Analytic Pricing of Volatility-Equity Options within Affine Models: an Efficient Conditioning Technique

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Abstract

We price for different affine stochastic volatility models some derivatives that recently appeared in the market. These products are characterised by payoffs depending on both stock and its volatility. Using a Fourier-analysis approach, we recover in a much simpler way some results already established in the literature for the single factor specification of the volatility and we provide closed-form solution for different products and two multivariate Wishart-based stochastic volatility models. The methodology turns out to be independent of the dimension of the problem thanks to a simple conditioning with respect to the subfiltration generated by the variance path. We implement the formulas for realistic model parameter values and put our results in the broader perspective of model risk. Overall, our results highlight the great flexibility and tractability of Wishart-based stochastic volatility models to develop multivariate extensions of the Heston model.

Keywords: Option Pricing, Target Volatility Options, Corridor Variance Swap, Double Digital Call, Wishart Stochastic Volatility Models.

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1 Introduction

The first generation of equity derivative products had payoffs depending on the stock price, like vanilla options, or the stock price path, like the look back or barrier options. During the nineties the volatility has become an asset class by itself, first by the creation of the volatility index VIX and almost ten years later (around 2004) by the emergence and steady growth of the VIX futures and VIX option markets. More recently, on a large number of indexes were built the corresponding volatility indexes and derivatives, like futures and options, started to be traded. These evolutions have led to the development of exotic volatility derivatives, whose payoffs depend on the volatility path[1], or equity-volatility derivatives, whose payoffs depend explicitly on both the stock and the volatility (path). The growing complexity of the equity-volatility derivative market has created new modelling and implementation challenges.

Among the recent equity-volatility products that have attracted some attention are the target volatility option (TVO), the corridor variance swap (CVS) and the double digital call option (DDC). Within the Heston (1993)'s model closed-form solutions were proposed by several authors[2]. A TVO is a European-type derivative contract whose value at maturity is given by the product of three terms: a vanilla European call, a target volatility parameter representing the investors expectation of the future realized volatility and the inverse of the realized volatility of the underlying. For this products, Di Graziano and Torricelli (2012) and Torricelli (2013) provide within the Heston (1993) framework a pricing methodology based on the Laplace transform, see also recent developments in Torricelli (2014). Grasselli and Marabel Romo (2014) considered in the 2-factor Heston model the pricing of vanilla and forward-starting TVO, that is, TVO where the strike is determined at a later date. A Corridor Variance Swap is a generalisation of a standard variance swap in that the volatility is accumulated only when the underlying stock is within a pre-specified band, see Carr and Lewis (2004). In Zheng and Kwok (2014) the pricing of discrete corridor variance swap is investigated within a jump-diffusion Heston model by using the Fourier transform approach. Lastly, in Torricelli (2013) the DDC is considered within the Heston model and a closed form solution is proposed. All these results crucially exploit the analytical tractability of the Heston model and more generally of the standard affine framework.

[1] Let us just mention, without pretending to be exhaustive, some works from this growing literature; Sepp (2008), Bao et al. (2012), Zhu and Zhang (2007), Shen and Siu (2013) and Lian et al. (2014).

[2] To be more precise some authors consider the Heston model with jumps on the stock and/or the volatility, with the unifying computational framework proposed in Duffie et al. (2000), but for the products considered here the jumps do not introduce any special difficulty. We refer to these jump-diffusion models as the Heston model although in its initial specification it has no jumps

These products illustrate the growing importance of equity-volatility derivatives and underline the need to understand whether they can be priced within more sophisticated frameworks that were recently developed.

The seminal work of Heston was extended to a multivariate stochastic volatility model using the vector affine process in Duffie et al. (2000) (see also Duffie and Kan (1996)) or by using two square root processes as in Christoffersen et al. (2009). Following the introduction in finance of the Wishart process, which is a matrix stochastic process, by Gouriéroux and Sufana (2010) more profound multivariate extensions of the affine model were proposed in Da Fonseca, Grasselli, and Tebaldi (2008) and Da Fonseca, Grasselli, and Tebaldi (2007). The first one is a multivariate stochastic volatility single-stock model while the second one is a multi-asset stochastic volatility and correlation model. These two models allow the computation in closed form of the characteristic function so that efficient option pricing through fast Fourier transform can be performed. However, extending results available for the Heston model to those more sophisticated models is far from being a straightforward task. Depending on the product at hand it may or may not be possible to price in closed form these products within Wishart-based stochastic volatility models. It is therefore of interest to understand when such extensions can be performed.

In this work we propose a general pricing framework for volatility derivatives based on a simple yet powerful approach which combines conditioning with respect to the subfiltration generated by the volatility path and Fourier techniques. This conditioning technique is standard in option pricing, see Leblanc (1996) or Henry-Labordère (2009), but our work will underline its importance for handling multi-factor or multi-asset stochastic volatility models. We provide closed-form solutions for the TVO price based on the Fourier transform much in the spirit of Torricelli (2013). For the corridor variance swap we develop a pricing formula in the spirit of Zheng and Kwok (2014) and for the Double Digital call option we show how a closed-form solution can be obtained. The essential contribution of our work is to explain how these techniques apply to Wishart based stochastic volatility models, either the WASC of Da Fonseca et al. (2007) or the WMSV of Da Fonseca et al. (2008). In these particular cases the closed form solution for the characteristic function turns out to be crucial for an efficient numerical implementation. Within the general affine framework we price these three products for typical parameter values (these values are obtained form a vanilla option calibration procedure). This will allow us to illustrate an important problem of exotic option pricing, namely the issue of model risk, and provide an integrated perspective of exotic option pricing and calibration on vanilla options with important consequences in terms of regulation of derivative products.

The structure of the paper is as follows. In Section 1 we present the different models; in Section 2 we focus on the pricing of TVO and corridor variance swap; in Section 3 we provides a numerical implementation; Section 4 contains the pricing of a digital call; Section 5 provide some open problems illustrating some intrinsic difficulties related to Wishart based models. The last section concludes and we gather all tables in the Appendix.

2 The Models

In this section we briefly review the stochastic volatility models that will be considered in the sequel together with their moment generating functions. We present the Heston, the BiHeston, the WMSV and the WASC models. The first two are well known but are given here for convenience as they will be involved in the numerical experiments. We could have unified the presentation of the Heston and BiHeston models but we prefer to avoid cumbersome notations.

