Supplement to "Liquidity based modeling of asset price bubbles via random matching"

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This is a supplement to the paper [1]. The supplement is organized as follows. First, we prove Theorem 3.13 in [1] which provides the existence of the dynamical system $\mathbb{D}$ introduced in Definition 3.6 in [1]. Second, we show some properties of $\mathbb{D}$ which are summarized in Theorem 3.14 in [1].
In the following, we only state the basic setting and refer to [1] for definitions.

1 Setting

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be a probability space and $(\hat{\Omega}, \hat{\mathcal{F}})$ another measurable space. We define the product space

$$(\Omega, \mathcal{F}) := (\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}). \tag{1.1}$$

Let $\hat{P}$ be a Markov kernel (or stochastic kernel) from $\tilde{\Omega}$ to $\hat{\Omega}$. Given $\tilde{\omega} \in \tilde{\Omega}$, we set $\hat{P}^{\tilde{\omega}} := \hat{P}(\tilde{\omega})$ with a slight notational abuse. We then introduce a probability measure P on $(\Omega, \mathcal{F})$ as the semidirect product of $\tilde{P}$ and $\hat{P}$, that is,

$$P(\tilde{A} \times \hat{A}) := (\tilde{P} \ltimes \hat{P})(\tilde{A} \times \hat{A}) = \int_{\tilde{A}} \hat{P}^{\tilde{\omega}}(\hat{A}) d\tilde{P}(\tilde{\omega}). \tag{1.2}$$

We fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents and let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. All agents in I can be classified according to their type. In particular, we let $S = \{1, 2, ..., K\}$ be a finite space of types and say that an agent has type J if he is not matched. We denote by $\hat{S} := S \times (S \cup \{J\})$ the extended type space. Moreover, we call $\hat{\Delta}$ the space of extended type distributions, which is the set of probability distributions p on $\hat{S}$ satisfying $p(k, l) = p(l, k)$ for any k and l in S. This space is endowed with the topology $\mathcal{T}^{\Delta}$ induced by the topology of the space of matrices with $|S|$ rows and $|S| + 1$ columns. We consider $(n)_{n \geq 1}$ time periods and denote by $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)$ the matrix valued processes, with $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n) = (\eta^n_{kl}, \theta^n_{kl}, \xi^n_{kl}, \sigma^n_{kl}[r, s], \varsigma^n_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}$ for $n \geq 1$, on $(\Omega, \mathcal{F}, P)$. For a detailed introduction of these processes we refer to Section 3 in [1]. Moreover, let $\hat{p} = (\hat{p}^n)_{n \geq 1}$ be a stochastic process on $(\Omega, \mathcal{F}, P)$ with values in $\hat{\Delta}$, representing the evolution of the underlying extended type distribution. We assume that $\hat{p}^0$ is deterministic.
Given the input processes $(\eta, \theta, \xi, \sigma, \varsigma)$ we denote by $\mathbb{D}$ a dynamical system on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ and

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by $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n\in\mathbb{N}\setminus\{0\}}$ the agent-type function, the random matching and the partner-type function, respectively, as introduced in Definition 3.6 in [1], which we recall in the following.

Definition 1.1. A *dynamical system* $\mathbb{D}$ defined on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a triple $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n\in\mathbb{N}\setminus\{0\}}$ such that for each integer period $n \geq 1$ we have:

1. $\alpha^n : I \times \Omega \to S$ is the $\mathcal{I} \boxtimes \mathcal{F}$-measurable agent type function. The corresponding end-of-period type of agent i under the realization $\omega \in \Omega$ is given by $\alpha^n(i, \omega) \in S$.

2. A random matching $\pi^n : I \times \Omega \to I$, describing the end-of-period agent $\pi^n(i)$ to whom agent i is currently matched, if agent i is currently matched. If agent i is not matched, then $\pi^n(i) = i$. The associated $\mathcal{I} \boxtimes \mathcal{F}$-measurable partner-type function $g^n : I \times \Omega \to S \cup \{J\}$ is given by

$$g^n(i, \omega) = \begin{cases} \alpha^n(\pi^n(i, \omega), \omega) & \text{if } \pi^n(i, \omega) \neq i \\ J & \text{if } \pi^n(i, \omega) = i, \end{cases}$$

providing the type of the agent to whom agent i is matched, if agent i is matched, or J if agent i is not matched.

Let the initial condition $\Pi^0 = (\alpha^0, \beta^0)$ of $\mathbb{D}$ be given. We now construct a dynamical system $\mathbb{D}$ defined on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ with input processes $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)_{n\geq 1}$. We assume that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1}, g^{n-1})$ is given for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n, g^n)$ by characterizing the three sub-steps of random change of types of agents, random matchings, break-ups and possible type changes after matchings and break-ups as follows.

Mutation: For $n \geq 1$ consider an $\mathcal{I} \boxtimes \mathcal{F}$-measurable post mutation function

$$\bar{\alpha}^n : I \times \Omega \to S.$$

In particular, $\bar{\alpha}_i^n(\omega) := \bar{\alpha}^n(i, \omega)$ is the type of agent i after the random mutation under the scenario $\omega \in \Omega$. The type of the agent to whom an agent is matched is identified by a $\mathcal{I} \boxtimes \mathcal{F}$-measurable function

$$\bar{g}^n : I \times \Omega \to S \cup \{J\},$$

given by

$$\bar{g}^n(i, \omega) = \bar{\alpha}^n(\pi^{n-1}(i, \omega), \omega)$$

for any $\omega \in \Omega$. In particular, $\bar{g}_i^n(\omega) := \bar{g}^n(i, \omega)$ is the type of the agent to whom an agent is matched under the scenario $\omega \in \Omega$. Given $\hat{p}^{n-1}$ and $\tilde{\omega} \in \tilde{\Omega}$, for any k_1, k_2, l_1 and l_2 in S, for any $r \in S \cup \{J\}$, for λ-almost every agent i, we set

$$\begin{aligned} &\hat{P}^{\tilde{\omega}} \left(\bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_2, \bar{g}_i^n(\tilde{\omega}, \cdot) = l_2 | \alpha_i^{n-1}(\tilde{\omega}, \cdot) = k_1, g_i^{n-1}(\tilde{\omega}, \cdot) = l_1, \hat{p}^{n-1}(\tilde{\omega}, \cdot) \right)(\hat{\omega}) \\ &\quad = \eta_{k_1,k_2}\left(\tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega})\right) \eta_{l_1,l_2}\left(\tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega})\right), \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\hat{P}^{\tilde{\omega}} \left(\bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_2, \bar{g}_i^n(\tilde{\omega}, \cdot) = r | \alpha_i^{n-1}(\tilde{\omega}, \cdot) = k_1, g_i^{n-1}(\tilde{\omega}, \cdot) = J, \hat{p}^{n-1}(\tilde{\omega}, \cdot) \right)(\hat{\omega}) \\ &\quad = \eta_{k_1,k_2}\left(\tilde{\omega}, n, \hat{p}^{n-1}(\tilde{\omega}, \hat{\omega})\right) \delta_J(r), \end{aligned} \tag{1.4}$$

