$. In this case, by Theorem 3.5 below it is possible that a large fraction of shares of assets is sold, but Corollary 3.3 ensures that the fraction of finally defaulted institutions stays small.

## 3.2 Non-Resilience

To a large degree we can also characterize non-resilient systems. Note, however, that in our model description we made the conservative assumption that each institution $i \in [n]$ in the system is exposed to the final price impact $h(\boldsymbol{\chi}_n)$ with its total initial asset holdings $\boldsymbol{x}_i$. One can argue that institutions sell off their assets gradually and are hence not exposed to the total price change. The following result considers non-resilience under this conservative assumption. For other scenarios it can serve as a first indication of non-resilience.

In this subsection, we restrict ourselves to initial shocks of the form $\ell_i \in \{0, 2c_i\}$ for all $i \in [n]$, where $\mathbb{P}(L = 2C) > 0$ and $L/C$ is independent of $(\boldsymbol{X}, C)$. That is, each institution $i$ defaults initially with positive probability. Rather than $\ell_i = 2c_i$, the first natural choice to model the default of institution $i$ would be $\ell_i = c_i$. Note, however, that in the setting of Section 2, even if $\mathbb{P}(L = C) > 0$, it is possible that no initial defaults occur since $(L, C)$ is defined as the weak limit of a sequence $(L_n, C_n)$ and it is possible that $L_n < C_n$ for all $n \in \mathbb{N}$ and still $L = C$ almost surely. In order to derive meaningful results one therefore has to choose $\ell_i = 2c_i$ (or any other multiple larger than 1). Note that this does not change the outcome of the fire sales process since $\rho(u) = \rho(1)$ for all $u \geq 1$.

We now call a financial system *non-resilient* if the fraction of finally defaulted institutions is lower bounded by some positive constant.

**Definition 3.4** (Non-Resilience)**.** An $(\boldsymbol{X}, C)$-system is said to be *non-resilient* if there exists $\Delta \in \mathbb{R}_+$ such that $\liminf_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| > \Delta$ for any initial shock $L$ with the above listed properties.

We derive the following lower bound on the final default fraction and finally sold shares.

**Theorem 3.5.** *If the initial shock $L$ satisfies above properties and $h^m(\boldsymbol{\chi})$ is strictly increasing in $\chi^m$ for all $m \in [M]$, then for any $\epsilon > 0$ it holds*

$$\liminf_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| > \mathring{g}(\boldsymbol{\chi}^*) - \epsilon \quad \text{and} \quad \liminf_{n\to\infty} \chi_{n,L}^m > (\chi^*)^m - \epsilon.$$

The assumption that $h^m(\boldsymbol{\chi})$ is strictly increasing in $\chi^m$ excludes a rather pathological case. It is satisfied by all the standard price impact functions in the literature such as linear price impact $h^m_{\text{lin}}(\boldsymbol{\chi}) = p^m \alpha \chi^m$ or log-linear price impact $h^m_{\text{loglin}}(\boldsymbol{\chi}) = p^m(1 - \exp(-\alpha\chi^m))$ for initial share price $p^m$ and some parameter $\alpha > 0$.

**Corollary 3.6** (Non-Resilience Criterion)**.** *If $h^m(\boldsymbol{\chi})$ is strictly increasing in $\chi^m$ for all $m \in [M]$, and $\mathring{g}(\boldsymbol{\chi}^*) > 0$, then the $(\boldsymbol{X}, C)$-system is non-resilient.*

As remarked earlier already, for most practical purposes it will hold that $\mathring{g}(\boldsymbol{\chi}^*) = g(\boldsymbol{\chi}^*)$. For the reasonable case that $h^m(\boldsymbol{\chi})$ is strictly increasing in $\chi^m$ for all $m \in [M]$, we can thus fully describe stability of an $(\boldsymbol{X}, C)$-system by the combination of Corollaries 3.3 and 3.6. Only for rather pathological cases we cannot decide if an $(\boldsymbol{X}, C)$-system is resilient or non-resilient.

### 3.3 Systemic Capital Requirements

Theorems 3.2 and 3.5 can be used to derive sufficient and necessary capital requirements to make a given system resilient with respect to initial shocks. That is, given the asset holdings of each institution, we want to determine sharp bounds for the capital that each institution must hold so that the system is (non-)resilient in the sense of Definitions 3.1 and 3.4. Recall that non-resilience according to Definition 3.4 is always under our conservative model assumption that price impact is applied to all initially held shares of an asset. The derived requirements will generally depend on the actual sale function $\rho$ and price impact $h$. In the following, we demonstrate the procedure of deriving capital requirements along a series of examples of ever increasing complexity culminating in a quite general setting with multiple assets.

**One Asset with Sales at Default** We start with considering a system with institutions investing in one asset only. The distribution $F_X$ of asset holdings is assumed to have a power law tail in the sense that there exist constants $B_1, B_2 \in (0, \infty)$ such that for $x$ large enough

$$B_1 x^{1-\beta} \leq 1 - F_X(x) \leq B_2 x^{1-\beta}, \tag{3.1}$$

for some $\beta > 2$. Whereas the reduction to one asset is a strong simplification, there is empirical evidence for power laws in investment volumes, see e. g. [18]. Moreover, we assume $\mathbb{P}(X \geq 1) = 1$ – institutions involved in the fire sales process hold at least one share. Further, assume $\rho(u) = \mathbf{1}\{u \geq 1\}$, i. e. institutions do not sell their assets until they default. Moreover, for the price impact we also assume a power-law, that is, there are $\nu, \mu_1, \mu_2 \in \mathbb{R}_+$ such that for small $\chi$

$$\mu_1 \chi^\nu \leq h(\chi) \leq \mu_2 \chi^\nu. \tag{3.2}$$

A typical assumption in the fire sales literature, see e. g. [6, 8, 9, 10, 11], is (log-)linear price impact, i. e. $\nu = 1$. This choice can for example be justified by the fact that limit order books' shape functions are approximately constant close to the bid price, see also [20].

We now derive necessary and sufficient requirements for the institutions' capital buffers to make the financial system resilient. That is, given the asset holdings $(x_1(n), \ldots, x_n(n))_{n\in\mathbb{N}}$ with empirical distribution converging to $F_X$ we want to determine a sequence of minimal capitals $(c_1(n), \ldots, c_n(n))_{n\in\mathbb{N}}$ sufficient for ensuring resilience of the system in the sense of Definition 3.1. It turns out that a natural description emerges when we choose the capitals in dependence on the asset holdings by the following power form: $c_i = \alpha x_i^\gamma$ for $\alpha \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}_{+,0}$.

**Corollary 3.7.** *Consider a system as specified above. Then,*

1. *if $\gamma > 1 - \nu(\beta - 2)$, then the system is resilient.*
2. *if $\gamma = 1 - \nu(\beta - 2)$ and $\alpha > \mu_2 \left(B_2 \frac{\beta-1}{\beta-2}\right)^\nu$, then the system is resilient.*
3. *if $\gamma = 1 - \nu(\beta - 2)$ and $\alpha < \mu_1 \left(B_1 \frac{\beta-1}{\beta-2}\right)^\nu$, then the system is non-resilient.*
4. *if $\gamma < 1 - \nu(\beta - 2)$, then the system is non-resilient.*

Typically one can choose $B_1$ and $B_2$ resp. $\mu_1$ and $\mu_2$ arbitrarily close as $x \to \infty$ resp. $\chi \to 0$. Corollary 3.7 hence states necessary and sufficient conditions on the capital to make the financial system resilient.

**One Asset with Intermediate Sales** In the previous example we considered the conservative case of sales at default only. Intermediate sales will make the system less resilient, however, and we consider an example of this kind here as well. We assume again (3.1) and (3.2), that is, the asset holdings are power-law distributed with parameter $\beta$ and the price impact has a power-law approximation with exponent $\nu$ when $\chi \to 0$. Moreover, capitals are again specified by $c_i = \alpha x_i^\gamma$. In contrast to the assumptions in the previous paragraph we now choose $\rho(u) = 1 \wedge u^q$ for some $q \in \mathbb{R}_+$. The parameter $q$ can be understood as a measure for the institutions' confidence in the asset, since it describes the speed at which they sell it. The outbreak of fire sales is then governed by $\nu$ (price impact) and $q$ (speed of selling). In fact, the product $\nu q$ is a crucial quantity in the decision whether a system is resilient. It is easy to show that if $\nu q < 1$ the system is always non-resilient, as the institutions sell overproportionally fast compared to the price fall. Note that this quantity cannot be influenced by regulations but is intrinsic to the market in our setting.

**Corollary 3.8.** *Consider a system as described above and assume that $\nu q > 1$. If $\gamma > 1 - \nu(\beta - 2)$, then the system is resilient. If $\gamma < 1 - \nu(\beta - 2)$, then the system is non-resilient.*

Similar as in the proof of Corollary 3.7 it is possible to derive sufficient capital requirements also at the critical values $\nu q = 1$ resp. $\gamma = 1 - \nu(\beta - 2)$; the details are omitted.

**Multiple Assets** The two previous examples have already given first important insights into the calculation of sufficient capital requirements for stability in a given system. Whereas these concentrated on systems with one asset only, however, in reality institutions are invested in a large number of assets $M$. We will derive sufficient capital requirements also in this setting. In practice, linear capital requirements seem reasonable and the previous examples have shown already that these are sufficient in the one-asset case. Furthermore, linear capitals allow for tractable calculations in the following.

We keep the same assumptions as in the previous example. In particular, $\rho^m(y) = 1 \wedge y^{q^m}$ for some $q^m \in \mathbb{R}_+$, $m \in [M]$ (recall the note in the beginning of Section 2 about different sale functions for different assets) and $h^m(\boldsymbol{\chi}) \leq \mu^m (\chi^m)^\nu$ for $\chi^m \to 0$ for some $\nu, \mu^m \in \mathbb{R}_+$. Consider then linear capital $c_i = \sum_{m\in[M]} \theta^m x_i^m$ for each institution $i \in [n]$, where $\theta^m, \in \mathbb{R}_+$, $m \in [M]$.

**Corollary 3.9.** *Consider a system with multiple assets as described above. Then the system is resilient if for each $m \in [M]$ one of the following holds:*

1. $q^m > \nu^{-1}$,
2. $q^m = \nu^{-1}$ *and* $\theta^m > \mu^m \mathbb{E}[X^m]$.

The first condition reflects the interplay of price impact and the speed of asset sales as for the one-asset case. The second condition gives an explicit linear fraction of the institutions' holdings of each asset that ensures resilience. It depends on the price impact function by $\mu^m$ and the number of shares of the asset held on average by each other institution in the system $\mathbb{E}[X^m]$. While the condition $q^m > \nu^{-1}$ is sufficient in the limit $n \to \infty$ by our theory, for real networks of finite size the quantity $\mu^m \mathbb{E}[X^m]$ is of big interest as it gives a proper scaling factor also for values of $q^m$ other than $\nu^{-1}$. The following sample calculations show that $\theta^m$ is of a reasonable magnitude also under our conservative model assumptions: Assume that the price impact is log-linear with $h^m(\chi) = 1 - e^{-\alpha^m \chi}$ for some $\alpha^m \in \mathbb{R}_+$ and that the sale of all assets in the considered system reduces the asset price by 50%. This implies $\alpha^m = \log(2)/\mathbb{E}[X^m]$. Hence $\mu^m = \alpha^m$ ($\nu = 1$) and $\theta^m > \log(2) \approx 0.69$ ensures resilience.