2.1 The Heston (1993) Model

We denote by s_t a stock whose dynamics are given by the following system of stochastic differential equations (SDEs in the sequel):

$$ds_t = s_t r dt + s_t \sqrt{v_t}(\rho dw_{1,t} + \sqrt{1-\rho^2} dw_{2,t}),\ s_0 > 0, \tag{1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dw_{1,t},\ v_0 > 0, \tag{2}$$

where $w_t = (w_{1,t}, w_{2,t})_{t\geq 0}$ is a two-dimensional Brownian motion, $\kappa \in \mathbb{R}$, $\kappa\theta \in \mathbb{R}_+$, $\sigma > 0$ and $\rho \in [-1, 1]$.

The joint moment-generating function, defined by $G_{\text{HES}}(t, z, \lambda_v, \Lambda_v) = \mathbb{E}\left[e^{z \ln s_t + \lambda_v v_t + \Lambda_v \int_0^t v_u du}\right]$[3], is known in closed form. In fact, the affinity of the model leads to the following lemma whose standard proof is omitted.

[3]Hereafter we consider the unconditional moment generating function and we will provide the price at time zero of a payoff maturing at time t. Of course in our homogeneous Markovian setting we can easily adapt the arguments to the conditional moment generating function $G_{\text{HES}}(t, z, \lambda_v, \Lambda_v) = \mathbb{E}_s\left[e^{z \ln s_t + \lambda_v v_t + \Lambda_v \int_0^t v_u du}\right]$, for $s \leq t$.

Lemma 2.1. *The moment generating function of* $(\ln s_t, v_t, \int_0^t v_u du)$ *is given by:*

$$
\begin{aligned}
G_{\text{HES}}(t, z, \lambda_v, \Lambda_v) &= \mathbb{E}\left[e^{z \ln s_t + \lambda_v v_t + \Lambda_v \int_0^t v_u du}\right] \\
&= e^{z \ln s_0 + zrt + A(t)v_0 + b(t)},
\end{aligned}
$$

with the deterministic functions $A(t), b(t)$ *defined as:*

$$
\begin{aligned}
A(t) &= \frac{\eta\lambda_+ e^{-\sqrt{\Gamma}t} + \lambda_-}{\frac{\sigma^2}{2}\left(\eta e^{-\sqrt{\Gamma}t} + 1\right)}, \\
b(t) &= \frac{2\kappa\theta}{\sigma^2}\left(t\lambda_- - \log\left(\frac{\eta e^{-\sqrt{\Gamma}t} + 1}{1+\eta}\right)\right),
\end{aligned}
$$

with

$$
\lambda_\pm = \frac{(\kappa - z\rho\sigma) \pm \sqrt{\Gamma}}{2}; \tag{3}
$$

$$
\Gamma = (\kappa - z\rho\sigma)^2 - \sigma^2(z^2 - z + 2\Lambda_v); \tag{4}
$$

$$
\eta = -\frac{\sigma^2\lambda_v - 2\lambda_-}{\sigma^2\lambda_v - 2\lambda_+}. \tag{5}
$$

2.2 The BiHeston Model

We consider here the Christoffersen et al. (2009) specification of a model where the diffusion term of the asset is described as a combination of two square root processes. This specification is also referred to as the Double Heston, or BiHeston model. The stock price dynamics are defined via the following set of stochastic differential equations:

$$
ds_t = s_t r dt + s_t\left(\sqrt{v_t^0}dZ_t^0 + \sqrt{v_t^1}dZ_t^1\right), \; s_0 > 0, \tag{6}
$$

$$
dv_t^0 = \kappa_0\left(\theta_0 - v_t^0\right)dt + \sigma_0\sqrt{v_t^0}dW_t^0, \; v_0^0 > 0, \tag{7}
$$

$$
dv_t^1 = \kappa_1\left(\theta_1 - v_t^1\right)dt + \sigma_1\sqrt{v_t^1}dW_t^1, \; v_0^1 > 0,, \tag{8}
$$

with $d\langle Z^0, W^0\rangle_t = \rho_0 dt, d\langle Z^1, W^1\rangle_t = \rho_1 dt$, while all other correlations are set to zero in order to grant the analytical tractability of the model [4]. The parameters in (7) and (8) satisfy the following restrictions: $\kappa_i \in \mathbb{R}$, $\kappa_i\theta_i \in \mathbb{R}_+$, $\sigma_i > 0$ and $\rho_i \in [-1, 1]$ for $i = \{0, 1\}$.

[4] In other words, $dW_t^0 dW_t^1 = dZ_t^0 dZ_t^1 = dW_t^0 dZ_t^1 = dW_t^1 dZ_t^0 = 0$ in order to grant the affinity of the infinitesimal generator, see e.g. Da Fonseca et al. (2008).

The joint moment generating function of the asset returns, the variance process $v_t = (v_t^0 + v_t^1)$ and the integrated variance process $V_t = \int_0^t v_u du = \int_0^t (v_u^0 + v_u^1) du$ is given by:

$$G_{2\text{HES}}(t, z, \lambda_{v^0}, \lambda_{v^1}, \Lambda_{v^0}, \Lambda_{v^1}) = \mathbb{E}\left[e^{z \ln s_t + \lambda_{v^0} v_t^0 + \lambda_{v^1} v_t^1 + \int_0^t (\Lambda_{v^0} v_u^0 + \Lambda_{v^1} v_u^1) du}\right].$$

Since the model is affine, it is natural to look for an exponentially affine form, and the next lemma gives the explicit expression for this function:

Lemma 2.2. *The joint moment generating function of* $(\ln s_t, v_t^0, v_t^1, \int_0^t v_u^0 du, \int_0^t v_u^1 du)$ *is given by:*

$$\begin{aligned} G_{2\text{HES}}(t, z, \lambda_{v^0}, \lambda_{v^1}, \Lambda_{v^0}, \Lambda_{v^1}) &= \mathbb{E}\left[e^{z \ln s_t + \lambda_{v^0} v_t^0 + \lambda_{v^1} v_t^1 + \int_0^t (\Lambda_{v^0} v_u^0 + \Lambda_{v^1} v_u^1) du}\right] \\ &= e^{z x_0 + z r t + A^0(t) v_0^0 + b^0(t) + A^1(t) v_0^1 + b^1(t)}, \end{aligned} \tag{9}$$

where the deterministic functions A^j, b^j, $j = 0, 1$, *satisfy:*

$$\begin{aligned} A^j(t) &= \frac{\eta_j \lambda_+^j e^{-\sqrt{\Gamma_j} t} + \lambda_-^j}{\frac{\sigma_j^2}{2}\left(\eta_j e^{-\sqrt{\Gamma_j} t} + 1\right)}, \\ b^j(t) &= \frac{2\kappa_j \theta_j}{\sigma_j^2}\left(t \lambda_-^j - \log\left(\frac{\eta_j e^{-\sqrt{\Gamma_j} t} + 1}{1 + \eta_j}\right)\right), \end{aligned}$$

with

$$\lambda_\pm^j = \frac{(\kappa_j - \rho_j \sigma_j z) \pm \sqrt{\Gamma_j}}{2}, \tag{10}$$

$$\Gamma_j = (\kappa_j - \rho_j \sigma_j z)^2 - \sigma_j^2 (z^2 - z + 2\Lambda_{v^j}), \tag{11}$$

$$\eta_j = -\frac{\sigma_j^2 \lambda_{v^j} - 2\lambda_-^j}{\sigma_j^2 \lambda_{v^j} - 2\lambda_+^j}. \tag{12}$$

This model constitutes a multivariate extension of the Heston model and uses two unrelated square root processes. It would be possible to use instead the vector affine process of Duffie et al. (2000) (see also Duffie and Kan (1996)). However, in that case the moment-generating function would involve Riccati ordinary differential equations that can not be computed in closed form (see Grasselli and Tebaldi (2008) for further details regarding the solvability of these equations) and require the use of numerical schemes implying a much higher computational complexity. Furthermore, if the derivative with respect to a model parameter or the argument of the moment-generating function is needed then the computational burden is even higher. As a consequence, multidmensional extensions of Heston's model can come with a significant (complete) loss of analytical tractability. The fact that the two square root processes cannot be correlated is an intrinsic constraint of the Duffie-Kan affine process and one of the main advantage of the next model is to remove that constraint.

2.3 The WMSV Model

We consider now the Wishart Multidimensional Stochastic Volatility Model (WMSV hereafter) proposed by Da Fonseca, Grasselli, and Tebaldi (2008). This stochastic volatility model extends the original Heston (1993) model to the case where the volatility is described by the Wishart process, a matrix-valued stochastic process introduced by Bru (1991). Within the WMSV model the dynamics for the stock price are given by the following SDE:

$$ds_t = s_t r dt + s_t \mathrm{Tr}\left[\sqrt{\Sigma_t}\left(dW_t R^\top + dB_t\sqrt{\mathbb{I} - RR^\top}\right)\right], \ s_0 > 0, \tag{13}$$

where Tr is the trace operator, $W_t, B_t \in M_n$ (the set of square matrices) are composed by n^2 independent Brownian motions under the risk-neutral measure (B_t and W_t are independent), $R \in M_n$ represents the correlation matrix and Σ_t belongs to the set of symmetric $n \times n$ positive semi-definite matrices. In this specification the volatility is multi-dimensional and depends on the elements of the matrix process Σ_t, which is assumed to satisfy the following dynamics:

$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right) dt + \sqrt{\Sigma_t} dW_t Q + Q^\top (dW_t)^\top \sqrt{\Sigma_t}, \tag{14}$$

with initial condition Σ_0 a strictly positive definite matrix and parameters $\Omega, M \in M_n$, and $Q \in GL(n)$, the set of invertible $n \times n$ matrices.

Equation (14) characterizes the Wishart process investigated by Bru (1991) and then introduced in finance by Gouriéroux and Sufana (2010) and many other authors including Gouriéroux et al. (2009), Grasselli and Tebaldi (2008), Da Fonseca et al. (2008), Da Fonseca et al. (2007)[5]. For an extension of the classical work of Bru (1991), see for example Cuchiero et al. (2011). Existence and uniqueness results for the SDE (14) are provided in Mayerhofer et al. (2011). The Wishart processes represents the matrix analogue of the square root mean-reverting process. In order to grant the the typical mean reverting feature of the volatility, the matrix M is assumed to be negative semi-definite. The constant drift part satisfies $\Omega\Omega^\top = \beta Q^\top Q$ with the real parameter $\beta \geq n - 1$ (see Cuchiero et al. (2011)). If β satisfies the stronger assumption $\beta \geq n + 1$ then the unique strong solution to the SDE (14) evolves as a strictly positive definite matrix, see Mayerhofer et al. (2011).

In this model the instantaneous variance of the asset returns is associated to the trace of the Wishart

[5]For other option pricing applications of this model see for example Benabid et al. (2008), Branger and Muck (2012), Leung et al. (2013) and Gnoatto and Grasselli (2014a).

matrix, that is:

$$d\langle \ln s_t \rangle = \mathrm{Tr}[\Sigma_t]dt,$$

which alone is not Markovian. Computing the expectation of this trace using a partial differential approach would also require consideration of the full state variable Σ.

Given a scalar z and two square (symmetric) matrices $\Lambda_\Sigma, \Lambda_I$, the joint moment generating function of the asset returns, the (Wishart) process Σ_t and the integrated Wishart process $\int_0^t \Sigma_u du$ is given by the function $G_{\text{WMSV}}(t, z, \Lambda_\Sigma, \Lambda_I) = \mathbb{E}\left[e^{z \ln s_t + \mathrm{Tr}[\Lambda_\Sigma \Sigma_t] + \mathrm{Tr}\left[\Lambda_I \int_0^t \Sigma_u du\right]}\right]$ which is known in closed form. Da Fonseca, Grasselli, and Tebaldi (2008) proved the following result:

Lemma 2.3. *The joint moment-generating function of* $(\ln s_t, \Sigma_t, \int_0^t \Sigma_u du)$ *is given by:*

$$G_{\text{WMSV}}(t, z, \Lambda_\Sigma, \Lambda_I) = e^{z \ln s_0 + zrt + \mathrm{Tr}[A(t)\Sigma_0] + b(t)}, \tag{15}$$

where the deterministic matrix function $A(t)$ *and the scalar function* $b(t)$ *satisfy the following ODE (ordinary differential equations)*[6]*:*

$$\frac{dA}{dt} = A\left(M + zQ^\top R^\top\right) + \left(M + zQ^\top R^\top\right)^\top A + 2AQ^\top QA + \frac{z(z-1)}{2}\mathbb{I} + \Lambda_I, \tag{16}$$

$$\frac{db}{dt} = \mathrm{Tr}[\Omega\Omega^\top A], \tag{17}$$

with boundary conditions $A(0) = \Lambda_\Sigma$ *and* $b(0) = 0$*, whose solution is:*

$$A(t) = (\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}(\Lambda_\Sigma A_{11}(t) + A_{21}(t)), \tag{18}$$

$$b(t) = -\frac{\beta}{2}\mathrm{Tr}\left[\log(\Lambda_\Sigma A_{12}(t) + A_{22}(t)) + t(M + zQ^\top R^\top)\right], \tag{19}$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = \exp t \begin{pmatrix} M + zQ^\top R^\top & -2Q^\top Q \\ \frac{z(z-1)}{2}\mathbb{I}_n + \Lambda_I & -\left(M + zQ^\top R^\top\right)^\top \end{pmatrix}. \tag{20}$$

In order to compute some derivative prices we need to be able to differentiate the moment generating function. Thanks to the strong analytical tractability of the WMSV model this quantity can be computed explicitly as shown in the following result.

Corollary 2.1. *The derivative of the function* $g(\alpha) := G_{\text{WMSV}}(t, z, \alpha\Lambda_\Sigma, \Lambda_I)$ *(with* $\alpha \in \mathbb{R}$*) is given by*

$$\frac{dg(\alpha)}{d\alpha} = \left(\mathrm{Tr}\left[\partial_\alpha A(t)\Sigma_0\right] + \partial_\alpha b(t)\right) G_{\text{WMSV}}(t, z, \alpha\Lambda_\Sigma, \Lambda_I), \tag{21}$$

[6] To simplify notations we omit the dependency of these functions on the time variable t in the ODE.

with

$$\partial_\alpha A(t) = -(\alpha\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}\Lambda_\Sigma A_{12}(t)A(t) + (\alpha\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}\Lambda_\Sigma A_{11}(t), \tag{22}$$

$$\partial_\alpha b(t) = -\frac{\beta}{2}\mathrm{Tr}\left[\Lambda_\Sigma A_{12}(t)(\alpha\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}\right]. \tag{23}$$

Proof. Consider an invertible matrix A of size $(n \times n)$ depending on the parameter α. Then taking the derivative of $AA^{-1} = \mathbb{I}_n$ gives $\partial_\alpha A^{-1} = -A^{-1}\partial_\alpha A A^{-1}$ and using this equality from (18) we obtain (22). Let us now consider the map $\alpha \to \mathrm{Tr}\left[\log(\alpha\Lambda_\Sigma A_{12}(t) + A_{22}(t))\right]$. Let be $C = \alpha\Lambda_\Sigma A_{12}(t) + A_{22}(t)$ and $B = C - \mathbb{I}_n$ then we have

$$\begin{aligned}\mathrm{Tr}\left[\partial_\alpha \ln C\right] = \mathrm{Tr}\left[\partial_\alpha \ln(\mathbb{I}_n + B)\right] = \mathrm{Tr}\left[\partial_\alpha\left\{B - \frac{B^2}{2} + \frac{B^3}{3}\right\}\right] = \mathrm{Tr}\left[\partial_\alpha B - \frac{\partial_\alpha BB + B\partial_\alpha B}{2} + \dots\right] \\ = \mathrm{Tr}\left[\partial_\alpha B - \partial_\alpha BB + \frac{\partial_\alpha BB^2}{2} + \dots\right] = \mathrm{Tr}\left[\partial_\alpha B\left\{\mathbb{I}_n - B + \frac{B^2}{2} + \dots\right\}\right] = \mathrm{Tr}\left[\partial_\alpha CC^{-1}\right].\end{aligned}$$

From this last equality we deduce the result. □

This corollary illustrates the high tractability of the WMSV model. Equations (16) and (17) are useful as they underline the importance of (18), (19) and (20). Had these last equations not been available, to compute the solution of Corollary 2.1 we would have had to discretize both the matrix ODE (16) and (17) as well as the sensitivity of these equations with respect to the parameters of interest. The computational cost would have been much higher.

2.4 The WASC Model

The Wishart Affine Stochastic Correlation (WASC hereafter) model of Da Fonseca, Grasselli, and Tebaldi (2007) consists in a n-dimensional risky asset $s_t = (s_t^1, .., s_t^n)^\top$ whose dynamics are given by:

$$ds_t = \mathrm{diag}[s_t]\left[r\mathbf{1} + \sqrt{\Sigma_t}dZ_t\right], \tag{24}$$

where $Z_t \in \mathbb{R}^n$ is a vector Brownian motion and $\mathbf{1}$ is a $n \times 1$ vector of ones, while the returns' variance-covariance matrix Σ_t evolves stochastically, according to the Wishart dynamics (14) introduced previously.

The leverage effects and the asymmetric correlation effects are modeled by introducing the following correlation structure among Brownian motions:

$$dZ_t = \sqrt{1 - \rho^\top\rho}\,dB_t + dW_t\rho,$$

where ρ is a vector of size n, with $\rho \in [-1, 1]^n$ and $\rho^\top \rho \leq 1$ (B_t is a vector Brownian motion under the risk-neutral measure and is independent of W_t). Remarkably, such correlation structure is the only one which is compatible with the affine property of the model, see Da Fonseca et al. (2007).

The instantaneous variance of the asset returns is associated to the diagonal terms of the Wishart matrix, that is:

$$d\langle \ln s_t^i \rangle = \Sigma_t^{ii} dt,$$

so that the integrated variance of the i-th asset is given by $V_t^i = \int_0^t \Sigma_u^{ii} du$. The instantaneous assets' covariance is given by $d\langle \ln s_t^i, \ln s_t^j \rangle = \Sigma_t^{ij} dt$. Notice that the volatility of the i^{th} asset when considered alone is not Markovian so that a partial differential equation approach involving this state variable must also consider all other components of the matrix Σ_t.