We then set

$$\bar{\beta}^n(\omega) = (\bar{\alpha}^n(\omega), \bar{g}^n(\omega)), \quad n \geq 1.$$

The post-mutation extended type distribution realized in the state of the world $\omega \in \Omega$ is denoted by $\check{p}(\omega) = (\check{p}^n(\omega)[k,l])_{k \in S, l \in S \cup J}$, where

$$\check{p}^n(\omega)[k,l] := \lambda(\{i \in I : \bar{\alpha}^n(i,\omega) = k, \bar{g}^n(i,\omega) = l\}). \tag{1.5}$$

Matching: We introduce a random matching $\bar{\pi}^n : I \times \Omega \to I$ and the associated post-matching partner type function $\bar{\bar{g}}^n$ given by

$$\bar{\bar{g}}^n(i,\omega) = \begin{cases} \bar{\alpha}^n(\bar{\pi}^n(i,\omega),\omega) & \text{if } \bar{\pi}^n(i,\omega) \neq i \\ J & \text{if } \bar{\pi}^n(i,\omega) = i, \end{cases}$$

satisfying the following properties:

1. $\bar{\bar{g}}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.
2. For any $\tilde{\omega} \in \tilde{\Omega}$, any $k, l \in S$ and any $r \in S \cup \{J\}$, it holds

   $$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n(\tilde{\omega},\cdot) = r | \bar{\alpha}_i^n(\tilde{\omega},\cdot) = k, \bar{g}_i^n(\tilde{\omega},\cdot) = l)(\hat{\omega}) = \delta_l(r).$$

   This means that

   $$\bar{\pi}_\omega^n(i) = \pi_\omega^{n-1}(i) \quad \text{ for any } i \in \{i : \pi^{n-1}(i,\omega) \neq i\}.$$

3. Given $\tilde{\omega} \in \tilde{\Omega}$ and the post-mutation extended type distribution $\check{p}^n$ in (1.5), an unmatched agent of type k is matched to a unmatched agent of type l with conditional probability $\theta_{kl}(\tilde{\omega}, n, \check{p}^n)$, that is for λ-almost every agent i and $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega}$, we define

   $$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n(\tilde{\omega},\cdot) = l | \bar{\alpha}_i^n(\tilde{\omega},\cdot) = k, \bar{g}_i^n(\tilde{\omega},\cdot) = J, \check{p}^n(\tilde{\omega},\cdot))(\hat{\omega}) = \theta_{kl}^n(\tilde{\omega}, \check{p}^n(\tilde{\omega},\hat{\omega})). \tag{1.6}$$

   This also implies that

   $$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n(\tilde{\omega},\cdot) = J | \bar{\alpha}_i^n(\tilde{\omega},\cdot) = k, \bar{g}_i^n(\tilde{\omega},\cdot) = J, \check{p}^n(\tilde{\omega},\cdot))(\hat{\omega}) = 1 - \sum_{l \in S} \theta_{kl}^n(\tilde{\omega}, \check{p}^n(\tilde{\omega},\hat{\omega})) = b^k(\tilde{\omega}, \check{p}^n(\tilde{\omega},\hat{\omega})). \tag{1.7}$$

The extended type of agent i after the random matching step is

$$\bar{\bar{\beta}}_i^n(\omega) = (\bar{\alpha}_i^n(\omega), \bar{\bar{g}}_i^n(\omega)), \quad n \geq 1.$$

We denote the post-matching extended type distribution realized in $\omega \in \Omega$ by $\check{\check{p}}^n(\omega) = (\check{\check{p}}^n(\omega)[k,l])_{k \in S, l \in S \cup J}$, where

$$\check{\check{p}}^n(\omega)[k,l] := \lambda(\{i \in I : \bar{\bar{\alpha}}^n(i,\omega) = k, \bar{g}^n(i,\omega) = l\}). \tag{1.8}$$

Type changes of matched agents with break-up: We now define a random matching π^n by

$$\pi^n(i) = \begin{cases} \bar{\pi}^n(i) & \text{if } \bar{\pi}^n(i) \neq i \\ i & \text{if } \bar{\pi}^n(i) = i. \end{cases} \tag{1.9}$$

We then introduce an $(\mathcal{I} \boxtimes \mathcal{F})$-measurable agent type function α^n and an $(\mathcal{I} \boxtimes \mathcal{F})$-measurable partner function g^n with

$$g^n(i,\omega) = \alpha^n(\pi^n(i,\omega),\omega), \quad n \geq 1,$$

for all $(i,\omega) \in I \times \Omega$. Given $\tilde{\omega} \in \tilde{\Omega}$, $\check{\check{p}}^n \in \hat{\Delta}$, for any $k_1, k_2, l_1, l_2 \in S$ and $r \in S \cup \{J\}$, for λ-almost every agent i, and for $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega}$, we set

$$\hat{P}^{\tilde{\omega}} \left(\alpha_i^n(\tilde{\omega},\cdot) = l_1, g_i^n(\tilde{\omega},\cdot) = r | \bar{\alpha}_i^n(\tilde{\omega},\cdot) = k_1, \bar{\bar{g}}_i^n(\tilde{\omega},\cdot) = J \right)(\hat{\omega}) = \delta_{k_1}(l_1)\delta_J(r), \tag{1.10}$$

$$\hat{P}^{\tilde{\omega}} \left(\alpha_i^n(\tilde{\omega},\cdot) = l_1, g_i^n(\tilde{\omega},\cdot) = l_2 | \bar{\alpha}_i^n(\tilde{\omega},\cdot) = k_1, \bar{\bar{g}}_i^n(\tilde{\omega},\cdot) = k_2, \check{\check{p}}^n(\tilde{\omega},\cdot) \right)(\hat{\omega})$$

$$= \left(1 - \xi_{k_1k_2}(\tilde{\omega}, n, \check{\tilde{p}}^n(\tilde{\omega}, \hat{\omega}))\right) \sigma_{k_1k_2}[l_1, l_2](\tilde{\omega}, n, \check{\tilde{p}}^n(\tilde{\omega}, \hat{\omega})), \tag{1.11}$$

$$\begin{aligned} &\hat{P}^{\tilde{\omega}} \left(\alpha_i^n(\tilde{\omega}, \cdot) = l_1, g_i^n(\tilde{\omega}, \cdot) = J | \bar{\alpha}_i^n(\tilde{\omega}, \cdot) = k_1, \bar{\bar{g}}_i^n(\tilde{\omega}, \cdot) = k_2, \check{\tilde{p}}^n(\tilde{\omega}, \cdot)\right)(\hat{\omega}) \\ &= \xi_{k_1k_2}(\tilde{\omega}, n, \check{\tilde{p}}^n(\tilde{\omega}, \hat{\omega})) \varsigma_{k_1k_2}^n[l_1](\tilde{\omega}, n, \check{\tilde{p}}^n(\tilde{\omega}, \hat{\omega})). \end{aligned} \tag{1.12}$$

The extended-type function at the end of the period is

$$\beta^n(\omega) = (\alpha^n(\omega), g^n(\omega)), \quad n \geq 1.$$

We denote the extended type distribution at the end of period n realized in $\omega \in \Omega$ by $\hat{p}^n(\omega) = (\hat{p}^n(\omega)[k, l])_{k \in S, l \in S \cup J}$, where

$$\hat{p}^n(\omega)[k, l] := \lambda(\{i \in I : \alpha^n(i, \omega) = k, g^n(i, \omega) = l\}). \tag{1.13}$$

Furthermore, the definition of Markov conditionally independent (MCI) dynamical system is provided in Definition 3.8 in [1]. We work under the following assumption, which is Assumption 3.9 in [1].