Corollary 3.9 thus derives linear capital requirements for institutions investing in more than one asset which are already used in Basel III for instance. We can explicitly determine the coefficients for these linear capital requirements in our model.

# 4 Applications and Simulations

In this section we apply the theory developed in Sections 2 and 3 to investigate which structures or properties of systems promote the emergence and spread of fire sales. To achieve this our route is as follows. In Subsection 4.1 we consider systems parametrized by two orthogonal characteristic quantities: *portfolio diversification* and *portfolio similarity*. We first analyze their effect in the setting from Sections 2 and 3, and we verify our findings also with simulations for finite systems of reasonable size. At this we assume initial shocks on institutions' capitals directly rather than shocks on certain asset prices. In Subsection 4.2, we concentrate on three fundamentally different system configurations and we test our derived capital requirements for shocks on asset prices. As we will show, it is beneficial to combine our capital requirements with classical risk capital in form of *value-at-risk*. While the *value-at-risk* part of the capital ensures for any institution that an initial shock can be absorbed with probability $1-\epsilon$ (for some small $\epsilon > 0$), the additional *systemic surcharge* in form of our capital requirements makes sure that also in the unlikely event of initial distress the spread of fire sales is locally confined.

## 4.1 The Effect of Portfolio Diversification and Similarity

For simplicity, throughout this section we assume that the limiting total asset holdings $X^{\text{tot}} = X^1 + \ldots + X^M$ are Pareto distributed with density $f_{X^{\text{tot}}}(x) = (\beta-1)x^{-\beta}\mathbf{1}\{x \geq 1\}$ for some exponent $\beta > 2$. One can generalize our results also to more general distributions. Further, we make the assumptions that $\rho(u) = \mathbf{1}\{u \geq 1\}$ and $h^m(\boldsymbol{\chi}) = 1 - e^{-\chi^m}$ to simplify calculations, but also for other sensible choices our observations below are applicable.

In a first example we consider a system of institutions whose investment in each asset $m \in [M]$ makes up a fraction $\lambda^m \in \mathbb{R}_+$ of their total asset holdings, where $\sum_{m\in[M]} \lambda^m = 1$. We show that perfect diversification ($\lambda^1 = \ldots = \lambda^M = M^{-1}$) maximizes stability of the system.

**Example 4.1.** *For a system as described above the functions* $f^m(\boldsymbol{\chi})$ *are given by*

$$f^m(\boldsymbol{\chi}) = \lambda^m \mathbb{E}\left[X^{tot}\mathbf{1}\left\{X^{tot}\sum_{\ell=1}^{M}\lambda^\ell\left(1-e^{-\chi^\ell}\right)\geq C\right\}\right]-\chi^m, \quad m\in[M].$$

*Let us write* $t=\sum_{1\leq\ell\leq M}\lambda^\ell(1-e^{-\chi^\ell})$ *for short. Now assume similar to Corollary 3.7 that* $C=\alpha(X^{tot})^\gamma$ *for some constants* $\alpha,\gamma\in\mathbb{R}_{+,0}$. *Then*

$$\begin{aligned}f^m(\boldsymbol{\chi}) &= \lambda^m\mathbb{E}\left[X^{tot}\mathbf{1}\left\{X^{tot}\geq(\alpha/t)^{\frac{1}{1-\gamma}}\right\}\right]-\chi^m\\ &= \lambda^m\int_{\max\left\{1,(\alpha/t)^{\frac{1}{1-\gamma}}\right\}}^{\infty}(\beta-1)x^{1-\beta}\,\mathrm{d}x-\chi^m = \lambda^m\frac{\beta-1}{\beta-2}\min\left\{1,(t\alpha^{-1})^{\frac{\beta-2}{1-\gamma}}\right\}-\chi^m.\end{aligned}$$

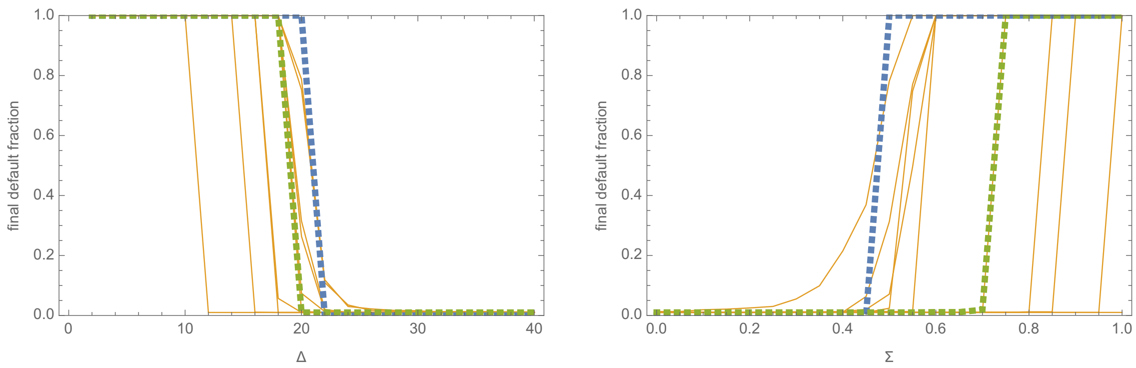
*Motivated by the symmetry of the functions, we consider* $f^m(\boldsymbol{\chi})$ *along direction* $\boldsymbol{v}\in\mathbb{R}_+^M$, *with* $v^m=(\lambda^m)^{-1}$. *Then*

$$\frac{f^m(\chi\boldsymbol{v})}{\lambda^m}=\frac{\beta-1}{\beta-2}\left(\alpha^{-1}\sum_{\ell=1}^{M}\lambda^\ell\left(1-e^{-\chi/\lambda^\ell}\right)\right)^{\frac{\beta-2}{1-\gamma}}-\frac{\chi}{(\lambda^m)^2}.$$

*Let* $\gamma_c:=3-\beta$ *and* $\alpha_c:=\sum_{1\leq m\leq M}(\lambda^m)^2(\beta-1)/(\beta-2)$. *We infer that if* $\chi\in\mathbb{R}_{+,0}$ *is small enough and* $\gamma>\gamma_c$ *or* $\gamma=\gamma_c$ *and* $\alpha>\alpha_c$, *then* $\frac{\mathrm{d}}{\mathrm{d}\chi}f^m(\chi\boldsymbol{v})<0$ *for all* $m\in[M]$. *That is,* $\boldsymbol{\chi}^*=\mathbf{0}$ *and the system is resilient by Corollary 3.3. On the other hand, if either* $\gamma<\gamma_c$ *or* $\gamma=\gamma_c$ *and* $\alpha<\alpha_c$, *then* $\frac{\mathrm{d}}{\mathrm{d}\chi}f^m(\chi\boldsymbol{v})>0$ *for all* $m\in[M]$ *and the system is non-resilient by Corollary 3.6, as* $\boldsymbol{\chi}^*\neq\mathbf{0}$ *and* $\mathring{g}(\boldsymbol{\chi}^*)=g(\boldsymbol{\chi}^*)>0$. *Since* $\gamma_c$ *does not depend on the choice of* $\{\lambda^m\}_{m\in[M]}$, *it makes sense to consider* $\alpha_c$ *as a measure for stability of the system (the smaller* $\alpha_c$, *the more stable the system). Clearly,* $\alpha_c$ *becomes minimized for* $\lambda^m=M^{-1}$ *for all* $m\in[M]$ *and hence a perfectly diversified system is the most stable.*

Next, we consider a financial system that comprises of $S\in\mathbb{N}$ subsystems of equal size $n/S$. For each subsystem $s\in[S]$ there shall exist a set of $D_s=D\in\mathbb{N}$ specialized assets that can only be invested in by institutions from subsystem $s$. In addition to these $S\cdot D$ specialized assets, there shall exist a set of $J\in\mathbb{N}$ joint assets that can be invested in by any institution of the whole system and that hence connect the different subsystems. Thus, each institution can choose from $\Delta:=D+J$ different assets to invest in. We call $\Delta$ the *diversification* of the system. Further, for each institution a fraction $\Sigma:=J/\Delta$ of its available assets is available also to every other institution in the system. We call $\Sigma$ the *(portfolio) similarity* in the system. Then, as in Example 4.1 we could compute the optimal investments for each institution (which is shifted towards investing in the specialized assets to avoid overlap with other subsystems). Instead we assume in the following example that each institution still perfectly diversifies its investment over the $D+J$ assets available to it. This is reasonable if the single institutions do not have a perfect overview of the whole financial system. The effect of diversification $\Delta$ and similarity $\Sigma$ is similar for the two different allocations.

**Example 4.2.** *Consider a system as described above consisting of* $S$ *subsystems and allowing each institution to invest in* $D$ *specialized assets and in* $J$ *joint assets in equal shares. Then the*




(a) (b)

Figure 4: (a) The effect of varying portfolio diversification $\Delta$ as $\Sigma = 0.5$ is fixed. (b) The effect of varying portfolio similarity as $\Delta = 20$ is fixed. In blue: the theoretical final default fraction. In orange: 10 exemplary simulations. In green: the median over $10^3$ simulations.

*system is described by the following functions:*

$$f^j(\boldsymbol{\chi}) := S^{-1}\sum_{s=1}^{S}\mathbb{E}\left[\frac{X^{tot}}{D+J}\mathbf{1}\left\{\frac{X^{tot}}{D+J}\left(\sum_{k=1}^{J}\left(1-e^{-\chi^k}\right)+\sum_{d=1}^{D}\left(1-e^{-\chi^{s,d}}\right)\right)\geq C\right\}\right]-\chi^j,$$

$$f^{s,d}(\boldsymbol{\chi}) := S^{-1}\mathbb{E}\left[\frac{X^{tot}}{D+J}\mathbf{1}\left\{\frac{X^{tot}}{D+J}\left(\sum_{j=1}^{J}\left(1-e^{-\chi^j}\right)+\sum_{e=1}^{D}\left(1-e^{-\chi^{s,e}}\right)\right)\geq C\right\}\right]-\chi^{s,d},$$

*where $j \in [J]$, $s \in [S]$, $d \in [D]$ and $\boldsymbol{\chi} = (\chi^1, \dots, \chi^J, \chi^{1,1}, \dots, \chi^{S,D}) \in \mathbb{R}_{+,0}^{J+SD}$ with small misuse of notation. Similar as in Example 4.1 we derive that*

$$\gamma_c = 3 - \beta \quad \textit{and} \quad \alpha_c = \frac{J + \frac{D}{S}}{(D+J)^2}\frac{\beta-1}{\beta-2} = \frac{1+(S-1)\Sigma}{\Delta S}\frac{\beta-1}{\beta-2}.$$

*From the formula it is obvious that $\alpha_c$ decreases (i. e. stability of the system increases) as $\Delta$ increases or $\Sigma$ decreases.*

Example 4.2 hence shows that diversification makes the system more stable (as already seen in Example 4.1) whereas stronger similarity between the institutions' portfolios makes the system more fragile.