Given a vector $z \in \mathbb{R}^n$ and two square (symmetric) matrices $\Lambda_\Sigma, \Lambda_I$, the joint moment generating function of the asset returns, the (Wishart) variance process Σ_t and the integrated variance process $\int_0^t \Sigma_u du$ is given by the function $G_{\text{WASC}}(t, z, \Lambda_\Sigma, \Lambda_I) = \mathbb{E}\left[e^{z^\top \ln s_t + \text{Tr}[\Lambda_\Sigma \Sigma_t] + \text{Tr}\left[\Lambda_I \int_0^t \Sigma_u du\right]} \right]$ which is known in closed form (see Da Fonseca, Grasselli, and Tebaldi (2007) for the proof of the following result).

Lemma 2.4. *The joint moment generating function of* $(\ln s_t, \Sigma_t, \int_0^t \Sigma_u du)$ *is given by:*

$$G_{\text{WASC}}(t, z, \Lambda_\Sigma, \Lambda_I) = e^{z^\top \ln s_0 + z^\top \mathbf{1} rt + \text{Tr}[A(t)\Sigma_0] + b(t)}, \tag{25}$$

where the deterministic matrix function $A(t)$ *and the scalar function* $b(t)$ *satisfy the following ODEs:*

$$\frac{dA}{dt} = A\left(M + Q^\top \rho z^\top\right) + \left(M + Q^\top \rho z^\top\right)^\top A + 2AQ^\top QA + \frac{1}{2} zz^\top - \frac{1}{2}\sum_{j=1}^n z^j e^{jj} + \Lambda_I, \tag{26}$$

$$\frac{db}{dt} = \text{Tr}[\Omega\Omega^\top A], \tag{27}$$

with boundary conditions $A(0) = \Lambda_\Sigma$ *and* $b(0) = 0$*, whose solution is:*

$$A(t) = (\Lambda_\Sigma A_{12}(t) + A_{22}(t))^{-1}(\Lambda_\Sigma A_{11}(t) + A_{21}(t)); \tag{28}$$

$$b(t) = -\frac{\beta}{2}\text{Tr}\left[\log(\Lambda_\Sigma A_{12}(t) + A_{22}(t)) + t(M + Q^\top \rho z^\top)\right], \tag{29}$$

with

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = \exp t \begin{pmatrix} M + Q^\top \rho z^\top & -2Q^\top Q \\ \frac{1}{2}\left(zz^\top - \sum_{j=1}^n z^j e^{jj}\right) + \Lambda_I & -\left(M + Q^\top \rho z^\top\right)^\top \end{pmatrix}. \tag{30}$$

In perfect analogy with the WMSV model, also in the WASC model it is possible to compute explicitly the derivative of the function $\alpha \to G_{\text{WASC}}(t, z, \alpha\Lambda_\Sigma, \Lambda_I)$. Since we arrive at the same expression we omit the result.

The remark at the end of section 2.3 applies *mutatis mutandis* here.

3 Stock-Volatility Derivative Products

In this section we provide a systematic pricing framework in order to price TVOs, Corridor Variance Swaps and Double Digital Calls within the previously introduced stochastic volatility models. For the TVO our method completes the one proposed e.g. by Torricelli (2013): in fact, it will be clear that a great advantage of our approach is that it is independent of the number of volatility factors. This will be crucial as we want to apply the methodology to the multi-factor Heston model as well as the Wishart-based stochastic volatility models.

We display separately the results for the different models although the results clearly suggest that we could have unified the presentation. This apparent unity is, however, misleading and we will show later examples for which the Wishart based models or even the BiHeston model introduce strong difficulties.

3.1 The Target Volatility Option

The payoff of a Target Volatility Option expiring at time t is given by:

$$c_{\text{TVO}} = \mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}\bar{\sigma}(s_t - K)_+\right], \tag{31}$$

with $\bar{V}_t = \frac{V_t}{t} = \frac{1}{t}\int_0^t v_u du$ and $\bar{\sigma}$ a positive constant. This contract, in essence, provides the right, but not the obligation, to buy a fractional amount of the stock at the prespecified strike price K. The fraction depends on the ratio between the fixed constant $\bar{\sigma}$ and the realized volatility; without loss of generality, we shall set $\bar{\sigma} \equiv 1$. The joint process $(s_t, v_t)_{t\geq 0}$ follows e.g. the dynamics given by equations (1) and (2) (we consider for ease of notation the Heston (1993) specification for the volatility process).

First, we express the option price as a function of the Fourier transform of the stock and its volatility, it leads to the following result.

Lemma 3.1. *The price of the Target Volatility Option can be expressed as:*

$$c_{\text{TVO}} = \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z)\mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}} e^{iz\ln s_t}\right] dz,$$

where $\hat{g}(z) = -\frac{1}{2\pi}\frac{K^{1-iz}}{iz(1-iz)}$ *and* $\gamma = \Im(z) < -1$.

Proof.

$$\begin{aligned}\mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}(s_t - K)_+\right] &= \mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}\mathbb{E}\left[(s_t - K)_+ | \mathcal{F}_v\right]\right] \\ &= \mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}\int_{-\infty}^{+\infty}(e^x - K)_+ f(x|v)dx\right],\end{aligned}$$

where $\mathcal{F}_v$ is the filtration generated by the volatility path. The density of the logarithm of the stock conditional on the volatility path is given by:

$$f(x|v) = \frac{1}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-izx}\mathbb{E}\left[e^{iz\ln s_t}|\mathcal{F}_v\right] dz.$$

Replacing this expression in the previous equation leads, after using Fubini's theorem, to the result. The computation of the Fourier transform $\hat{g}(z)$ is easily done as we have:

$$\begin{aligned}\hat{g}(z) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-izx}(e^x - K)_+ dx \\ &= \frac{-1}{2\pi}\frac{K^{1-iz}}{iz(1-iz)},\end{aligned}$$

provided that $\Im(z) < -1$, which leads to the constraint on γ (see Lewis (2000)). ☐

Hereafter, we will use the well-known relation valid for any $x, \alpha > 0$:

$$\frac{1}{x^\alpha} = \frac{1}{\alpha\Gamma(\alpha)}\int_0^{+\infty} e^{-u^{\frac{1}{\alpha}}x} du$$

from which we will deduce the price of the TVO for the different models.

Note that this computational trick can be applied with same purpose to certain non affine models, see Leblanc (1996).

TVO in the Heston model For the Heston model we have the following lemma.