Assumption 1.2. *Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be the probability space introduced. We assume that there exists its corresponding hyperfinite internal probability space, which we denote from now on also by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by a slight notational abuse.*

As already pointed out in [1], the proofs of the results below follow by analogous arguments as in [2] which is possible due to the product structure of the space Ω in (1.1) and the Markov kernel P in (1.2). As in [2] we use some concepts and notations from nonstandard analysis. Note here that an object with an upper left star means the transfer of a standard object to the nonstandard universe. For a detailed overview of the necessary tools of nonstandard analysis, we refer to Appendix D.2. in [2].

2 Proof of Theorem 3.13 in [1]

From now on, we fix the hyperfinite internal space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, along with the input functions $(\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}$ from $\tilde{\Omega} \times \mathbb{N} \times \Delta$ to $[0, 1]$ introduced above. Given this framework we prove the existence of a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, on which a dynamical system $\mathbb{D}$ described in Definition 1.1 for such input probabilities is defined. More specifically, we are going to construct the space $\hat{\Omega}$ and the probability measure $\hat{P}$ such that $\Omega = \tilde{\Omega} \times \hat{\Omega}$ and $P = \tilde{P} \ltimes \hat{P}$ is a Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$.
We now present and prove Theorem 3.13 in [1]. The proof is based on Proposition 3.12 in [1], which focuses on the random matching step and shows the existence of a suitable hyperfinite probability space and partial matching, generalizing Lemma 7 in [2].

Theorem 2.1. *Let Assumption 3.9 in [1] hold and $(\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}$ be the input functions from $\tilde{\Omega} \times \mathbb{N} \times \hat{\Delta}$ defined in Section 3 in [1]. Then for any extended type distribution $\ddot{p} \in \hat{\Delta}$ and any deterministic initial condition $\Pi^0 = (\alpha^0, \pi^0)$ there exists a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which a discrete dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$ as in Definition 3.6 in [1] can be constructed with discrete time input processes $(\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)_{n \geq 1}$ coming from $(\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k,l,r,s \in S \times S \times S \times S}$ as stated in Section 2 in [1]. In particular,*

$$\Omega = \tilde{\Omega} \times \hat{\Omega}, \quad \mathcal{F} = \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}, \quad P = \tilde{P} \ltimes \hat{P},$$

where $(\hat{\Omega}, \hat{\mathcal{F}})$ is a measurable space and $\hat{P}$ a Markov kernel from $\tilde{\Omega}$ to $\hat{\Omega}$. The dynamical system $\mathbb{D}$ is also MCI according to Definition 3.8 in [1] and with initial cross-sectional extended type distribution $\hat{p}^0$ equal to $\ddot{p}^0$ with probability one.

Proof. At each time period we construct three internal measurable spaces with internal transition probabilities taking into account the following steps:

1. random mutation
2. random matching
3. random type changing with break-up.

Let M be a limited hyperfinite number in ${}^*\mathbb{N}_\infty$. Let $\{n\}_{n=0}^{M}$ be the hyperfinite discrete time line and $(I, \mathcal{I}_0, \lambda_0)$ the agent space, where $I = \{1, ..., \hat{M}\}$, $\mathcal{I}_0$ is the internal power set on I, λ_0 is the internal counting probability measure on $\mathcal{I}_0$, and $\hat{M}$ is an unlimited hyperfinite number in ${}^*\mathbb{N}_\infty$.
We start by transferring the deterministic functions[1] $\eta(0,\cdot), \theta(0,\cdot), \xi(0,\cdot), \sigma(0,\cdot), \varsigma(0,\cdot) : \hat{\Delta} \to [0,1]$ to the nonstandard universe. In particular, we denote by ${}^*\theta_{kl}^0$ for any $k,l \in S$ and by ${}^*f^0$ for $f = \eta, \xi, \sigma, \varsigma$ the internal functions from ${}^*\hat{\Delta}$ to $[0,1]$. We also let $\hat{\theta}_{kl}^0(\hat{\rho}) = {}^*\hat{\theta}_{kl}^0(\hat{\rho})$ and $\hat{b}_k^0 = 1 - \sum_{l \in S} \hat{\theta}_{kl}^0(\hat{\rho})$ for any $k,l \in S$ and $\hat{\rho} \in {}^*\hat{\Delta}$, with $1 \in {}^*\mathbb{N}$.
We start at $n = 0$. To do so, we introduce the trivial probability space over the single set $\{0\}$ denoted by $(\bar{\Omega}_0, \bar{\mathcal{F}}_0, \bar{Q}_0)$. Let $\{A_{kl}\}_{(k,l)\in\hat{S}}$ be an internal partition of I such that $\frac{|A_{kl}|}{\hat{M}} \simeq \ddot{p}_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$, such that $|A_{kk}|$ is even for any $k,l \in S$ and $|A_{kl}| = |A_{lk}|$ for any $k,l \in S$. Let α^0 be an internal function from $(I, \mathcal{I}_0, \lambda_0)$ to S such that $\alpha^0(i) = k$ if $i \in \bigcup_{l \in S\cup\{J\}} A_{kl}$. Let π^0 be an internal partial matching from I to I such that $\pi^0(i) = i$ on $\bigcup_{k\in S} A_{kJ}$, and the restriction $\pi^0|_{A_{kl}}$ is an internal bijection from A_{kl} to A_{lk} for any $k,l \in S$. Let

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i \\ J & \text{if } \pi^0(i) = i. \end{cases}$$

It is clear that $\lambda_0(\{i : \alpha^0(i) = k, g^0(i) = l\}) \simeq \ddot{p}_{kl}^0$ for any $k \in S$ and $l \in S \cup \{J\}$.
Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ be the hyperfinite internal space. Since the intensities are supposed to be deterministic at initial time, the Markov kernel from $\tilde{\Omega}$ is trivial and we define the initial internal product probability space as

$$(\Omega_0, \mathcal{F}_0, Q_0) := (\tilde{\Omega} \times \bar{\Omega}_0, \tilde{\mathcal{F}} \otimes \bar{\mathcal{F}}_0, \tilde{P} \otimes \bar{Q}_0).$$