Note that all previous conclusions build on the (asymptotic) theory from Sections 2 and 3. To verify and back up the result for finite systems, however, we also give a simulation based verification for a series of moderate size ($n = 10^4$) systems. We chose $\beta = 3$ and $S = 2$. For $D = J = 10$, we then derive $\Delta = 20$, $\Sigma = 0.5$, $\gamma_c = 0$ and $\alpha_c = 0.075$. We therefore assigned to each institution the capital $c_i = \alpha_c$. Further, we drew the total asset holdings $x_i^{\text{tot}}$ for each institution $i \in [n]$ as random numbers according to the above described Pareto distribution. Finally, we randomly chose a set of initially defaulted institutions of size $0.01n$ and equally distributed across the $S$ subsystems. To see the effect of diversification, we first fix $\Sigma = 0.5$ and let $D = J$ vary from 1 to 20 (i. e. $\Delta \in [40]$). The results are plotted in Figure 4(a). Since we calibrated the capitals $c_i = \alpha_c$ to the values $\Delta = 20$ and $\Sigma = 0.5$, the theoretical (asymptotic) final default fraction is 1 for $\Delta \leq 20$ and 0 otherwise. This curve is shown in blue. In orange we exemplarily illustrate 10 of the $10^3$ simulations. One can see that in each simulation the final default fraction rapidly decreases at a certain value for $\Delta$ close to the theoretical value of 20. In green finally, we plot the median over all $10^3$ simulations which is very close to the theoretical curve despite the finite system size and hence verifies that systems become more resilient as $\Delta$ increases. Deviations from the theoretical curve become smaller as $n$ increases.

Furthermore, in the same setting we conducted simulations for systems of fixed diversification $\Delta = 20$ and varying similarity $\Sigma$ between 0 and 1 ($J \in [0, 20]$ and $D = 20 - J$). The results are shown in Figure 4(b). Again, in blue we plot the theoretically predicted curve which is 0 for $\Sigma < 0.5$ and 1 otherwise. In orange 10 exemplary simulations are shown. For these, one can see that either there exists an individual threshold for $\Sigma$ close to 0.5 at which the final default fraction rapidly increases or the final fraction stays constant at 1%. The median over the $10^3$ simulations for each $\Sigma$ can be seen in green and it verifies that the system becomes less resilient as the similarity $\Sigma$ increases. Again deviations from the theoretical curve are due to finite size effects and become smaller as $n$ increases.

## 4.2 Testing the Capital Requirements by Simulations

In contrast to the previous subsection, we will consider initial stress in a system in form of shocks on asset prices. There are then two dimensions to consider regarding stability of a system. First, for each institution the probability of initial default should be small. Second, in the rare event that some institutions become initially distressed the remaining capital of the institutions still needs to be high enough to stop the spread of fire sales. The latter of the two is precisely the systemic risk capital derived in Subsection 3.3. To further ensure rare initial distress we increase capital $c_i$ by $i$'s *value-at-risk* with respect to some level $\epsilon > 0$ which is a classical risk capital for example used in the *Basel III* framework. In that sense, our systemic risk capital becomes a *systemic risk surcharge* to the classical risk capital:

$$c_i = \text{value-at-risk}(i) + \text{systemic risk surcharge}(i), \qquad i \in [n] \tag{4.1}$$

The aim of this subsection is to verify by simulations that capitals of the form as in (4.1) indeed ensure resilience of a system to initial asset shocks.

In this analysis, we further want to demonstrate the effect of different system characteristics. For simplicity we choose to consider two subsystems $S_1 = [n/2] \subset [n]$ and $S_2 = [n] \backslash S_1$; the considerations extend readily to a larger number of subsystems. Further, there are two assets $A$ and $B$ (i. e. $M = 2$) with uncorrelated price changes. Denote by $x_i^{\text{tot}} = x_i^A + x_i^B$ the total number of shares held by institution $i \in [n]$ and denote the limiting random variable by $X^{\text{tot}}$. We then consider the following three scenarios, see also Figure 5:

(a) The two subsystems invest in different assets.
(b) Each institution's portfolio is perfectly diversified.
(c) All institutions in the system invest in the same asset.

More formally, if we denote by $\pi^{jA}$ resp. $\pi^{jB}$ the proportion invested in assets $A$ and $B$ by an institution in subsystem $S_j$, $j = 1, 2$, we can express the three scenarios as follows:

(a) $\pi^{1A} = \pi^{2B} = 1$ and $\pi^{1B} = \pi^{2A} = 0$.
(b) $\pi^{1A} = \pi^{1B} = \pi^{2A} = \pi^{2B} = 1/2$.
(c) $\pi^{1A} = \pi^{2A} = 1$ and $\pi^{1B} = \pi^{2B} = 0$.

**Remark 4.3.** In terms of the dimensions $\Delta$ and $\Sigma$ from the previous subsection we can characterize the three cases by

(a) $\Delta = 1,\ \Sigma = 0,$ (b) $\Delta = 2,\ \Sigma = 1,$ (c) $\Delta = 1,\ \Sigma = 1.$

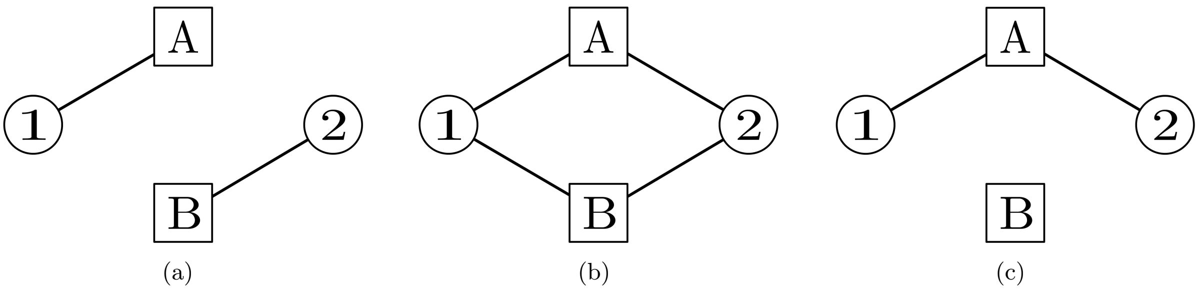



Figure 5: Illustrations of different system configurations for subsystems 1 and 2, and assets $A$ and $B$. (a) Separated undiversified subsystems. (b) Connected diversified subsystems. (c) Connected undiversified subsystems.

From the previous results we would therefore expect that configurations (a) and (b) are more stable than configuration (c), while we cannot make any statements about the relation between (a) and (b). Note, however, that in the previous subsection we considered the initial shock to consist of initial defaults of some institutions independent of the asset holdings. In this subsection, we will consider initial shocks on the financial system by reducing asset prices (see below). It will turn out that diversification can then have a positive effect on stability of the system as in [17] because the variance of each institution's initial loss becomes smaller, or it can have a negative effect as in [5, 23, 28] because the system is exposed to more assets or becomes more connected via assets.

Again, we assume that $\rho(u) = \mathbf{1}\{u \geq 1\}$ and $h^m(\boldsymbol{\chi}) = 1 - e^{-\chi^m}$, $m = A, B$. Further, let

$$f_{X^{\text{tot}}}(x) = (\beta - 1)x^{-\beta}\left(\kappa\mathbf{1}\{1 \leq x \leq b\} + \mu^{\beta-1}\mathbf{1}\{x > b\}\right), \tag{4.2}$$

where $\beta = 2.5$, $\mu = 0.25$, $b = 10^{1/(\beta-1)}\mu$ and $\kappa = (1 - \mu^{\beta-1}b^{1-\beta})/(1 - b^{1-\beta})$. That is the tail of the distribution resembles a Pareto distribution with exponent $\beta$ and $b$ is chosen such that the tail describes 10% of the probability mass. For $x \leq b$ the exponent $\beta$ stays the same but the coefficient $\kappa$ is chosen such that the remaining mass of 90% is distributed on the interval $[1, b]$ (instead of $[\mu, b]$). Note that for the computations of $\gamma_c$ and $\alpha_c$ in the previous subsection only the tail of the distribution was relevant and so we know that $\alpha_c = \frac{1}{2}\frac{\beta-1}{\beta-2}\mu^{\beta-1}$ for cases (a) and (b) resp. $\alpha_c = \frac{\beta-1}{\beta-2}\mu^{\beta-1}$ for case (c) and $\gamma_c = 3-\beta$ in all cases. We consider these as the systemic risk surcharges as discussed above. Further the *value-at-risk* capital for some institution $i \in [n]$ is given by $\theta x_i^{\text{tot}}$ for some global parameter $\theta \in [0, 1]$ that needs to be calibrated to the confidence level $\epsilon$ and the initial shock distribution (see below). The piecewise form of $f_{X^{\text{tot}}}$ in (4.2) rather than for example a perfect Pareto distribution ensures that capital $c_i$ is actually smaller then the maximum potential loss $x_i^{\text{tot}}$ also for institutions with small investments (note that the systemic surcharge $\alpha_c$ is determined from the tail of the distribution).

Finally, we model the initial shock on asset $m = A, B$ as $e^{-R^m}$, where $R^A \stackrel{d}{=} R^B$ are independent random variables such that $\mathbb{P}(R^A = 0) = 90\%$ and with the remaining probability of 10% it holds $R^A = |T|/10$ for $T$ having Student's $t$-distribution with 1.5 degrees of freedom (Student's $t$-distribution is a popular choice in market models since heavy tails can be modeled by less than 2 degrees of freedom).

For each realization of $(R^A, R^B)$ we can then numerically determine the asymptotic final default fraction as in Section 2. We choose $\epsilon = 5\%$ which yields $\theta \approx 8.36\%$ in cases (a) and (c) resp. $\theta \approx 8.47\%$ in case (b) as the parameter for the *value-at-risk* capital. The numerical average final default fraction for the three configurations is listed in Table 1. The most stable

| Case | ∅ initially infected fraction | ∅ finally infected fraction | amplification |
|---|---|---|---|
| (a) | 1.35% | 3.82% | 1.83 |
| (b) | 0.80% | 3.26% | 3.06 |
| (c) | 0.56% | 2.13% | 2.84 |
| (c’) | 1.35% | 5.00% | 2.70 |

Table 1: Simulation results for capital allocation determined by the value-at-risk plus the systemic risk surcharge

| Case | ∅ initially infected fraction | ∅ finally infected fraction | amplification |
|---|---|---|---|
| (a) | 2.38% | 8.95% | 2.76 |
| (b) | 1.76% | 15.1% | 7.53 |
| (c) | 0.84% | 5.58% | 5.66 |
| (c’) | 2.38% | 10.0% | 3.21 |

Table 2: Simulation results for capital allocation determined by the systemic risk surcharge

| Case | ∅ initially infected fraction | ∅ finally infected fraction | amplification |
|---|---|---|---|
| (a) | 5.00% | 5.00% | 0.00 |
| (b) | 5.00% | 5.00% | 0.00 |
| (c) | 5.00% | 5.00% | 0.00 |

Table 3: Simulation results for capital allocation determined by the value-at-risk

configuration is (c). Recall, however, that in this case $\alpha_c$ is the double amount than in cases (a) and (b). Therefore, we further included case (c’) where we adjusted the value of $\alpha_c$ accordingly. It can then be seen that this configuration is in fact the least stable one and (b) becomes the most stable one. Diversification is thus beneficial for the capital allocation value-at-risk plus systemic risk surcharge. In comparison to this, we performed the same simulations with capitals determined solely by the systemic risk charge. The results are listed in Table 2. Clearly, the system becomes less stable. In particular, the least stable configuration is now (b), the diversified one. For completeness also consider Table 3 listing the average fractions for systems equipped exactly with the value-at-risk as capital. In this case, all final infections are already initial infections and there is no amplification (final infection divided by initial infection minus 1). So rather counterintuitively our systemic risk surcharge increases the amplification. This is because due to the additional capital some institutions are initially saved from infection but become infected in the course of the fire sales process. Overall, the combination of value-at-risk with the systemic risk surcharge significantly increases stability of the financial system.