Lemma 3.2. *In the Heston model of Lemma 2.1 the target volatility option price is:*

$$c_{\text{TVO}} = \frac{2}{\Gamma(\frac{1}{2})}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_0^{+\infty} \hat{g}(z) G_{\text{HES}}\left(t, iz, 0, -\frac{u^2}{t}\right) du dz$$

where $\hat{g}(z)$ *is given in Lemma 3.1 and* $\gamma = \Im(z) < -1$.

Proof.

$$\mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}e^{iz\ln s_t}\right] = \frac{2}{\Gamma(\frac{1}{2})}\int_0^{+\infty} e^{-rt}\mathbb{E}\left[e^{-u^2\bar{V}_t+iz\ln s_t}\right]du.$$

The result directly follows by observing that the integrand above depends on the moment generating function G_{HES} computed in Lemma 2.1. □

TVO in the BiHeston model In the BiHeston model we have $\bar{V}_t = \frac{V_t}{t} = \frac{1}{t}\int_0^t (v_u^0 + v_u^1)du$, with the following result:

Lemma 3.3. *In the BiHeston model of Lemma 2.2 the option price is given by:*

$$c_{\text{TVO}} = \frac{2}{\Gamma(\frac{1}{2})}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_0^{+\infty}\hat{g}(z)G_{2\text{HES}}\left(t, iz, 0, 0, -\frac{u^2}{t}, -\frac{u^2}{t}\right)dudz$$

where $G_{2\text{HES}}$ is the joint Laplace transform defined in Lemma 2.2, $\hat{g}(z)$ is given in Lemma 3.1 and $\gamma = \Im(z) < -1$.

TVO in the WMSV model In the WMSV model we have $\bar{V}_t = \frac{V_t}{t} = \frac{1}{t}\int_0^t \mathrm{Tr}\left[\Sigma_u\right]du$ and easily deduce the following result.

Lemma 3.4. *In the WMSV model of Lemma 2.3 the option price is given by:*

$$c_{\text{TVO}} = \frac{2}{\Gamma(\frac{1}{2})}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_0^{+\infty}\hat{g}(z)G_{\text{WMSV}}\left(t, iz, 0_n, -\frac{u^2}{t}\mathbb{I}_n\right)dudz,$$

where G_{WMSV} is the moment generating function defined in Lemma 2.3, $\hat{g}(z)$ is given in Lemma 3.1 and $\gamma = \Im(z) < -1$.

TVO in the WASC model Lastly, for the WASC model, let us consider the payoff of a Target Volatility Option on the first asset s_t^1, such that the option price writes as follows:

$$c_{\text{TVO}} = \mathbb{E}\left[\frac{e^{-rt}}{\sqrt{\bar{V}_t}}(s_t^1 - K)_+\right],$$

with $\bar{V}_t = \frac{V_t}{t} = \frac{1}{t}\int_0^t \Sigma_u^{11}du$. In this case, following the same computations we arrive at the following result.

Lemma 3.5. *In the WASC model of Lemma 2.4 the option price is given by:*

$$c_{\text{TVO}} = \frac{2}{\Gamma(\frac{1}{2})}\int_{-\infty+i\gamma}^{+\infty+i\gamma}\int_0^{+\infty}\hat{g}(z)G_{\text{WASC}}\left(t, ize_1, 0_n, -\frac{u^2}{t}e_{11}\right)dudz,$$

where G_{WASC} is defined in Lemma 2.4, $\hat{g}(z)$ is given in Lemma 3.1, e_1 (resp. e_{11}) represents the first element of the canonical basis in $\mathbb{R}^n$ (resp. in M_n) and $\gamma = \Im(z) < -1$.

3.2 The Corridor Variance Swap

In this sub-section we focus on the pricing of Corridor Variance Swaps, see e.g. Zheng and Kwok (2014), Albanese and Osseiran (2007) and the early work of Carr and Lewis (2004). The payoff at time t is given by

$$\mathrm{VS}(t) = \mathbb{E}\left[\frac{1}{t}\int_0^t \mathcal{V}_u 1_{\{L \leq s_u \leq H\}} du - K\right], \tag{32}$$

where $\mathcal{V}_t$ is equal to v_t, $v_t^0 + v_t^1$, $\mathrm{Tr}[\Sigma_t]$ or Σ_t^{11} depending on which model is considered. The Corridor Variance Swap coincides with a classic Variance Swap provided that the underlying remains in a given corridor defined by the interval $[L, H]$.

The building block for pricing Corridor Variance Swaps is the computation of the term $\mathbb{E}\left[\mathcal{V}_t 1_{\{x_t \leq h\}}\right]$ where $x_t = \ln s_t$ and $h = \ln H$. We have the following result.

Lemma 3.6. *Consider $I_{t,h} = \mathbb{E}\left[\mathcal{V}_t 1_{\{x_t \leq h\}}\right]$ with $x_t = \ln s_t$ and $\mathcal{V}_t$ defined above. Then we have*

$$I_{t,h} = \frac{1}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma} \frac{e^{-ihz_1}}{-iz_1} \partial_\alpha \mathbb{E}\left[e^{iz_1 x_t + \alpha \mathcal{V}_t}\right]_{|\alpha=0} dz_1 \tag{33}$$

with $\gamma = \Im(z) > 0$.

Proof. We denote by $f(x|\mathcal{V}_t)$ the density of x_t conditional to $\mathcal{V}_t$ and its Fourier transform by $\phi(z|\mathcal{V}_t)$ then we have:

$$\begin{aligned}
I_{t,h} &= \mathbb{E}\left[\mathcal{V}_t 1_{\{x_t \leq h\}}\right] \\
&= \mathbb{E}\left[\mathcal{V}_t \int_{-\infty}^{+\infty} 1_{\{x \leq h\}} f(x|\mathcal{V}_t) dx\right] \\
&= \mathbb{E}\left[\mathcal{V}_t \int_{-\infty}^{+\infty} 1_{\{x \leq h\}} \frac{1}{2\pi}\int_{\mathbb{C}} e^{-ixz_1} \phi(z_1|\mathcal{V}_t) dz_1 dx\right] \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \mathbb{E}\left[\mathcal{V}_t \phi(z_1|\mathcal{V}_t)\right] \frac{e^{-ihz_1}}{-iz_1} dz_1 \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \mathbb{E}\left[\mathcal{V}_t e^{iz_1 x_t}\right] \frac{e^{-ihz_1}}{-iz_1} dz_1
\end{aligned}$$

with $\Im(z_1) > 0$. As we have $\mathbb{E}\left[\mathcal{V}_t e^{iz_1 x_t}\right] = \partial_\alpha \mathbb{E}\left[e^{iz_1 x_t + \alpha \mathcal{V}_t}\right]_{|\alpha=0}$ we deduce immediately the result. □

Thanks to the previous lemma we are able to provide the price of the corridor variance swap.