Suppose now that the dynamical system $\mathbb{D}$ has been constructed up to time $n - 1 \in {}^*\mathbb{N}$ for $n \geq 1$, i.e., that the sequences $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=0}^{3n-3}$ and $\{\alpha^l, \pi^l\}_{l=0}^{n-1}$ have been constructed. In particular, we assume to have introduced the spaces $(\hat{\Omega}_m, \hat{\mathcal{F}}_m)$ and the Markov kernel $\hat{P}_m$ from $\tilde{\Omega}$ to $\hat{\Omega}_m$ for any $m = 1, \ldots, n-3$, so that we can define $\Omega_m := \tilde{\Omega} \times \hat{\Omega}_m$ as a hyperfinite internal set with internal power set $\mathcal{F}_m := \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}_m$ and $Q_m := \tilde{P} \ltimes \hat{P}_m$ as an internal transition probability from Ω^{m-1} to $(\Omega_m, \mathcal{F}_m)$, where

$$\Omega^m := \tilde{\Omega} \times \hat{\Omega}^m, \quad \hat{\Omega}^m := \bar{\Omega}_0 \times \prod_{j=1}^{m} \hat{\Omega}_j, \quad \hat{\mathcal{F}}^m := \bar{\mathcal{F}}_0 \otimes \left(\otimes_{j=1}^m \hat{\mathcal{F}}_j\right) \quad \text{and} \quad \mathcal{F}^m = \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}^m. \tag{2.1}$$

[1] Note that at initial time, the functions are supposed to be deterministic and in particular independent of $\tilde{\Omega}$.

In this setting, α^l is an internal type function from $I \times \Omega^{3l-1}$ to the space S, and π^l an internal random matching from $I \times \Omega^{3l}$ to I, such that

$$\alpha^l(i,(\tilde{\omega},\hat{\omega}^{3l-1})) = \alpha^l(i,\hat{\omega}^{3l-1}), \quad \text{for any } (\tilde{\omega},\hat{\omega}^{3l-1}) \in \Omega^{3l-1}$$

and

$$\pi^l(i,(\tilde{\omega},\hat{\omega}^{3l})) = \pi^l(i,\hat{\omega}^{3l}), \quad \text{for any } (\tilde{\omega},\hat{\omega}^{3l}) \in \Omega^{3l}.$$

Given $\omega^{3l} \in \Omega^{3l}$ we denote by $\pi^l_{\hat{\omega}^{3l}} : I \to I$ the function given by

$$\pi^l_{\hat{\omega}^{3l}}(i) := \pi^l(i,(\tilde{\omega},\hat{\omega}^{3l})) = \pi^l(i,\hat{\omega}^{3l}).$$

A similar notation will be used for $\alpha^l_{\hat{\omega}^{3l}} : I \to S$. We now have the following.

(i) Random mutation step:

We let $\hat{\Omega}_{3n-2} := S^I$, which is the space of all internal functions from I to S, and denote its internal power set by $\hat{\mathfrak{F}}_{3n-2}$. For each $i \in I$ and $\omega^{3n-3} = (\tilde{\omega},\hat{\omega}^{3n-3}) \in \Omega^{3n-3}$, if $\alpha^{n-1}(i,\omega^{3n-3}) = \alpha^{n-1}\left(i,\hat{\omega}^{3n-3}\right) = k$, define a probability measure $\gamma_i^{\tilde{\omega},\hat{\omega}^{3n-3}}$ on S by letting $\gamma_i^{\tilde{\omega},\hat{\omega}^{3n-3}}(l) := \theta_{kl}(\tilde{\omega},n,\hat{\rho}^{n-1}_{\hat{\omega}^{3n-3}})$ for each $l \in S$ with

$$\hat{\rho}^{n-1}_{\hat{\omega}^{3n-3}}[k,r] := \lambda_0(\{i \in I : \alpha^n_{\hat{\omega}^{3n-3}}(i) = k, \alpha^n_{\hat{\omega}^{3n-3}}(\pi^n_{\hat{\omega}^{3n-3}}(i)) = r\}), \quad k,r \in S$$

and

$$\hat{\rho}^{n-1}_{\hat{\omega}^{3n-3}}[k,J] := \lambda_0(\{i \in I : \alpha^n_{\hat{\omega}^{3n-3}}(i) = k, \pi^n_{\hat{\omega}^{3n-3}}(i) = i\}), \quad k \in S.$$

Define a Markov kernel $\hat{P}^{\hat{\omega}^{3n-3}}_{3n-2}$ from $\tilde{\Omega}$ to $\hat{\Omega}_{3n-2}$ by letting $\hat{P}^{\hat{\omega}^{3n-3}}_{3n-2}(\tilde{\omega})$ be the internal product measure $\prod_{i\in I}\gamma_i^{\tilde{\omega},\hat{\omega}^{3n-3}}$. Define $\bar{\alpha}^n : \left(I \times \Omega^{3n-2}\right) \to S$ by

$$\bar{\alpha}^n(i,(\tilde{\omega},\hat{\omega}^{3n-2})) := \bar{\alpha}^n(i,\hat{\omega}^{3n-2}) =: \hat{\omega}_{3n-2}(i)$$

and $\bar{g}^n : \left(I \times \Omega^{3n-2}\right) \to S \cup \{J\}$ by

$$\bar{g}^n(i,(\tilde{\omega},\hat{\omega}^{3n-2})) := \bar{g}^n(i,\hat{\omega}^{3n-2}) := \begin{cases} \bar{\alpha}^n(\pi^{n-1}(i,\hat{\omega}^{3n-3}),\hat{\omega}^{3n-2}) & \text{if } \pi^{n-1}(i,\hat{\omega}^{3n-3}) \neq i \\ J & \text{if } \pi^{n-1}(i,\hat{\omega}^{3n-3})) = i. \end{cases}$$

Moreover, we introduce the notation

$$\bar{\alpha}^n_{\hat{\omega}^{3n-2}}(\cdot) : I \to S, \quad \bar{\alpha}^n_{\hat{\omega}^{3n-2}}(i) := \bar{\alpha}^n(i,(\tilde{\omega},\hat{\omega}^{3n-2})) {:=} \bar{\alpha}^n(i,\hat{\omega}^{3n-2})$$

for the type function. We then define $\pi^{n-1}_{\hat{\omega}^{3n-3}}(\cdot) : I \to I$ and $g^n_{\hat{\omega}^{3n-2}} : I \to S \cup \{J\}$ analogously. Finally, we define the cross-internal extended type distribution after random mutation $\check{\rho}^n_{\hat{\omega}^{3n-2}}$ by

$$\check{\rho}^n_{\hat{\omega}^{3n-2}}[k,l] := \lambda_0(\{\in I : \bar{\alpha}^n_{\hat{\omega}^{3n-2}}(i) = k, \bar{g}^n_{\hat{\omega}^{3n-2}}(i) = l\}), \quad k,l \in S.$$

(ii) Directed random matching:

Let $(\hat{\Omega}_{3n-1},\hat{\mathfrak{F}}_{3n-1})$ and $\hat{P}^{\hat{\omega}^{3n-2}}_{3n-1}$ be the measurable space and the Markov kernel, respectively, provided by Proposition 3.12 in [1], with type function $\bar{\alpha}^n_{\hat{\omega}^{3n-2}}(\cdot)$ and partial matching function $\pi^{n-1}_{\hat{\omega}^{3n-3}}(\cdot)$, for fixed matching probability function $\theta\left(\cdot,n,\check{\rho}^n_{\hat{\omega}^{3n-2}}\right)$. Proposition 3.12 in [1] also provides the directed random matching

$$\pi_{\theta^n\left(\cdot,\check{\rho}^n_{\hat{\omega}^{3n-2}}\right),\bar{\alpha}^n_{\hat{\omega}^{3n-2}},\pi^{n-1}_{\hat{\omega}^{3n-3}}},$$

which is a function defined on $(\Omega_{3n-1}, \mathcal{F}_{3n-1})$ by

$$\pi_{\theta^n(\cdot,\check{\rho}^n_{\hat{\omega}^{3n-2}}),\bar{\alpha}^n_{\hat{\omega}^{3n-2}},\pi^{n-1}_{\hat{\omega}^{3n-3}}}(i,(\tilde{\omega},\hat{\omega}_{3n-1})) := \pi_{\theta^n(\cdot,\check{\rho}^n_{\hat{\omega}^{3n-2}}),\bar{\alpha}^n_{\hat{\omega}^{3n-2}},\pi^{n-1}_{\hat{\omega}^{3n-3}}}(i,\hat{\omega}_{3n-1}).$$

We then define $\bar{\pi}^n : \left(I \times \Omega^{3n-1}\right) \to I$ by

$$\bar{\pi}^n(i,(\tilde{\omega},\hat{\omega}_{3n-1})) := \bar{\pi}^n(i,\hat{\omega}^{3n-1}) := \pi_{\theta^n(\cdot,\check{\rho}^n_{\hat{\omega}^{3n-2}}),\bar{\alpha}^n_{\hat{\omega}^{3n-2}},\pi^{n-1}_{\hat{\omega}^{3n-3}}}(i,\hat{\omega}_{3n-1})$$

and

$$\bar{g}^n(i,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{g}^n(i,\hat{\omega}^{3n-1}) := \begin{cases} \bar{\alpha}^n(\bar{\pi}^n(i,\hat{\omega}^{3n-1}),\hat{\omega}^{3n-2}) & \text{if } \bar{\pi}^n(i,\hat{\omega}^{3n-1}) \neq i \\ J & \text{if } \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = i. \end{cases}$$

Define now the cross-internal extended type distribution after the random matching $\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}$ by

$$\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}[k,l] := \lambda_0(\{\in I : \bar{\alpha}^n_{\hat{\omega}^{3n-1}}(i) = k, \bar{g}^n_{\hat{\omega}^{3n-1}}(i) = l\}).$$

(iii) Random type changing with break-up for matched agents:
Introduce $\hat{\Omega}_{3n} := (S \times \{0,1\})^I$ with internal power set $\hat{\mathcal{F}}_{3n}$, where 0 represents "unmatched" and 1 represents "paired"; each point $\hat{\omega}_{3n} = (\hat{\omega}^1_{3n}, \hat{\omega}^2_{3n}) \in \hat{\Omega}_{3n}$ represents an internal function from I to $S \times \{0,1\}$. Define a new type function $\alpha^n : (I \times \Omega^{3n}) \to S$ by letting $\alpha^n(i,(\tilde{\omega},\hat{\omega}^{3n})) := \alpha^n(i,\hat{\omega}^{3n}) = \hat{\omega}^1_{3n}(i)$. Fix $(\tilde{\omega},\hat{\omega}^{3n-1}) \in \Omega^{3n-1}$. For each $i \in I$, we proceed in the following way.

1. If $\bar{\pi}^n(i,\hat{\omega}^{3n-1}) = i$ (i is not paired after the matching step at time n), let $\tau_i^{\tilde{\omega},\hat{\omega}^{3n-1}}$ be the probability measure on the type space $S \times \{0,1\}$ that gives probability one to the type $(\bar{\alpha}^n(i,(\tilde{\omega},\hat{\omega}^{3n-2})),0) = (\bar{\alpha}^n(i,\hat{\omega}^{3n-2}),0)$ and zero to the rest

2. If $\bar{\pi}^n(i,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = j \neq i$ (i is paired after the matching step at time n), $\bar{\alpha}^n(i,(\tilde{\omega},\hat{\omega}^{3n-2})) = \bar{\alpha}^n(i,\hat{\omega}^{3n-2}) = k$, $\bar{\pi}^n(i,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = j$ and $\bar{\alpha}^n(j,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{\alpha}^n(j,\hat{\omega}^{3n-1}) = l$, define a probability measure $\tau_{ij}^{\tilde{\omega},\hat{\omega}^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ as

$$\tau_{ij}^{\tilde{\omega},\hat{\omega}^{3n-1}}((k',0),(l',0)) := \left(1 - \xi_{kl}(\tilde{\omega},n,\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}})\right) \varsigma_{kl}[k'] \left(\tilde{\omega},n,\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}\right) \varsigma_{lk}[l'] \left(\tilde{\omega},n,\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}\right)$$

and

$$\tau_{ij}^{\tilde{\omega},\hat{\omega}^{3n-1}}\left((k',1),(l',1)\right) := \xi_{kl}\left(\tilde{\omega},n,\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}\right) \sigma_{kl}[k',l'] \left(\tilde{\omega},n,\check{\check{\rho}}^n_{\hat{\omega}^{3n-1}}\right)$$

for $k', l' \in S$, and zero for the rest.

Let $A^n_{\hat{\omega}^{3n-1}} = \{(i,j) \in I \times I : i < j, \bar{\pi}^n(i,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = j\}$ and $B^n_{\hat{\omega}^{3n-1}} = \{i \in I : \bar{\pi}^n(i,(\tilde{\omega},\hat{\omega}^{3n-1})) = \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = i\}$. Define a Markov kernel $\hat{P}_{3n}^{\hat{\omega}^{3n-1}}$ from $\tilde{\Omega}$ to $\hat{\Omega}^{3n}$ by

$$\hat{P}_{3n}^{\hat{\omega}^{3n-1}}(\tilde{\omega}) := \prod_{i \in B^n_{\tilde{\omega},\hat{\omega}^{3n-1}}} \tau_i^{\tilde{\omega},\hat{\omega}^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\hat{\omega}^{3n-1}}} \tau_{ij}^{\tilde{\omega},\hat{\omega}^{3n-1}}.$$

Let

$$\begin{aligned} \pi^n(i,(\tilde{\omega},\hat{\omega}^{3n})) &= \pi^n(i,\hat{\omega}^{3n}) \\ &:= \begin{cases} J & \text{if } \bar{\pi}^n(i,\hat{\omega}^{3n-1}) = J \text{ or } \hat{\omega}^2_{3n}(i) = 0 \text{ or } \hat{\omega}^2_{3n}(\bar{\pi}^n(i,\hat{\omega}^{3n-1})) = 0 \\ \bar{\pi}^n(i,\hat{\omega}^{3n-1}) & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$g^n(i,(\tilde{\omega},\hat{\omega}^{3n})) = g^n(i,\hat{\omega}^{3n}) := \begin{cases} \alpha^n(\pi^n(i,\hat{\omega}^{3n}),\hat{\omega}^{3n}) & \text{if } \pi^n(i,\hat{\omega}^{3n}) \neq i \\ J & \text{if } \pi^n(i,\hat{\omega}^{3n}) = i. \end{cases}$$

Define $\hat{\rho}^n_{\hat{\omega}^{3n}} = \lambda_0(\alpha^n_{\hat{\omega}^{3n}}, \pi^n_{\hat{\omega}^{3n}})^{-1}$.