# 5 Proofs

## 5.1 Proofs for Section 2

*Proof of Lemma 2.4.* Existence of $\hat{\boldsymbol{\chi}}$ follows from the Knaster-Tarski theorem. We now construct a joint root $\mathring{S}_0 \ni \bar{\boldsymbol{\chi}} \leq \hat{\boldsymbol{\chi}}$ such that we can conclude $\hat{\boldsymbol{\chi}} = \bar{\boldsymbol{\chi}} \in \mathring{S}_0$.

It holds $\mathring{f}^m(\hat{\boldsymbol{\chi}}) = 0$ for all $m \in [M]$ and thus (for any fixed $m \in [M]$) $\mathring{f}^m(\boldsymbol{\chi}) \leq 0$ for all $\hat{\boldsymbol{\chi}} \geq \boldsymbol{\chi} \in \mathbb{R}_{+,0}^M$ such that $\chi^m = \hat{\chi}^m$ by monotonicity of $\mathring{f}^m$. Consider then the following sequence $(\boldsymbol{\chi}_{(k)})_{k\in\mathbb{N}} \subset \mathbb{R}_{+,0}^M$:

- $\boldsymbol{\chi}_{(0)} = \mathbf{0} \in \mathring{S}_0$

- $\boldsymbol{\chi}_{(1)} = (\chi^1_{(1)}, 0, \dots, 0)$, where $0 \le \chi^1_{(1)} \le \hat{\chi}^1$ is the smallest possible value such that $\mathring{f}^1(\boldsymbol{\chi}_{(1)}) = 0$. It is possible to find such $\chi^1_{(1)}$ since $\mathring{f}^1(\boldsymbol{\chi}) + \chi^1$ is monotonically increasing in $\chi^1$, $\mathring{f}^1(\mathbf{0}) \ge 0$ and $\mathring{f}^1(\hat{\chi}^1, 0, \dots, 0) \le 0$. By monotonicity of $\mathring{f}^m$ with respect to $\chi^1$ for all $m \in [M]\backslash\{1\}$, it then holds $\mathring{f}^m(\boldsymbol{\chi}_{(1)}) \ge \mathring{f}^m(\mathbf{0}) \ge 0$ for all $1 \neq m \in [M]$ and in particular $\boldsymbol{\chi}_{(1)} \in \mathring{S}_0$.
- $\boldsymbol{\chi}_{(2)} = \boldsymbol{\chi}_{(1)} + (0, \chi^2_{(2)}, 0, \dots, 0)$, where $0 \le \chi^2_{(2)} \le \hat{\chi}^2$ is the smallest value such that $\mathring{f}^2(\boldsymbol{\chi}_{(2)}) = 0$. Again it is possible to find such $\chi^2_{(2)}$ since $\mathring{f}^2(\boldsymbol{\chi}) + \chi^2$ is monotonically increasing in $\chi^2$, $\mathring{f}^2(\boldsymbol{\chi}_{(1)}) \ge 0$ and $\mathring{f}^2(\boldsymbol{\chi}_{(1)} + (0, \hat{\chi}^2, 0, \dots, 0)) \le 0$. By monotonicity of $\mathring{f}^m$ with respect to $\chi^2$ for all $m \in [M]\backslash\{2\}$, it then holds $\mathring{f}^m(\boldsymbol{\chi}_{(2)}) \ge \mathring{f}^m(\boldsymbol{\chi}_{(1)}) \ge 0$ for all $2 \neq m \in [M]$ and in particular $\boldsymbol{\chi}_{(2)} \in \mathring{S}_0$.
- $\boldsymbol{\chi}_{(i)}$, $i \in \{3, \dots, M\}$, are found analogously, changing only the corresponding coordinate.
- $\boldsymbol{\chi}_{(M+1)} = \boldsymbol{\chi}_{(M)} + (\chi^1_{(M+1)} - \chi^1_{(M)}, 0, \dots, 0)$, where $\chi^1_{(M)} \le \chi^1_{(M+1)} \le \hat{\chi}^1$ is the smallest value such that $\mathring{f}^1(\boldsymbol{\chi}_{(M+1)}) = 0$, which is again possible by monotonicity of $\mathring{f}^1(\boldsymbol{\chi}) + \chi^1$, $\mathring{f}^1(\boldsymbol{\chi}_{(M)}) \ge 0$ and $\mathring{f}^1(\boldsymbol{\chi}_{(M)} + (\hat{\chi}^1 - \chi^1_{(M)}, 0, \dots, 0)) \le 0$. Further, it still holds $\boldsymbol{\chi}_{(M+1)} \in \mathring{S}_0$.
- Continue for $\boldsymbol{\chi}_i$, $i \ge M + 2$.

The sequence $(\boldsymbol{\chi}_{(k)})_{k \in \mathbb{N}}$ constructed this way has the following properties: It is non-decreasing in each coordinate and bounded inside $[\mathbf{0}, \hat{\boldsymbol{\chi}}]$. Hence by monotone convergence, each coordinate of $\boldsymbol{\chi}_{(k)}$ converges and so $\bar{\boldsymbol{\chi}} = \lim_{k\to\infty} \boldsymbol{\chi}_{(k)}$ exists. Since the convergence is from below, it holds

$$\begin{aligned}\mathring{f}^m(\bar{\boldsymbol{\chi}}) &= \mathbb{E}\left[X^m \mathring{\rho}\left(\frac{L + \boldsymbol{X} \cdot h(\lim_{k\to\infty} \boldsymbol{\chi}_{(k)})}{C}\right)\right] - \lim_{k\to\infty} \chi^m_{(k)} \\ &= \lim_{k\to\infty} \mathbb{E}\left[X^m \mathring{\rho}\left(\frac{L + \boldsymbol{X} \cdot h(\boldsymbol{\chi}_{(k)})}{C}\right)\right] - \chi^m_{(k)} = \lim_{k\to\infty} \mathring{f}^m(\boldsymbol{\chi}_{(k)}) \ge 0\end{aligned}$$

and thus $\bar{\boldsymbol{\chi}} \in \mathring{S}_0$. Now suppose there is $m \in [M]$ such that $\mathring{f}^m(\bar{\boldsymbol{\chi}}) > 0$. By lower semi-continuity of $\mathring{f}^m$ then also $\mathring{f}^m(\boldsymbol{\chi}_{(k)}) > \epsilon$ for some $\epsilon > 0$ and $k$ large enough. This, however, is a contradiction to the construction of the sequence $(\boldsymbol{\chi}_{(k)})_{k\in\mathbb{N}}$ since $\mathring{f}^m(\boldsymbol{\chi}_{(k)}) = 0$ in every $M$-th step. Hence $\mathring{f}^m(\bar{\boldsymbol{\chi}}) = 0$ for all $m \in [M]$ and $\bar{\boldsymbol{\chi}}$ is a joint root of all functions $\mathring{f}^m$, $m \in [M]$.

Now turn to the proof that $\boldsymbol{\chi}^* \in S_0$: We first consider the case that $\rho$ is continuous. We approximate $\boldsymbol{\chi}^* \in S_0$ by the sequence $(\hat{\boldsymbol{\chi}}(\epsilon))_{\epsilon>0}$ of smallest fixpoints for the functions $f^m(\boldsymbol{\chi}) + \epsilon$. This allows us to apply the Knaster-Tarski Theorem and the monotonicity properties of $f^m + \epsilon$ similar as above. Simple topological arguments will then allow us to conclude that $\boldsymbol{\chi}^* \in S_0$. Let for $\epsilon > 0$

$$S(\epsilon) := \bigcap_{m\in[M]} \{\boldsymbol{\chi} \in \mathbb{R}^M_{+,0} \,:\, f^m(\boldsymbol{\chi}) \ge -\epsilon\}$$

and denote by $S_0(\epsilon)$ the connected component of $\mathbf{0}$ in $S(\epsilon)$. By the same procedure as for $\hat{\boldsymbol{\chi}}$ above, we now derive that there exists a smallest (componentwise) point $\hat{\boldsymbol{\chi}}(\epsilon) \in S_0(\epsilon)$ such that $f^m(\hat{\boldsymbol{\chi}}(\epsilon)) = -\epsilon$ for all $m \in [M]$. Clearly, $\hat{\boldsymbol{\chi}}(\epsilon)$ is non-decreasing (componentwise) in $\epsilon$ and hence $\tilde{\boldsymbol{\chi}} := \lim_{\epsilon\to0+} \hat{\boldsymbol{\chi}}(\epsilon)$ exists (we will show that $\tilde{\boldsymbol{\chi}} = \boldsymbol{\chi}^*$ in fact).

Now by monotonicity of $S_0(\epsilon)$, we derive that $\hat{\boldsymbol{\chi}}(\delta) \in S_0(\delta) \subseteq S_0(\epsilon)$ for all $\delta \le \epsilon$. Since $S_0(\epsilon)$ is a closed set, it must thus hold that also $\tilde{\boldsymbol{\chi}} = \lim_{\delta\to0+} \hat{\boldsymbol{\chi}}(\delta) \in S_0(\epsilon)$ for all $\epsilon > 0$ and in particular, $\tilde{\boldsymbol{\chi}} \in \bigcap_{\epsilon>0} S_0(\epsilon)$. Further, we derive that $\bigcap_{\epsilon>0} S_0(\epsilon) \subseteq \bigcap_{\epsilon>0} S(\epsilon) \subseteq S$. Moreover,

$\bigcap_{\epsilon>0} S_0(\epsilon)$ is the intersection of a chain of connected, compact sets in the Hausdorff space $\mathbb{R}^M$ and it is hence a connected, compact set itself. Since it further contains $\mathbf{0}$, we can then conclude that $\bigcap_{\epsilon>0} S_0(\epsilon) \subseteq S_0$ and thus $\tilde{\boldsymbol{\chi}} \in S_0$.

Consider now an arbitrary $\boldsymbol{\chi} \in S_0$. We want to show that $\boldsymbol{\chi} \leq \tilde{\boldsymbol{\chi}}$ componentwise and thus $\tilde{\boldsymbol{\chi}} = \boldsymbol{\chi}^*$. It suffices to show that $S_0 \subset [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon)]$ for all $\epsilon$. Then $\boldsymbol{\chi} \leq \hat{\boldsymbol{\chi}}(\epsilon)$ and $\boldsymbol{\chi} \leq \lim_{\epsilon\to 0+} \hat{\boldsymbol{\chi}}(\epsilon) = \tilde{\boldsymbol{\chi}}$. Hence assume that $S_0 \not\subset [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon)]$. By connectedness of $S_0$ we find $\bar{\boldsymbol{\chi}} \in S_0$ with $\bar{\chi}^m \leq \hat{\chi}^m(\epsilon)$ for all $m \in [M]$ and equality for at least one coordinate (otherwise $S_0 \cap \partial[\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon)] = \emptyset$ and $S_0 = \left(S_0 \cap \left(\mathbb{R}^M_{+,0} \backslash [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon)]\right)\right) \cup (S_0 \cap [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon)))$ is the union of two open non-empty sets and hence not connected). W. l. o. g. let this coordinate be $\bar{\chi}^1$. By monotonicity of $f^1$ with respect to $\chi^m$ for every $1 \neq m \in [M]$, we thus derive that $f^1(\bar{\boldsymbol{\chi}}) \leq f^1(\hat{\boldsymbol{\chi}}(\epsilon)) = -\epsilon < 0$ which is a contradiction to $\bar{\boldsymbol{\chi}} \in S_0$.