Corollary 3.1. *The price of the Corridor Variance Swap is given by*

$$\mathrm{VS}(t) = \frac{1}{t}\int_0^t I_{u,h} - I_{u,l} du - K, \tag{34}$$

where the quantity $I_{t,h}$ is given in Lemma 3.6.

Finally, the next result expresses the derivative of the moment-generating function according to the different model specification.

Lemma 3.7. *The quantity* $\partial_\alpha \mathbb{E}\left[e^{iz_1 x_t + \alpha \mathcal{V}_t}\right]$ *is given by:*

$$\begin{cases} \partial_\alpha G_{\mathrm{HES}}(t, iz, \alpha, 0) \\ \partial_\alpha G_{\mathrm{2HES}}(t, iz, \alpha, \alpha, 0, 0) \\ \partial_\alpha G_{\mathrm{WMSV}}(t, iz, \alpha \mathbb{I}_n, 0_n) \\ \partial_\alpha G_{\mathrm{WASC}}(t, (iz_1, 0)^\top, \alpha e_{11}, 0_n). \end{cases}$$

Let us stress again the fact that the pricing of the corridor variance swap requires the computation of the derivative of the moment-generating function. For all models we consider in this paper (Heston, double Heston, WMSV and WASC) this function is known in closed form. For a standard affine Duffie and Kan (1996) model, for which the Riccati ODEs cannot be explicitly computed (and therefore need to be simulated using a Runge-Kutta scheme for example), it will involve the discretized version of the sensitivity with respect to the initial condition of these Riccati ODEs. In that case the computational burden increases significantly and it underlines the analytical advantages of the Wishart based stochastic volatility models when it comes to build multidimensional extensions. This computational improvement already appears in the pricing of Range Notes; see for example Chiarella et al. (2014) that should be compared with Jang and Yoon (2010). This result also emphasizes the importance of developing alternative expressions for the moment generating function for the WASC and WMSV models[7], along these lines see Gnoatto and Grasselli (2014b).

To further illustrate the problem related to the dimension of the state variables let us explain some important differences. In this work we consider the integrated volatility in equation (32) (and follow the definition proposed by Carr and Lewis (2004)) but in practice it is in fact a discretely sampled variance that is traded and its value requires the computation of the quantity:

$$\mathbb{E}\left[(\ln(s_{t_k}) - \ln(s_{t_{k-1}}))^2 1_{\{s_{t_{k-1}} \in [L;H]\}}\right]$$

with $t_{k-1} < t_k$. In Zheng and Kwok (2014), to compute such expectation the authors derive twice the moment-generating function with respect to the argument of the stock (its logarithm in fact) and conclude hastily that their "analytic procedure can be applied to any affine model of the underlying asset price and payoff structures of higher moments swaps.". An inspection of the moment-generating

[7]The formulas for these two models are obtained through linearization of Riccati's equations, as suggested by Grasselli and Tebaldi (2008).

functions (15) and (25) shows that their solution applied to these models will lead to more than tedious computations.

3.3 The Double Digital Call

In this section we investigate the pricing of a Double Digital Call, see e.g. Torricelli (2013), whose payoff is the indicator function of the event $\{s_t \geq K_1, \bar{V}_t \geq K_2\}$, so that the price is given by

$$c_{\text{DDC}}(s_0, K, t) = \mathbb{E}\left[e^{-rt} 1_{\{s_t \geq K_1, \bar{V}_t \geq K_2\}}\right], \tag{35}$$

where as usual $\bar{V}_t = \frac{V_t}{t} = \frac{1}{t}\int_0^t v_u du$ denotes the integrated variance. We start with the Heston model and check that the results remain valid for multidimensional volatility extensions.

Lemma 3.8. *The Double Digital Call option price can be expressed as:*

$$c_{\text{DDC}} = \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z)\mathbb{E}\left[1_{\{\bar{V}_t \geq K_2\}} e^{iz \ln s_t}\right] dz,$$

where $\hat{g}(z) = \frac{1}{2iz\pi} e^{-iz \ln K_1}$ *and* $\gamma = \Im(z) < 0$.

Proof.

$$\begin{aligned} c_{\text{DDC}}(s_0, K, t) &= \mathbb{E}\left[e^{-rt} 1_{\{s_t \geq K_1, \bar{V}_t \geq K_2\}}\right] \\ &= \mathbb{E}\left[e^{-rt} 1_{\bar{V}_t \geq K_2} \mathbb{E}\left[1_{\{s_t \geq K_1\}} | \mathcal{F}_v\right]\right] \\ &= \mathbb{E}\left[e^{-rt} 1_{\{\bar{V}_t \geq K_2\}} \int_{-\infty}^{+\infty} 1_{\{e^x \geq K_1\}} f(x|v) dx\right], \end{aligned}$$

where as usual $\mathcal{F}_v$ stands for the filtration generated by the volatility path. The density of the logarithm of the stock conditional on the volatility path is given by:

$$f(x|v) = \frac{1}{2\pi} \int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-izx} \mathbb{E}\left[e^{iz \ln s_t} | \mathcal{F}_v\right] dz,$$

therefore using Fubini's theorem we get:

$$\begin{aligned} c_{\text{DDC}}(s_0, K, t) &= \mathbb{E}\left[e^{-rt} 1_{\{\bar{V}_t \geq K_2\}} \int_{-\infty}^{+\infty} 1_{\{e^x \geq K_1\}} \frac{1}{2\pi} \int_{\mathbb{C}} e^{-izx} \mathbb{E}\left[e^{iz \ln s_t} | \mathcal{F}_v\right] dz dx\right] \\ &= e^{-rt} \int_{-\infty}^{+\infty} \hat{g}(z) \mathbb{E}\left[1_{\{\bar{V}_t \geq K_2\}} e^{iz \ln s_t}\right] dz \end{aligned}$$

where

$$\begin{aligned} \hat{g}(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} 1_{\{e^x \geq K_1\}} e^{-izx} dx \\ &= \frac{1}{2iz\pi} e^{-iz \ln K_1}, \end{aligned}$$

provided that $\Im(z) < 0$ in order to grant the convergence of the previous integral. □