By repeating this procedure, we construct a hyperfinite sequence of internal transition probability spaces $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=0}^{3M}$ and a hyperfinite sequence of internal type functions and internal random matchings $\{(\alpha^n,\pi^n)\}_{n=0}^{M}$. Moreover, define $(\Omega^m,\mathcal{F}^m)$ as in (2.1), and

$$\hat{P}^m := \prod_{i=1}^{m} \hat{P}_i, \qquad Q^m := \tilde{P} \ltimes \hat{P}^m,$$

where the product of the Markov kernels is $\tilde{\omega}$-wise.
Let $(I\times\Omega^{3M}, \mathcal{I}_0\otimes\mathcal{F}^{3M}, \lambda_0\otimes Q^{3M})$ be the internal product probability space of $(I,\mathcal{I}_0,\lambda_0)$ and $(\Omega^{3M},\mathcal{F}^{3M},Q^{3M})$. Denote the Loeb spaces of $(\Omega^{3M},\mathcal{F}^{3M},Q^{3M})$ and the internal product $(I\times\Omega^{3M}, \mathcal{I}_0\otimes\mathcal{F}^{3M}, \lambda_0\otimes Q^{3M})$ by $(\Omega^{3M},\mathcal{F},P)$ and $(I\times\Omega^{3M},\mathcal{I}\boxtimes\mathcal{F},\lambda\boxtimes P)$, respectively. For simplicity, let Ω^{3M} be denoted by Ω and $\hat{\Omega}^{3M}$ by $\hat{\Omega}$. Denote now Q^{3M} by P and the Markov kernel $\hat{P}^{3M}$ by $\hat{P}$.
The properties of a dynamical system as well as the independence conditions follow now by applying similar arguments as in the proof of Theorem 5 in [2] for any fixed $\tilde{\omega}\in\tilde{\Omega}$. The only difference is that in our setting the input processes for the random mutation step and the break-up step also depend on the extended type distribution. Furthermore, these arguments are similar to the ones in the proof of Lemma 3.2 and can be found there with all details. □

3 Proof of Theorem 3.14 in [1]

We now prove Theorem 3.14 in [1] which is a generalization of the results in Appendix C in [2]. For $n \geq 1$ we define the mapping Γ^n from $\tilde{\Omega}\times\hat{\Delta}$ to $\hat{\Delta}$ by

$$\begin{aligned}\Gamma^n_{kl}(\tilde{\omega},\hat{p}) = &\sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^n))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^n\right)\tilde{p}^n_{k_1l_1} \\ &+\sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^n))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^n\right)\theta_{k_1l_1}(\tilde{\omega},n,\tilde{p}^n)\tilde{p}^n_{k_1J}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\Gamma^n_{kJ}(\tilde{\omega},\hat{p}) = b_k(\tilde{\omega},n,\tilde{p}^n)\tilde{p}^n_{kJ} + &\sum_{k_1,l_1\in S}\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^n)\varsigma_{k_1l_1}[k](\tilde{\omega},\tilde{\tilde{p}}^n)\tilde{p}^n_{k_1l_1} \\ +&\sum_{k_1,l_1\in S}\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^n)\varsigma_{k_1l_1}[k](\tilde{\omega},n,\tilde{\tilde{p}}^n)\theta_{k_1l_1}(\tilde{\omega},n,\tilde{p}^n)\tilde{p}^n_{k_1J} \end{aligned} \tag{3.2}$$

with

$$\begin{aligned}\tilde{p}^n_{kl} &= \sum_{k_1,l_1\in S}\eta_{k_1k}(\tilde{\omega},n,\hat{p})\eta_{l_1l}(\tilde{\omega},n,\hat{p})\hat{p}_{k_1l_1} \\ \tilde{p}^n_{kJ} &= \sum_{l\in S}\hat{p}_{lJ}\eta_{lk}(\tilde{\omega},n,\hat{p})\end{aligned}$$

and

$$\tilde{\tilde{p}}_{kl}^n = \tilde{p}_{kl}^n + \theta_{kl}(\tilde{\omega}, n, \tilde{p}^n)\tilde{p}_{kJ}^n$$
$$\tilde{\tilde{p}}_{kJ}^n = b_k(\tilde{\omega}, n, \tilde{p}^n)\tilde{p}_{kJ}^n.$$

Theorem 3.14 in [1] is proven with the help of the following lemmas.

Lemma 3.1. *Assume that the discrete dynamical system $\mathbb{D}$ defined in Definition 3.6 in [1] is Markov conditionally independent given $\tilde{\omega}$ as defined in Definition 3.8 in [1]. Then given $\tilde{\omega} \in \tilde{\Omega}$, the discrete time processes $\{\beta_i^n\}_{n=0}^{\infty}, i \in I$, are essentially pairwise independent on $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\tilde{\omega}})$. Moreover, for fixed $n = 1, ..., M$ also $(\bar{\beta}_i^n)_{n=0}^{\infty}$ and $(\bar{\bar{\beta}}_i^n)_{n=0}^{\infty}, i \in I$, are essentially pairwise independent on $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathcal{F}}, \lambda \boxtimes \hat{P}^{\tilde{\omega}})$.*

Proof. This can be proven by the same arguments used in the proof of Lemma 3 in [2]. □

We now derive a result which shows how to compute for a fixed $\tilde{\omega} \in \tilde{\Omega}$ the expected cross-sectional distributions $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n]$, $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}^n]$ and $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n]$.

Lemma 3.2. *The following holds for any fixed $\tilde{\omega} \in \tilde{\Omega}$.*

1. *For each $n \geq 1$, $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n] = \Gamma^n(\tilde{\omega}, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])$, with Γ defined in (3.1).*

2. *For each $n \geq 1$, we have*

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kl}^n] = \sum_{k_1, l_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l_1,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}_{k_1,l_1}^{n-1}]$$

and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n] = \sum_{k_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}_{k_1,J}^{n-1}].$$

3. *For each $n \geq 1$, we have*

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}_{kl}^n] = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kl}^n] + \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n]$$

and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}_{kJ}^n] = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}_{kJ}^n].$$