Now consider the general case that $\rho$ is right-continuous and let $(\rho_r(u))_{r\in\mathbb{N}}$ be a sequence of continuous sale functions approximating $\rho$ from above. Denoting by $S^r$ the analogue of $S$ for the sale function $\rho_r$, we derive that $S = \bigcap_{r\in\mathbb{N}} S^r$ since clearly $S^r \supseteq S$ for all $r \in \mathbb{N}$ and further by dominated convergence for every $\boldsymbol{\chi} \in \bigcap_{r\in\mathbb{N}} S^r$,

$$\chi^m \leq \mathbb{E}\left[X^m \rho_r\left(\frac{L + \boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right] \to \mathbb{E}\left[X^m \rho\left(\frac{L + \boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right], \quad \text{as } r \to \infty,$$

so that $\bigcap_{r\in\mathbb{N}} S^r \subseteq S$. If we further let $S_0^r$ denote the largest connected subset of $S^r$ containing $\mathbf{0}$, then $S_0^r$ is compact and connected for every $r \in \mathbb{N}$ and hence so is $\bigcap_{r\in\mathbb{N}} S_0^r$. Since further $\mathbf{0} \in \bigcap_{r\in\mathbb{N}} S_0^r$, we derive that $\bigcap_{r\in\mathbb{N}} S_0^r = S_0$. Let now $\boldsymbol{\chi}_r^*$ denote the analogue of $\boldsymbol{\chi}^*$ for the sale function $\rho_r$. Then $\lim_{r\to\infty} \boldsymbol{\chi}_r^* \in S_0^R$ for all $R \in \mathbb{N}$ and hence $\lim_{r\to\infty} \boldsymbol{\chi}_r^* \in \bigcap_{R\in\mathbb{N}} S_0^R = S_0$. Now suppose there existed a vector $\boldsymbol{\chi} \in S_0$ and $m \in [M]$ such that $\chi^m > \lim_{r\to\infty}(\chi_r^*)^m$. Then also for $R$ large enough, $\chi^m > (\chi_R^*)^m$ and hence $\boldsymbol{\chi} \notin S_0^R$. This, however, contradicts the assumption that $\boldsymbol{\chi} \in S_0 = \bigcap_{R\in\mathbb{N}} S_0^R$. Hence there exists no such $\boldsymbol{\chi} \in S_0$ and $\boldsymbol{\chi}^* = \lim_{r\to\infty} \boldsymbol{\chi}_r^* \in S_0$.

Finally, we show that $\boldsymbol{\chi}^*$ is a joint root of $f^m$, $m \in [M]$: Since $\boldsymbol{\chi}^* \in S_0$, it holds that $f^m(\boldsymbol{\chi}^*) \geq 0$ for all $m \in [M]$. Assume now that $f^m(\boldsymbol{\chi}^*) > 0$ for some $m \in [M]$. We can then gradually increase the $m$-coordinate of $\boldsymbol{\chi}^*$ (until $f^m(\boldsymbol{\chi}^*) = 0$). By monotonicity of $f^k(\boldsymbol{\chi})$ with respect to $\chi^m$ for every $m \neq k \in [M]$, however, we can be sure that we do not leave the set $S_0$ by this procedure which is a contradiction to the definition of $\boldsymbol{\chi}^*$. Hence $\boldsymbol{\chi}^*$ is a joint root of $f^m$, $m \in [M]$. □

**Remark 5.1.** In the proof of Lemma 2.4, for the case that $\rho$ is continuous, we constructed $\boldsymbol{\chi}^*$ as the limit of a sequence $(\hat{\boldsymbol{\chi}}(\epsilon))_{\epsilon>0}$ such that $f^m(\hat{\boldsymbol{\chi}}(\epsilon)) = -\epsilon$ for all $m \in [M]$. For non-continuous $\rho$ by the Knaster-Tarski theorem we still know that there exists a smallest vector $\hat{\boldsymbol{\chi}}(\epsilon)$ such that $f^m(\hat{\boldsymbol{\chi}}(\epsilon)) = -\epsilon$, but the construction of $\hat{\boldsymbol{\chi}}(\epsilon)$ as for $\hat{\boldsymbol{\chi}}$ in the proof of Lemma 2.4 fails and we can hence not be sure a priori that $\hat{\boldsymbol{\chi}}(\epsilon) \in S_0(\epsilon)$. Hence let further $\tilde{\boldsymbol{\chi}}(\epsilon)$ be defined as the smallest vector in $S_0(\epsilon)$ such that $f^m(\tilde{\boldsymbol{\chi}}(\epsilon)) = -\epsilon$. This vector exists again by the Knaster-Tarski theorem noting that analogue to Lemma 2.4 $S_0(\epsilon)$ contains its componentwise supremum $\boldsymbol{\chi}^*(\epsilon)$. Then by the same means as above, we derive that $\boldsymbol{\chi}^* = \lim_{\epsilon\to 0} \tilde{\boldsymbol{\chi}}(\epsilon)$.

In Theorem 2.5 we are considering a sequence of financial systems. The following lemma shows the convergence of the smallest joint roots under certain assumptions:

**Lemma 5.2.** *Let a sequence (for $r \in \mathbb{N}$) of financial systems be described by functions $\mathring{f}_r^m$, $m \in [M]$, with smallest joint root $\hat{\boldsymbol{\chi}}_r$. If $\liminf_{r\to\infty} \mathring{f}_r^m(\boldsymbol{\chi}) \geq \mathring{f}^m(\boldsymbol{\chi})$ pointwise for every $m \in [M]$, then $\liminf_{r\to\infty} \hat{\boldsymbol{\chi}}_r \geq \hat{\boldsymbol{\chi}}$, where $\hat{\boldsymbol{\chi}}$ denotes the smallest joint root of the functions $\mathring{f}^m$, $m \in [M]$.*

*Proof.* The main difficulty in showing the result is that we have $\liminf_{r\to\infty} \mathring{f}_r^m(\boldsymbol{\chi}) \geq \mathring{f}^m(\boldsymbol{\chi})$ only pointwise but not uniformly in $\boldsymbol{\chi}$. A further difficulty is the multidimensionality. The main idea is to construct a path in analogy to the construction in Lemma 2.4 that leads to a point $\tilde{\boldsymbol{\chi}}(\epsilon)$ smaller but close to $\hat{\boldsymbol{\chi}}$. On this path the functions $\mathring{f}_r^m$, $m \in [M]$ are all positive for $r$ large. It can then be compared componentwise with a path leading to $\hat{\boldsymbol{\chi}}_r$.

For this consider the construction of $\hat{\boldsymbol{\chi}}$ in Lemma 2.4 and change it in such a way that in each step $k = LM + m$ (where $L \in \mathbb{N}_0$ and $m \in [M]$) a point $\boldsymbol{\chi}_{(k)}(\epsilon)$ is chosen such that $\mathring{f}^m(\boldsymbol{\chi}_{(k)}(\epsilon)) \leq \epsilon$ for some fixed $\epsilon > 0$ (choose $\chi_{(k)}^m(\epsilon) \geq \chi_{(k-1)}^m(\epsilon)$ as the smallest possible value such that this inequality holds; it will then either be $\mathring{f}^m(\boldsymbol{\chi}_{(k)}(\epsilon)) = \epsilon$ or $\boldsymbol{\chi}_{(k)}(\epsilon) = \boldsymbol{\chi}_{(k-1)}(\epsilon)$). Note that $\mathring{f}^m(\boldsymbol{\chi}_{(k)}(\epsilon)) < \epsilon$ can only happen if $\mathring{f}^m(\mathbf{0}) < \epsilon$ in which case there exists $k_0 \in \mathbb{N}_\infty$ such that $\chi_{(k)}^m = 0$ and $\mathring{f}^m(\boldsymbol{\chi}_{(k)}) < \epsilon$ for all $k \leq k_0$ but $\mathring{f}^m(\boldsymbol{\chi}_{(k)}) \geq \epsilon$ and $\chi_{(k)}^m > 0$ for $k > k_0$. Then $(\boldsymbol{\chi}_{(k)}(\epsilon))_{k\in\mathbb{N}}$ is a non-decreasing (componentwise) sequence bounded by $\hat{\boldsymbol{\chi}}$ and hence $\tilde{\boldsymbol{\chi}}(\epsilon) = \lim_{k\to\infty} \boldsymbol{\chi}_{(k)}(\epsilon)$ exists. Further, it holds that $\mathring{f}^m(\tilde{\boldsymbol{\chi}}(\epsilon)) \leq \liminf_{k\to\infty} \mathring{f}^m(\boldsymbol{\chi}_{(k)}(\epsilon)) \leq \epsilon$. Finally, $\tilde{\boldsymbol{\chi}}(\epsilon)$ is non-increasing componentwise in $\epsilon$ and bounded inside $[\mathbf{0}, \hat{\boldsymbol{\chi}}]$ and thus $\tilde{\boldsymbol{\chi}} = \lim_{\epsilon\to 0+} \tilde{\boldsymbol{\chi}}(\epsilon)$ exists. Moreover, $\mathring{f}^m(\tilde{\boldsymbol{\chi}}) \leq \liminf_{\epsilon\to 0+} \mathring{f}^m(\tilde{\boldsymbol{\chi}}(\epsilon)) \leq \liminf_{\epsilon\to 0+} \epsilon = 0$ and hence in particular $\tilde{\boldsymbol{\chi}} = \hat{\boldsymbol{\chi}}$.

Fix now $\delta > 0$ and choose $\epsilon > 0$ small enough such that $\tilde{\chi}^m(\epsilon) > \hat{\chi}^m(1-\delta)^{1/2}$ for all $m \in [M]$. Further, choose $K = K(\epsilon) \in \mathbb{N}$ large enough such that $\chi_{(K)}^m(\epsilon) > \tilde{\chi}^m(\epsilon)(1-\delta)^{1/2}$ for all $m \in [M]$. In particular, $\chi_{(K)}^m(\epsilon) > \hat{\chi}^m(1-\delta)$. Now note that $\hat{\boldsymbol{\chi}}_r$ can be constructed by a sequence $(\boldsymbol{\chi}_{(k,r)})_{k\in\mathbb{N}}$ analogue to $\hat{\boldsymbol{\chi}}$ in the proof of Lemma 2.4 as well. We can then in each step $k \in \mathbb{N}$ cap the element of the constructing sequence $\boldsymbol{\chi}_{(k,r)}$ at $\boldsymbol{\chi}_{(k)}(\epsilon)$, which clearly does not increase the limit of the sequence. We want to make sure that in fact the cap is used in every step $k \leq K$ if we only choose $r$ large enough. Then we can conclude that $\hat{\boldsymbol{\chi}}_r \geq \boldsymbol{\chi}_{(K)}(\epsilon) \geq \hat{\boldsymbol{\chi}}(1-\delta)$ and hence letting $\delta \to 0$, $\liminf_{r\to\infty} \hat{\boldsymbol{\chi}}_r \geq \hat{\boldsymbol{\chi}}$.