Proof. Fix $\tilde{\omega} \in \tilde{\Omega}$ and $k, l \in S$. By Lemma 3.1 we know that the processes $(\beta_i^n)_{n=0}^{\infty}, i \in I$, are essentially pairwise independent. Then the exact law of large numbers in Lemma 1 in [2] implies that $\hat{p}^{n-1}(\hat{\omega}) = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\lambda(\beta^{n-1})^{-1}]$ for $\hat{P}$-almost all $\hat{\omega} \in \hat{\Omega}$. Thus equations (1.3) and (1.4) are equivalent to

$$\hat{P}^{\tilde{\omega}}\left(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = l_2 | \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1\right) = \eta_{k_1,k_2}\left(\tilde{\omega}, n, \mathbb{E}^{P^{\tilde{\omega}}}[\hat{p}^{n-1}]\right)\eta_{l_1,l_2}\left(\tilde{\omega}, n, \mathbb{E}^{P^{\tilde{\omega}}}[\hat{p}^{n-1}]\right) \tag{3.3}$$
$$\hat{P}^{\tilde{\omega}}\left(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = r | \alpha_i^{n-1} = k_1, g_i^{n-1} = J\right) = \eta_{k_1,k_2}\left(\tilde{\omega}, n, \mathbb{E}^{P^{\tilde{\omega}}}[\hat{p}^{n-1}]\right)\delta_J(r). \tag{3.4}$$

Therefore, for any $k_1, l_1 \in S$ we have

$$\hat{P}^{\tilde{\omega}}\left(\bar{\beta}_i^n = (k, J) | \beta_i^{n-1} = (k_1, l_1)\right) = 0 \tag{3.5}$$
$$\hat{P}^{\tilde{\omega}}\left(\bar{\beta}_i^n = (k, l) | \beta_i^{n-1} = (k_1, J)\right) = 0. \tag{3.6}$$

Then with the same calculations as in the proof of Lemma 4 in [2] we get that

$$\begin{aligned}
\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kl}] &= \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\lambda(i \in I : \bar{\beta}^n_{\omega}(i) = (k,l))] \\
&= \int_I \hat{P}^{\tilde{\omega}}(\bar{\beta}^n_i = (k,l)) d\lambda(i) \\
&= \sum_{k_1,l_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l_1,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k_1 l_1}] \qquad (3.7)
\end{aligned}$$

and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kJ}] = \sum_{k_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{kJ}]. \qquad (3.8)$$

By Lemma 3.1 we know that $\bar{\beta}^n$ is essentially pairwise independent. Again it follows by the exact law of large numbers that $\breve{p}^n(\hat{\omega}) = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]$ for $\hat{P}^{\tilde{\omega}}$-almost all $\hat{\omega} \in \hat{\Omega}$. Then (1.6) and (1.7) are equivalent to

$$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n = l | \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]) \qquad (3.9)$$

$$\hat{P}^{\tilde{\omega}}(\bar{\bar{g}}^n = J | \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]). \qquad (3.10)$$

By the same calculations as in the proof of Lemma 4 in [2] we have

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n_{kl}] = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kl}] + \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kJ}] \qquad (3.11)$$

and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n_{kJ}] = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kJ}]. \qquad (3.12)$$

By Lemma 3.1, $\bar{\bar{\beta}}^n$ is essentially pairwise independent and thus $\breve{\breve{p}}^n(\hat{\omega}) = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]$ for $\hat{P}$-almost all $\hat{\omega} \in \hat{\Omega}$. Then (1.11) and (1.12) are equivalent to

$$\hat{P}^{\tilde{\omega}}(\alpha^n_i = l_1, g^n_i = l_2 | \alpha^n_i = k_1, \bar{\bar{g}}^n_i = k_2) = (1 - \xi_{k_1 k_2}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]))\sigma_{k_1 k_2}[l_1, l_2]\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]\right)$$

and

$$\hat{P}^{\tilde{\omega}}(\alpha^n_i = l_1, g^n_i = J | \alpha^n_i = k_1, \bar{\bar{g}}^n_i = k_2) = \xi_{k_1 k_2}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n])\varsigma_{k_1 k_2}[l_1, l_2]\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]\right),$$

respectively. Thus

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kl}] = \sum_{k_1,l_1 \in S} (1 - \xi_{k_1 l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]))\sigma_{k_1 l_1}[k,l]\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]\right)\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n_{k_1 l_1}] \qquad (3.13)$$

and

$$\begin{aligned}
\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kJ}] &= \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n_{kJ}] \\
&\quad + \sum_{k_1,l_1 \in S} \xi_{k_1 l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n])\varsigma_{k_1 l_1}[k]\left(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n]\right)\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{\breve{p}}^n_{k_1 l_1}]. \qquad (3.14)
\end{aligned}$$

By plugging (3.8) in (3.13) we get

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kl}]$$

$$= \sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]\right)\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{k_1l_1}]$$

$$+\sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]\right)\eta_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{k_1J}]. \tag{3.15}$$

By using (3.12) and (3.13), it follows that

$$\begin{aligned}\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kJ}] &= b_k(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{kJ}]\\ &\quad+\sum_{k_1,l_1\in S}\xi_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n])\varsigma_{k_1l_1}[k]\left(\tilde{\omega},\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]\right)\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{k_1l_1}]\\ &\quad+\sum_{k_1,l_1\in S}\xi_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n])\varsigma_{k_1l_1}[k]\left(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n]\right)\theta_{k_1l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\breve{p}^n_{k_1J}].\end{aligned} \tag{3.16}$$

□

Lemma 3.3. *Assume that the discrete dynamical system $\mathbb{D}$ defined in Definition 3.6 in [1] is Markov conditionally independent given $\tilde{\omega}\in\tilde{\Omega}$ according to Definition Definition 3.8 in [1]. Then for fixed $\tilde{\omega}\in\tilde{\Omega}$ the following holds:*

1. *For λ-almost all $i\in I$, the extended type process $\{\beta^n_i\}_{n=0}^\infty$ for agent i is a Markov chain on $(I\times\hat{\Omega},\mathcal{I}\boxtimes\hat{\mathcal{F}},\lambda\boxtimes\hat{P}^{\tilde{\omega}})$ with transition matrix z^n after time $n-1$.*
2. *$\{\beta^n\}_{n=0}^\infty$ is also a Markov chain with transition matrix z^n at time $n-1$.*

Proof. Fix $\tilde{\omega}\in\tilde{\Omega}$.
1. The Markov property of $\{\beta^n_i\}_{n=0}^\infty$ on $(I\times\hat{\Omega},\mathcal{I}\boxtimes\hat{\mathcal{F}},\lambda\boxtimes\hat{P}^{\tilde{\omega}})$ follows by using the same arguments as in the proof of Lemma 5 in [2], for λ-almost all $i\in I$. We now derive the transition matrix with similar arguments as in [2]. By putting together (3.7), (3.8) and (3.15), we get

$$\begin{aligned}&\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kl}]\\ &=\sum_{k_1,l_1,k',l'\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}\right)\eta_{k'k_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\\ &\qquad\cdot\eta_{l'l_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k'l'}]\\ &\quad+\sum_{k_1,l_1,k'\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}\right)\theta_{k_1l_1}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\\ &\qquad\cdot\eta_{k'k_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k'J}].\end{aligned}$$

Thus we have

$$\begin{aligned}z^n_{(k'J)(kl)}(\tilde{\omega})&=\sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}\right)\theta_{k_1l_1}(\tilde{\omega},n,\tilde{p}^{\tilde{\omega},n})\\ &\quad\cdot\eta_{k'k_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\end{aligned} \tag{3.17}$$

and

$$z^n_{(k'l')(kl)}(\tilde{\omega})=\sum_{k_1,l_1\in S}(1-\xi_{k_1l_1}(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}))\sigma_{k_1l_1}[k,l]\left(\tilde{\omega},n,\tilde{\tilde{p}}^{\tilde{\omega},n}\right)\eta_{k'k_1}(\tilde{\omega},n,\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])$$

$$\cdot\,\eta_{l'l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]). \tag{3.18}$$

Similarly, equations (3.7), (3.8) and (3.16) yield to

$$\begin{aligned}
\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kJ}] = & \sum_{k'\in S} b_k(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n})\eta_{k'k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k'J}] \\
& + \sum_{k_1,l_1,k',l'\in S} \xi_{k_1l_1}(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n})\varsigma_{k_1l_1}[k]\left[\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right] \\
& \cdot\,\eta_{k'k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l'l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k'l'}] \\
& + \sum_{k_1,l_1,k'\in S} \xi_{k_1l_1}(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n})\varsigma_{k_1l_1}[k]\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right)\theta_{k_1l_1}(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n}) \\
& \cdot\,\eta_{k'k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k'J}].
\end{aligned}$$

Therefore, the transition probabilities from time $n-1$ to time n can be written as

$$\begin{aligned}
z^n_{(k'l')(kJ)}(\tilde{\omega}) = & \sum_{k_1,l_1\in S} \xi_{k_1l_1}(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n})\varsigma_{k_1l_1}[k]\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right) \\
& \cdot\,\eta_{k'k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l'l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
z^n_{(k'J)(kJ)}(\tilde{\omega}) = & b_k(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n})\eta_{k'k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]) \\
& + \sum_{k_1,l_1\in S} \xi_{k_1l_1}(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n})\varsigma_{k_1l_1}[k]\left(\tilde{\omega}, n, \tilde{\tilde{p}}^{\tilde{\omega},n}\right)\theta_{k_1l_1}(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n})\eta_{k'k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]).
\end{aligned} \tag{3.20}$$

2. The transition matrix of $\{\beta^n\}_{n=0}^{\infty}$ at time $n-1$ can be derived by using (3.17)-(3.20) and the Fubini property applied to $\lambda \boxtimes \hat{P}^{\tilde{\omega}}$ for every fixed $\tilde{\omega} \in \tilde{\Omega}$ as in the proof of Lemma 6 in [2]. □

We are now able to prove Theorem 3.14 in [1], which we present here.

Theorem 3.4. *Assume that the discrete dynamical system* $\mathbb{D}$ *introduced in Definition 3.6 in [1] is Markov conditionally independent given* $\tilde{\omega} \in \tilde{\Omega}$ *according to Definition 3.8 in [1]. Given* $\tilde{\omega} \in \tilde{\Omega}$*, the following holds:*

1. *For each* $n \geq 1$, $\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n] = \Gamma^n(\tilde{\omega}, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])$.
2. *For each* $n \geq 1$*, we have*

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kl}] = \sum_{k_1,l_1\in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l_1,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k_1,l_1}]$$

   and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kJ}] = \sum_{k_1\in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}_{k_1,J}].$$

3. *For each* $n \geq 1$*, we have*

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}^n_{kl}] = \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kl}] + \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kJ}]$$

   and

$$\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}^n_{kJ}] = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n])\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kJ}].$$

4. *For λ-almost every agent i, the extended-type process $\{\beta_i^n\}_{n=0}^\infty$ is a Markov chain in $\hat{S}$ on $(I \times \hat{\Omega}, \mathcal{I} \boxtimes \hat{\mathfrak{F}}, \lambda \boxtimes \hat{P}^{\tilde{\omega}})$, whose transition matrix z^n at time $n-1$ is given by*

$$
\begin{aligned}
z^n_{(k'J)(kl)}(\tilde{\omega}) = &\sum_{k_1,l_1,k' \in S} (1 - \xi_{k_1 l_1}(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}))\sigma_{k_1 l_1}[k,l] \left(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}\right) \theta_{k_1 l_1}(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n}) \\
&\cdot \eta_{k' k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
z^n_{(k'l')(kl)}(\tilde{\omega}) = &\sum_{k_1,l_1,k',l' \in S} (1 - \xi_{k_1 l_1}(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}))\sigma_{k_1 l_1}[k,l] \left(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}\right) \eta_{k' k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]) \\
&\cdot \eta_{l' l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
z^n_{(k'l')(kJ)}(\tilde{\omega}) = &\sum_{k_1,l_1 \in S} \xi_{k_1 l_1}(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n})\varsigma_{k_1 l_1}[k] \left(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}\right) \\
&\cdot \eta_{k' k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])\eta_{l' l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}])
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
z^n_{(k'J)(kJ)}(\tilde{\omega}) = &\, b_k(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n})\eta_{k'k}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]) \\
&+ \sum_{k_1,l_1 \in S} \xi_{k_1 l_1}(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n})\varsigma_{k_1 l_1}[k] \left(\tilde{\omega}, n, \tilde{\check{p}}^{\tilde{\omega},n}\right) \theta_{k_1 l_1}(\tilde{\omega}, n, \tilde{p}^{\tilde{\omega},n}) \\
&\cdot \eta_{k' k_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^{n-1}]).
\end{aligned} \tag{3.24}
$$

5. *For λ-almost every i and every λ-almost every j, the Markov chains $\{\beta_i^n\}_{n=0}^\infty$ and $\{\beta_j^n\}_{n=0}^\infty$ are independent on $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P}^{\tilde{\omega}})$.*

6. *For $\hat{P}^{\tilde{\omega}}$-almost every $\hat{\omega} \in \hat{\Omega}$, the cross sectional extended type process $\{\beta_{\hat{\omega}}^n\}_{n=0}^\infty$ is a Markov chain on $(I, \mathcal{I}, \lambda)$ with transition matrix z^n at time $n-1$, which is defined in* (3.21)- (3.24).

7. *We have $\hat{P}^{\tilde{\omega}}$-a.s. that*

$$
\mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{p}^n_{kl}] = \check{p}^n_{kl} \quad \text{and} \quad \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\check{\check{p}}^n_{kl}] = \check{\check{p}}^n_{kl} \quad \text{and} \quad \mathbb{E}^{\hat{P}^{\tilde{\omega}}}[\hat{p}^n_{kl}] = \hat{p}^n_{kl}.
$$

Proof. Fix $\tilde{\omega} \in \tilde{\Omega}$. Points 1. to 5. of Theorem 3.14 in [1] follow directly by Lemma 3.1, 3.2 and 3.3. Moreover, Points 6. and 7. can be proven by using the same arguments as in the proof of Theorem 4 in [2]. □

References

[1] Francesca Biagini, Andrea Mazzon, Thilo Meyer-Brandis, and Katharina Oberpriller. Liquidity based modeling of asset price bubbles via random matching. *preprint*, 2022. URL `http://arxiv.org/abs/2210.13804`.

[2] Darrell Duffie, Lei Qiao, and Yeneng Sun. Dynamic directed random matching. *Journal of Economic Theory*, 174:124–183, 2018.