We now show that the cap is applied in every step $k \leq K$ for $r$ large enough by an induction argument. For $k = 0$, clearly $\boldsymbol{\chi}_{(k,r)} = \boldsymbol{\chi}_{(k)}(\epsilon) = \mathbf{0}$ and the cap is applied. Now lets assume it holds for $k \leq k_0 < K$. If $\boldsymbol{\chi}_{(k_0+1)}(\epsilon) = \boldsymbol{\chi}_{(k_0)}(\epsilon)$, then of course the cap is also applied in step $k_0+1$ as the sequence $\boldsymbol{\chi}_{(k,r)}$ is increasing. Otherwise, note that by definition of $\boldsymbol{\chi}_{(k_0+1)}(\epsilon)$, it holds $\mathring{f}^m(\boldsymbol{\chi}) \geq \epsilon$ for all $\boldsymbol{\chi} \in \mathbb{R}_{+,0}^M$ such that $\chi^m \in [\chi_{(k_0)}^m(\epsilon), \chi_{(k_0+1)}^m(\epsilon)]$ and $\chi^\ell = \chi_{(k_0)}^\ell(\epsilon) = \chi_{(k_0+1)}^\ell(\epsilon)$ for all $\ell \in [M]\backslash\{m\}$. Now choose a discretization $\{\chi_j\}_{0\leq j\leq J}$ of $[\chi_{(k_0)}^m(\epsilon), \chi_{(k_0+1)}^m(\epsilon)]$ for $J < \infty$ such that $\chi_0 = \chi_{(k_0)}^m(\epsilon)$, $\chi_J = \chi_{(k_0+1)}^m(\epsilon)$ and $\chi_{j-1} < \chi_j < \chi_{j-1} + \epsilon/3$ for all $j \in [J]$. We now use the assumption that $\liminf_{r\to\infty} \mathring{f}_r^m(\boldsymbol{\chi}_j) \geq \mathring{f}^m(\boldsymbol{\chi}_j)$ for every $0 \leq j \leq J$, where $\chi_j^m = \chi_j$ and $\chi_j^\ell = \chi_{(k_0)}^\ell(\epsilon)$ for $\ell \in [M]\backslash\{m\}$. Then for $r$ large enough, $\mathring{f}_r^m(\boldsymbol{\chi}_j) \geq \mathring{f}^m(\boldsymbol{\chi}_j) - \epsilon/3 \geq 2\epsilon/3$. Finally, for any linear interpolation $\boldsymbol{\chi} = \alpha\boldsymbol{\chi}_{j-1} + (1-\alpha)\boldsymbol{\chi}_j$ between $\boldsymbol{\chi}_{j-1}$ and $\boldsymbol{\chi}_j$ ($\alpha \in [0,1]$), it holds

$$\mathring{f}_r^m(\boldsymbol{\chi}) \geq \mathring{f}_r^m(\boldsymbol{\chi}_{j-1}) + \chi_{j-1}^m - \chi_j^m \geq 2\epsilon/3 - \epsilon/3 = \epsilon/3.$$

Hence the cap is applied in step $k_0 + 1$. As there are only finitely many steps $k \leq K$, this finishes the proof. □

*Proof of Theorem 2.5.* We start by proving the lower bound. Recall from Proposition 2.2 that $\boldsymbol{\chi}_n \geq \hat{\boldsymbol{\chi}}_n$. Using weak convergence of the random vector $(\boldsymbol{X}_n, C_n, L_n)$ and approximating $\mathring{\rho}$ from below by a sequence of continuous sale functions $(\rho_r)_{r\in\mathbb{N}}$, we derive for $U \in \mathbb{R}_+$ that pointwise

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}\left[X_n^m \mathring{\rho}\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi}))}{C_n}\right)\right] &\geq \lim_{n\to\infty} \mathbb{E}\left[(X_n^m \wedge U)\, \rho_r\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi}))}{C_n}\right)\right] \\ &= \mathbb{E}\left[(X^m \wedge U)\, \rho_r\left(\frac{L + \boldsymbol{X} \cdot h(\boldsymbol{\chi}))}{C}\right)\right].\end{aligned}$$

Hence as $U \to \infty$ and $r \to \infty$ by monotone convergence,

$$\liminf_{n\to\infty} \mathbb{E}\left[X_n^m \mathring{\rho}\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi})}{C_n}\right)\right] - \chi^m \geq \mathring{f}^m(\boldsymbol{\chi}) \tag{5.1}$$

and we can use Lemma 5.2 to derive that $\liminf_{n\to\infty} \boldsymbol{\chi}_n \geq \liminf_{n\to\infty} \hat{\boldsymbol{\chi}}_n \geq \hat{\boldsymbol{\chi}}$.

We now want to show the lower bound on the final default fraction. Fix some $\delta > 0$ and choose $n$ large enough such that $\boldsymbol{\chi}_n \geq \hat{\boldsymbol{\chi}}_n \geq (1-\delta)\hat{\boldsymbol{\chi}}$ componentwise. Then

$$n^{-1}|\mathcal{D}_n| = \mathbb{P}\left(L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi}_n) \geq C_n\right) \geq \mathbb{P}\left(L_n + \boldsymbol{X}_n \cdot h((1-\delta)\hat{\boldsymbol{\chi}}) > C_n\right).$$

However, using weak convergence of $(\boldsymbol{X}_n, C_n, L_n)$ and approximating the indicator function $\mathbf{1}\{y > 1\}$ from below by continuous functions $(\phi_t)_{t\in\mathbb{N}}$, we derive that

$$\liminf_{n\to\infty} n^{-1}|\mathcal{D}_n| \geq \lim_{n\to\infty} \mathbb{E}\left[\phi_t\left(\frac{L_n + \boldsymbol{X}_n \cdot h((1-\delta)\hat{\boldsymbol{\chi}})}{C_n}\right)\right] = \mathbb{E}\left[\phi_t\left(\frac{L + \boldsymbol{X} \cdot h((1-\delta)\hat{\boldsymbol{\chi}})}{C}\right)\right]$$

and as $t \to \infty$,

$$\liminf_{n\to\infty} n^{-1}|\mathcal{D}_n| \geq \mathbb{P}\left(L + \boldsymbol{X} \cdot h((1-\delta)\hat{\boldsymbol{\chi}}) > C\right) = \mathring{g}((1-\delta)\hat{\boldsymbol{\chi}}).$$

This quantity now tends to $\mathring{g}(\hat{\boldsymbol{\chi}})$ as $\delta \to 0$ by lower semi-continuity of $\mathring{g}$.

Now we approach the second part of the theorem. Recall from Proposition 2.2 that $\boldsymbol{\chi}_n \leq \overline{\boldsymbol{\chi}}_n$. By the construction of $\boldsymbol{\chi}^*$ in the proof of Lemma 2.4, we have a non-increasing (as $\epsilon \to 0$) sequence $(\hat{\boldsymbol{\chi}}(\epsilon))_{\epsilon>0}$ such that $\lim_{\epsilon\to 0+} \hat{\boldsymbol{\chi}}(\epsilon) = \boldsymbol{\chi}^*$. (See Remark 5.1 for non-continuous $\rho$.) In particular, $\boldsymbol{\chi}^* \leq \hat{\boldsymbol{\chi}}(\epsilon)$ for every $\epsilon > 0$ and $f^m(\hat{\boldsymbol{\chi}}(\epsilon)) = -\epsilon$. Using weak convergence of $(\boldsymbol{X}_n, C_n, L_n)$ we derive for $U \in \mathbb{R}_+$ and $(\rho_s)_{s\in\mathbb{N}}$ an approximation of $\rho$ from above by continuous sale functions that

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{E}\left[X_n^m \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right] &= \mathbb{E}[X^m] - \liminf_{n\to\infty} \mathbb{E}\left[X_n^m\left(1 - \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right)\right] \\
&\leq \mathbb{E}[X^m] - \liminf_{n\to\infty} \mathbb{E}\left[(X_n^m \wedge U)\left(1 - \rho_s\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right)\right] \\
&= \mathbb{E}[X^m] - \mathbb{E}\left[(X^m \wedge U)\left(1 - \rho_s\left(\frac{L + \boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C}\right)\right)\right]
\end{aligned}$$

and as $U \to \infty$, $s \to \infty$, by monotone convergence

$$\limsup_{n\to\infty} \mathbb{E}\left[X_n^m \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right] \leq f^m(\hat{\boldsymbol{\chi}}(\epsilon)) + \hat{\chi}^m(\epsilon) = \hat{\chi}^m(\epsilon) - \epsilon.$$

Hence for $n$ large enough it holds

$$\mathbb{E}\left[X_n^m \rho\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right] - \hat{\chi}^m(\epsilon) \leq -\epsilon/2 < 0$$

for all $m \in [M]$. In particular, we know that $\overline{\boldsymbol{\chi}}_n \leq \hat{\boldsymbol{\chi}}(\epsilon)$. Letting $\epsilon \to 0$, this shows that $\limsup_{n\to\infty} \chi_n^m \leq \limsup_{n\to\infty} \overline{\chi}_n^m \leq (\chi^*)^m$ for all $m \in [M]$ and hence completes the proof of the upper bound on finally sold assets.

For the upper bound on the final default fraction $n^{-1}|\mathcal{D}_n| = \mathbb{P}(L_n + \boldsymbol{X}_n \cdot h(\boldsymbol{\chi}_n) \geq C_n)$, approximate the indicator function $\mathbf{1}\{y \geq 1\}$ from above by continuous functions $(\psi_t)_{t\in\mathbb{N}}$ and use weak convergence of $(\boldsymbol{X}_n, C_n, L_n)$ to derive

$$\limsup_{n\to\infty} n^{-1}|\mathcal{D}_n| \le \lim_{n\to\infty} \mathbb{E}\left[\psi_t\left(\frac{L_n + \boldsymbol{X}_n \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C_n}\right)\right] = \mathbb{E}\left[\psi_t\left(\frac{L + \boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon))}{C}\right)\right]$$

and as $t \to \infty$, $\limsup_{n\to\infty} n^{-1}|\mathcal{D}_n| \le g(\hat{\boldsymbol{\chi}}(\epsilon))$. Letting $\epsilon \to 0$, thus shows the second part of the theorem by upper semi-continuity of $g$. □

## 5.2 Proofs for Section 3

As in Section 3 we use the notation $g$, $f^m$, $\mathring{g}$, $\mathring{f}^m$, $\hat{\boldsymbol{\chi}}$ and $\boldsymbol{\chi}^*$ for an unshocked $(\boldsymbol{X}, C)$-system. If instead we index these quantities by $\cdot_L$, we mean the system shocked by $L$.

*Proof of Theorem 3.2.* By Remark 5.1, there exists a sequence of vectors $\tilde{\boldsymbol{\chi}}(\gamma) \in \mathbb{R}^M_{+,0}$ such that $f^m(\tilde{\boldsymbol{\chi}}(\gamma)) = -\gamma$ for all $m \in [M]$ and arbitrary $\gamma \in \mathbb{R}_+$. Now for arbitrary $\alpha \in \mathbb{R}_+$ it holds that

$$\begin{aligned} f^m_L(\boldsymbol{\chi}) &= \mathbb{E}\left[X^m \rho\left(\frac{L + \boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right] - \chi^m \\ &\le \mathbb{E}\left[X^m \mathbf{1}\left\{\frac{L}{C} \ge \alpha\right\}\right] + \mathbb{E}\left[X^m \rho\left(\frac{\alpha C + \boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right] - \chi^m. \end{aligned}$$

Since $\mathbb{E}[L/C] < \delta$, by Markov's inequality it holds that $\mathbb{P}(L/C \ge \alpha) \le \delta/\alpha$ and hence for $\delta > 0$ small enough, we have $\mathbb{E}[X^m \mathbf{1}\{L/C \ge \alpha\}] \le \gamma/3$ (recall that $\mathbb{E}[X^m] < \infty$). By dominated convergence and right-continuity of $\rho$, it thus holds that $f^m_L(\boldsymbol{\chi}) \le f^m(\boldsymbol{\chi}) + 2\gamma/3$ for $\alpha > 0$ small enough such that

$$\mathbb{E}\left[X^m \rho\left(\frac{\alpha C + \boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right] \le \mathbb{E}\left[X^m \rho\left(\frac{\boldsymbol{X} \cdot h(\boldsymbol{\chi})}{C}\right)\right] + \gamma/3.$$

In particular, $f^m_L(\tilde{\boldsymbol{\chi}}(\gamma)) \le -\gamma/3 < 0$ and hence $\boldsymbol{\chi}^*_L < \tilde{\boldsymbol{\chi}}(\gamma)$ for $\delta$ small enough. By similar means, we further derive that for $\delta$ small enough it holds $g_L(\tilde{\boldsymbol{\chi}}(\gamma)) \le g(\tilde{\boldsymbol{\chi}}(\gamma)) + \epsilon/3$. Together with Theorem 2.5, we thus derive that

$$\limsup_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| \le g_L(\boldsymbol{\chi}^*_L) + \epsilon/3 \le g_L(\tilde{\boldsymbol{\chi}}(\gamma)) + \epsilon/3 \le g(\tilde{\boldsymbol{\chi}}(\gamma)) + 2\epsilon/3.$$

Now since $\tilde{\boldsymbol{\chi}}(\gamma) \to \boldsymbol{\chi}^*$ and by upper semi-continuity of $g$, we can choose $\gamma > 0$ small enough such that $g(\tilde{\boldsymbol{\chi}}(\gamma)) \le g(\boldsymbol{\chi}^*) + \epsilon/3$ and conclude that $\limsup_{n\to\infty} n^{-1}|\mathcal{D}_{n,L}| \le g(\boldsymbol{\chi}^*) + \epsilon$.

For the bound on $\chi^m_{n,L}$ choose $\gamma$ and $\delta$ small enough such that $(\chi^*_L)^m \le \tilde{\chi}^m(\gamma) + \epsilon/3 \le (\chi^*)^m + 2\epsilon/3$ and conclude by Theorem 2.5 that

$$\limsup_{n\to\infty} \chi^m_{n,L} \le (\chi^*_L)^m + \epsilon/3 \le (\chi^*)^m + \epsilon.$$ □

*Proof of Theorem 3.5.* For any $\epsilon > 0$ and any subset $I \subset [M]$ let

$$T(\epsilon, I) := \bigcap_{m\in I} \left\{\boldsymbol{\chi} \in \mathbb{R}^M_{+,0} \,:\, \mathring{f}^m(\boldsymbol{\chi}) \le -\epsilon\right\} \cap \bigcap_{k\in I^c} \left\{\boldsymbol{\chi} \in \mathbb{R}^M_{+,0} \,:\, \chi^k \ge \mathbb{E}[X^k]\right\},$$

where $I^c := [M]\backslash I$. Analogously to the construction of $\hat{\boldsymbol{\chi}}$ in the proof of Lemma 2.4, we find the smallest (componentwise) point $\hat{\boldsymbol{\chi}}(\epsilon, I) \in \mathbb{R}^M_{+,0}$ such that $\mathring{f}^m(\boldsymbol{\chi}) = -\epsilon$ for $m \in I$ and $\chi^k = \mathbb{E}[X^k]$ for $k \in I^c$. Clearly, $\hat{\boldsymbol{\chi}}(\epsilon, I) \in T(\epsilon, I)$ and $\hat{\boldsymbol{\chi}}(\epsilon, I) \le \boldsymbol{\chi}$ for any other $\boldsymbol{\chi} \in T(\epsilon, I)$ (choose $\boldsymbol{\chi}$ as an upper bound in the construction).

In particular, $\hat{\boldsymbol{\chi}}(\epsilon, I)$ is non-decreasing. As it is bounded by $\mathbb{E}[\boldsymbol{X}]$, we therefore know that it is continuous for almost every $\epsilon > 0$. As moreover, $\mathbb{E}[X^m \rho(\mathbf{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))/C)]$ is bounded and increasing in $\epsilon$, we derive that for almost every $\epsilon > 0$ and arbitrary $\delta > 0$, we can find $\gamma > 0$ small enough such that

$$\mathbb{E}\left[X^m \rho\left(\frac{\boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))}{C}\right)\right] \leq \mathbb{E}\left[X^m \rho\left(\frac{\boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon - \gamma, I))}{C}\right)\right] + \delta.$$

As $\hat{\chi}^m(\epsilon, I)$ is strictly increasing for $m \in I$ and by the assumption of $h^m(\boldsymbol{\chi})$ being strictly increasing in $\chi^m$, we derive on $\{X^m > 0\}$ that $\rho(\mathbf{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon - \gamma, I))/C) \leq \mathring{\rho}(\mathbf{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))/C)$ and hence

$$\mathbb{E}\left[X^m \mathring{\rho}\left(\frac{\boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))}{C}\right)\right] \leq \mathbb{E}\left[X^m \rho\left(\frac{\boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))}{C}\right)\right] \leq \mathbb{E}\left[X^m \mathring{\rho}\left(\frac{\boldsymbol{X} \cdot h(\hat{\boldsymbol{\chi}}(\epsilon, I))}{C}\right)\right] + \delta.$$

Choosing $\delta$ arbitrarily small, we thus conclude that $f^m(\hat{\boldsymbol{\chi}}(\epsilon, I)) = \mathring{f}^m(\hat{\boldsymbol{\chi}}(\epsilon, I)) = -\epsilon$ for $m \in I$.

Suppose now there was some $\boldsymbol{\chi} \in S_0 \backslash [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon, I)]$. As $S_0 \subset [\mathbf{0}, \mathbb{E}[\boldsymbol{X}]]$ and by monotonicity of $f^m$, we could then find some $m \in I$ and $\tilde{\boldsymbol{\chi}} \in S_0$ such that $\tilde{\boldsymbol{\chi}} \leq \hat{\boldsymbol{\chi}}(\epsilon, I)$ and $\tilde{\chi}^m = \hat{\chi}^m(\epsilon, I)$. This on the other hand would imply $f^m(\tilde{\boldsymbol{\chi}}) \leq f^m(\hat{\boldsymbol{\chi}}(\epsilon, I)) = -\epsilon$, which contradicts $\tilde{\boldsymbol{\chi}} \in S_0$. We can thus conclude that $\boldsymbol{\chi}^* \in S_0 \subset [\mathbf{0}, \hat{\boldsymbol{\chi}}(\epsilon, I)]$.

Consider now a certain $L$ and let

$$I := \{m \in [M] \,:\, \hat{\chi}_L^m < \mathbb{E}[X^m]\}.$$

Then for $m \in I$, we have

$$\mathring{f}^m(\hat{\boldsymbol{\chi}}_L) = \frac{\mathring{f}_L^m(\hat{\boldsymbol{\chi}}_L) - \mathbb{P}(L = 2C)(\mathbb{E}[X^m] - \hat{\chi}_L^m)}{\mathbb{P}(L = 0)} < 0$$

since $\mathring{f}_L^m(\hat{\boldsymbol{\chi}}_L) = 0$ by definition. Let now

$$\epsilon := -\max_{m \in I} \mathring{f}^m(\hat{\boldsymbol{\chi}}_L) > 0.$$

By construction, $\hat{\boldsymbol{\chi}}_L \in T(\epsilon, I)$ and thus $\hat{\boldsymbol{\chi}}_L \geq \hat{\boldsymbol{\chi}}(\epsilon, I) \geq \boldsymbol{\chi}^*$. By Theorem 2.5 we can thus conclude that

$$n^{-1}|\mathcal{D}_{n,L}| \geq \mathring{g}_L(\hat{\boldsymbol{\chi}}_L) + o(1) \geq \mathring{g}(\boldsymbol{\chi}^*) + o(1)$$

and

$$\chi_{n,L}^m \geq \hat{\chi}_L^m + o(1) \geq (\chi^*)^m + o(1). \qquad \square$$

*Proof of Corollary 3.7.* First, let $\gamma \geq 1$. Then

$$f(\chi) = \mathbb{E}[X \mathbf{1}\{X h(\chi) \geq \alpha X^\gamma\}] - \chi \leq \mathbb{E}[X \mathbf{1}\{h(\chi) \geq \alpha\}] - \chi = -\chi$$

for $\chi$ small enough such that $h(\chi) < \alpha$. Hence $\chi^* = 0$ and $g(\chi^*) = \mathbb{P}(X h(\chi^*) \geq \alpha X^\gamma) = 0$. The system is hence resilient by Corollary 3.3.

Now assume that $\gamma < 1$. Then for $\chi$ small enough

$$\begin{aligned}
f(\chi) &\leq \mathbb{E}\left[X\mathbf{1}\left\{X \geq \left(\frac{\alpha}{\mu_2}\chi^{-\nu}\right)^{\frac{1}{1-\gamma}}\right\}\right] - \chi \\
&= \left(1 - F_X\left(\left(\frac{\alpha}{\mu_2}\chi^{-\nu}\right)^{\frac{1}{1-\gamma}}\right)\right)\left(\frac{\alpha}{\mu_2}\chi^{-\nu}\right)^{\frac{1}{1-\gamma}} + \int_{\left(\frac{\alpha}{\mu_2}\chi^{-\nu}\right)^{\frac{1}{1-\gamma}}}^{\infty}(1 - F_X(t))\mathrm{d}t - \chi \\
&\leq B_2\frac{\beta-1}{\beta-2}\left(\frac{\mu_2}{\alpha}\chi^{\nu}\right)^{\frac{\beta-2}{1-\gamma}} - \chi
\end{aligned}$$

and $\liminf_{\chi\to 0+} f(\chi)\chi^{-1} < 0$ for $\gamma > 1-\nu(\beta-2)$ or $\gamma = 1-\nu(\beta-2)$ and $\alpha > \mu_2\left(B_2\frac{\beta-1}{\beta-2}\right)^{\nu}$. This implies $\chi^* = 0$ and hence resilience as above.

On the other hand, for $\chi$ small enough also

$$f(\chi) \geq B_1\frac{\beta-1}{\beta-2}\left(\frac{\mu_1}{\alpha}\chi^{\nu}\right)^{\frac{\beta-2}{1-\gamma}} - \chi$$

and hence $\chi^* > 0$ for $\gamma < 1-\nu(\beta-2)$ or $\gamma = 1-\nu(\beta-2)$ and $\alpha < \mu_1\left(B_1\frac{\beta-1}{\beta-2}\right)^{\nu}$. Then

$$\mathring{g}(\chi^*) = \mathbb{P}\left(X > \left(\frac{\alpha}{h(\chi^*)}\right)^{\frac{1}{1-\gamma}}\right) \geq B_1\left(\frac{h(\chi^*)}{\alpha}\right)^{\frac{\beta-1}{1-\gamma}} > 0$$

and the system is non-resilient by Corollary 3.6. □

*Proof of Corollary 3.8.* Non-resilience for $\gamma < 1-\nu(\beta-2)$ is trivial from Corollary 3.7 noting that the intermediate sales only make the system even less resilient.

So assume in the following that $\gamma > 1-\nu(\beta-2)$: First, let $\gamma \geq 1$. Then for $\chi$ small enough (cf. the proof of Corollary 3.7) it holds

$$f(\chi) \leq \mathbb{E}\left[X\left(\frac{Xh(\chi)}{\alpha X^{\gamma}}\right)^{q}\mathbf{1}\left\{Xh(\chi) < \alpha X^{\gamma}\right\}\right] - \chi \leq \frac{\mathbb{E}[X]\mu_2^q}{\alpha}\chi^{\nu q} - \chi$$

and by $\nu q > 1$, we derive $\chi^* = 0$ and resilience of the system by Corollary 3.3.

Now let $\gamma < 1$. Using $B_1x^{1-\beta} \leq 1 - F_X(x) \leq B_2x^{1-\beta}$ for $x \geq x_0$, we derive

$$\begin{aligned}
&\mathbb{E}\left[X^{1+q(1-\gamma)}\mathbf{1}\left\{X < \left(\frac{\alpha}{h(\chi)}\right)^{\frac{1}{1-\gamma}}\right\}\right] \\
&= \int_0^{\left(\frac{\alpha}{h(\chi)}\right)^{\frac{1}{1-\gamma}}}(1+q(1-\gamma))t^{q(1-\gamma)}(1-F_X(t))\mathrm{d}t - \left(\frac{\alpha}{h(\chi)}\right)^{\frac{1}{1-\gamma}+q}\left(1 - F_X\left(\left(\frac{\alpha}{h(\chi)}\right)^{\frac{1}{1-\gamma}}\right)\right) \\
&\leq \left(B_2\frac{1+q(1-\gamma)}{2-\beta+q(1-\gamma)} - \tilde{B}_1\right)\left(\frac{\alpha}{h(\chi)}\right)^{\frac{2-\beta}{1-\gamma}+q} + \kappa,
\end{aligned}$$

where $\kappa > 0$ accounts for the lower part of the integral from 0 to $x_0$ and $\tilde{B}_1$ is chosen such that $\tilde{B}_1x^{1-\beta} \leq 1 - F_X(x)$ for all $x \geq 0$. For $\chi$ small enough (cf. the proof of Corollary 3.7) it then holds

$$
\begin{aligned}
f(\chi) &\leq B_2 \frac{\beta-1}{\beta-2}\left(\frac{\mu_2}{\alpha}\chi^{\nu}\right)^{\frac{\beta-2}{1-\gamma}} + \mathbb{E}\left[X^{1+q(1-\gamma)}\mathbf{1}\left\{X < \left(\frac{\alpha}{h(\chi)}\right)^{\frac{1}{1-\gamma}}\right\}\right]\left(\frac{h(\chi)}{\alpha}\right)^{q} - \chi \\
&\leq B_2 \frac{\beta-1}{\beta-2}\left(\frac{\mu_2}{\alpha}\chi^{\nu}\right)^{\frac{\beta-2}{1-\gamma}} + \left(B_2 \frac{1+q(1-\gamma)}{2-\beta+q(1-\gamma)} - \tilde{B}_1\right)\left(\frac{\mu_2}{\alpha}\chi^{\nu}\right)^{\frac{\beta-2}{1-\gamma}} + \kappa\left(\frac{\mu_2}{\alpha}\chi^{\nu}\right)^{q} - \chi
\end{aligned}
$$

and by $\nu(\beta-2)/(1-\gamma) > 1$ as well as $\nu q > 1$, we derive that $\chi^* = 0$ and the system is resilient by Corollary 3.3. □

*Proof of Corollary 3.9.* Let $\boldsymbol{v} \in \mathbb{R}_+^M$ be defined by $v^m := \theta^m/\mu^m$. The functions $f^m$, $m \in [M]$, are given by

$$
f^m(\boldsymbol{\chi}) = \mathbb{E}\left[X^m \rho^m\left(\frac{\sum_{m\in[M]} X^m h^m(\boldsymbol{\chi})}{\sum_{m\in[M]} \theta^m X^m}\right)\right] - \chi^m
$$

and thus for $\chi \in \mathbb{R}_{+,0}$ we have

$$
f^m(\chi\boldsymbol{v}) = \mathbb{E}[X^m \rho^m(\chi^{\nu})] - \chi\frac{\theta^m}{\mu^m} = \mathbb{E}[X^m]\left(\mathbf{1}\{\chi \geq 1\} + \chi^{\nu q^m}\mathbf{1}\{\chi < 1\}\right) - \chi\frac{\theta^m}{\mu^m}.
$$

As $\chi \to 0$,

$$
\liminf_{\chi\to 0+} f^m(\chi\boldsymbol{v})\chi^{-1} = \begin{cases} -\theta^m/\mu^m, & \text{if } q^m > \nu^{-1}, \\ \mathbb{E}[X^m] - \theta^m/\mu^m, & \text{if } q^m = \nu^{-1}, \\ \infty, & \text{if } q^m < \nu^{-1}. \end{cases}
$$

In particular, both 1 and 2 imply $\liminf_{\chi\to 0+} f^m(\chi\boldsymbol{v})\chi^{-1} < 0$ and since this holds for all $m \in [M]$, we can conclude that $\boldsymbol{\chi}^* = \mathbf{0}$ and the system is resilient by Corollary 3.3. □

## References


[1] V. V. Acharya, L. H. Pedersen, T. Philippon, and M. Richardson. Measuring Systemic Risk. *Review of Financial Studies*, 30(1):2–47, 2017.

[2] T. Adrian and M. K. Brunnermeier. CoVaR. *American Economic Review*, 106(7):1705–1741, 2016.

[3] H. Amini, R. Cont, and A. Minca. Resilience to Contagion in Financial Networks. *Mathematical Finance*, 26(2):329–365, 2016.

[4] Basel Committee on Banking Supervision. Basel III: A Global Regulatory Framework for More Resilient Banks and Banking Systems, 2011.

[5] N. Beale, D. G. Rand, H. Battey, K. Croxson, R. M. May, and M. A. Nowak. Individual versus Systemic Risk and the Regulator's Dilemma. *Proceedings of the National Academy of Sciences*, 108(31):12647–52, 2011.

[6] A. Braverman and A. Minca. Networks of Common Asset Holdings: Aggregation and Measures of Vulnerability. *Preprint*, 2014.

[7] C. Brownlees and R. F. Engle. SRISK: A Conditional Capital Shortfall Measure of Systemic Risk. *The Review of Financial Studies*, 30(1):48–79, 2017.

[8] F. Caccioli, M. Shrestha, C. Moore, and J. D. Farmer. Stability Analysis of Financial Contagion Due to Overlapping Portfolios. *Journal of Banking & Finance*, 46:233–245, 2014.

[9] R. Cifuentes, G. Ferrucci, and H. S. Shin. Liquidity Risk and Contagion. *Journal of the European Economic Association*, 3(2-3):556–566, 2005.

[10] R. Cont and E. Schaanning. Fire Sales, Indirect Contagion and Systemic Stress Testing. *Preprint*, 2017.

[11] R. Cont and E. Schaanning. Monitoring Indirect Contagion. *Preprint*, 2018.

[12] R. Cont and L. Wagalath. Running for the Exit: Distressed Selling and Endogenous Correlation in Financial Markets. *Mathematical Finance*, 23(4):718–741, 2013.

[13] R. Cont and L. Wagalath. Fire Sales Forensics: Measuring Endogenous Risk. *Mathematical Finance*, 26(4):835–866, 2016.

[14] N. Detering, T. Meyer-Brandis, K. Panagiotou, and D. Ritter. Managing Default Contagion in Inhomogeneous Financial Networks. *arXiv:1610.09542*, 2017.

[15] N. Detering, T. Meyer-Brandis, K. Panagiotou, and D. Ritter. Financial Contagion in a Generalized Stochastic Block Model. *arXiv:1803.08169*, 2018.

[16] L. Eisenberg and T. H. Noe. Systemic Risk in Financial Systems. *Management Science*, 47(2):236–249, 2001.

[17] R. Frey and J. Hledik. Diversification and Systemic Risk: A Financial Network Perspective. *Risks*, 6(2), 2018.

[18] D. Garlaschelli, S. Battiston, M. Castri, V. D. Servedio, and G. Caldarelli. The Scale-Free Topology of Market Investments. *Physica A: Statistical Mechanics and its Applications*, 350(2):491–499, 2005.

[19] G. Girardi, K. W. Hanley, S. Nikolova, L. Pelizzon, and M. Getmansky Sherman. Portfolio Similarity and Asset Liquidation in the Insurance Industry. *SSRN working paper*, 2018.

[20] G.-F. Gu, W. Chen, and W.-X. Zhou. Empirical Shape Function of Limit-Order Books in the Chinese Stock Market. *Physica A*, 387:5182–5188, 2008.

[21] W. Guo, A. Minca, and L. Wang. The Topology of Overlapping Portfolio Networks. *Statistics & Risk Modeling*, 33(3-4):139–155, 2016.

[22] T. R. Hurd. *Contagion! Systemic Risk in Financial Networks.* Springer, 2016.

[23] R. Ibragimov, D. Jaffee, and J. Walden. Diversification Disasters. *Journal of Financial Economics*, 99(2):333–348, 2011.

[24] A. E. Khandani and A. W. Lo. What Happened to the Quants in August 2007? Evidence from Factors and Transactions Data. *Journal of Financial Markets*, 14(1):1–46, 2011.

[25] O. Kley, C. Klüppelberg, and G. Reinert. Risk in a Large Claims Insurance Market with Bipartite Graph Structure. *Operations Research*, 64(5):1159–1176, 2016.

[26] M. Kritzman, Y. Li, S. Page, and R. Rigobon. Principal Components as a Measure of Systemic Risk. *The Journal of Portfolio Management*, 37(4):112–126, 2011.

[27] L. C. G. Rogers and L. A. M. Veraart. Failure and Rescue in an Interbank Network. *Management Science*, 59(4):882–898, 2013.

[28] W. Wagner. Diversification at Financial Institutions and Systemic Crises. *Journal of Financial Intermediation*, 19(3):373–386, 2010.

[29] S. Weber and K. Weske. The Joint Impact of Bankruptcy Costs, Fire Sales and Cross-Holdings on Systemic Risk in Financial Networks. *Probability, Uncertainty and Quantitative Risk*, 2(9), 2017.