Now the expression $I(s_0, v_0, t, z) = \mathbb{E}\left[1_{\{\bar{V}_t \geq K_2\}} e^{iz \ln s_t}\right]$ can be computed by using the Laplace-Fourier transform method as presented, among others, in Carr and Madan (1999), Lewis (2000) or Petrella (2004). More precisely, let us consider the Fourier transform of $I(s_0, v_0, t, z)$ with respect to K_2:

$$\begin{aligned}
\hat{I}(z, \hat{z}, t) &= \int_{-\infty}^{+\infty} e^{i\hat{z}K_2} \mathbb{E}\left[1_{\{\bar{V}_t \geq K_2\}} e^{iz \ln s_t}\right] dK_2 \\
&= \mathbb{E}\left[e^{iz \ln s_t} \int_{-\infty}^{+\infty} e^{i\hat{z}K_2} 1_{\{\bar{V}_t \geq K_2\}} dK_2\right] \\
&= \mathbb{E}\left[e^{iz \ln s_t} \frac{1}{i\hat{z}} e^{i\hat{z}\bar{V}_t}\right] \\
&= \frac{1}{i\hat{z}} \mathbb{E}\left[e^{iz \ln s_t} e^{\frac{i\hat{z}}{t} \int_0^t v_u du}\right].
\end{aligned}$$

The expression $\hat{I}(z, \hat{z}, t)$ (with $\Im(\hat{z}) < 0$) can be computed explicitly using the joint moment generating function under the different models.

Double Digital Call in the Heston model The expression $\mathbb{E}\left[e^{iz \ln s_t} e^{\frac{\pm i\hat{z}}{t} \int_0^t v_u du}\right]$ can be computed explicitly using the joint moment generating function $G_{\text{HES}}(t, z, \lambda_v, \Lambda_v) = \mathbb{E}\left[e^{z \ln s_t + \lambda_v v_t + \Lambda_v \int_0^t v_u du}\right]$ defined in Lemma 3, thus giving

$$\hat{I}(z, \hat{z}, t) = \frac{1}{i\hat{z}} G_{\text{HES}}\left(t, iz, 0, \frac{i\hat{z}}{t}\right).$$

Finally, the expression $I(s_0, v_0, t, z)$ is given by

$$\begin{aligned}
I(s_0, v_0, t, z) &= \frac{1}{2\pi} \int_{\mathbb{C}} e^{-i\hat{z}K_2} \hat{I}(z, \hat{z}, t) d\hat{z} \\
&= \frac{1}{2\pi} \int_{\mathbb{C}} e^{-i\hat{z}K_2} \frac{1}{i\hat{z}} G\left(t, iz, 0, \frac{i\hat{z}}{t}\right) d\hat{z}
\end{aligned}$$

and we get the following result.

Lemma 3.9. *Under the Heston model of Lemma 2.1 the price of a Double Digital Option is given by:*

$$c_{\text{DDC}}(s_0, K, t) = e^{-rt} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z) \frac{1}{2\pi} \int_{-\infty+i\hat{\gamma}}^{+\infty+i\hat{\gamma}} e^{-i\hat{z}K_2} \frac{1}{i\hat{z}} G_{\text{HES}}\left(t, iz, 0, \frac{i\hat{z}}{t}\right) d\hat{z} dz,$$

with $\gamma = \Im(z) < 0$, $\hat{\gamma} = \Im(\hat{z}) < 0$.

Double Digital Call in the BiHeston model The expression $\mathbb{E}\left[e^{iz \ln s_t} e^{\frac{\pm i\hat{z}}{t} \int_0^t (v_u^0 + v_u^1) du}\right]$ can be computed explicitly using the moment generating function $G_{2\text{HES}}$ defined in Lemma 2.2, thus giving

$$\hat{I}(z, \hat{z}, t) = \frac{1}{i\hat{z}} G_{2\text{HES}}\left(t, iz, 0, 0, \frac{i\hat{z}}{t}, \frac{i\hat{z}}{t}\right).$$

Lemma 3.10. *Under the BiHeston model of Lemma 2.2 the price of a Double Digital Option is given by:*

$$c_{\text{DDC}}(s_0, K, t) = e^{-rt} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z) \frac{1}{2\pi} \int_{-\infty+i\hat{\gamma}}^{+\infty+i\hat{\gamma}} e^{-i\hat{z}K_2} \frac{1}{i\hat{z}} G_{2\text{HES}} \left(t, iz, 0, 0, \frac{i\hat{z}}{t}, \frac{i\hat{z}}{t} \right) d\hat{z}dz,$$

with $\gamma = \Im(z) < 0$, $\hat{\gamma} = \Im(\hat{z}) < 0$.

Double Digital Call in the WMSV model The expression $\mathbb{E}\left[e^{iz\ln s_t} e^{\frac{i\hat{z}}{t}\int_0^t \text{Tr}[\Sigma_u]du}\right]$ can be computed explicitly using the moment generating function $G_{\text{WMSV}}(t, z, \Lambda_\Sigma, \Lambda_I)$ defined in Lemma 2.3, thus giving the following result.

Lemma 3.11. *Under the WMSV model Lemma 2.3 the price of a Double Digital Option is given by:*

$$c_{\text{DDC}}(s_0, K, t) = e^{-rt} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z) \frac{1}{2\pi} \int_{-\infty+i\hat{\gamma}}^{+\infty+i\hat{\gamma}} e^{-i\hat{z}K_2} \frac{1}{i\hat{z}} G_{\text{WMSV}} \left(t, iz, 0_n, \frac{i\hat{z}}{t}\mathbb{I}_n \right) d\hat{z}dz,$$

with $\gamma = \Im(z) < 0$, $\hat{\gamma} = \Im(\hat{z}) < 0$.

Double Digital Call in the WASC model The expression $\mathbb{E}\left[e^{iz\ln s_t^1} e^{\frac{i\hat{z}}{t}\int_0^t \Sigma_u^{11}du}\right]$ can be computed explicitly using the moment generating function $G_{\text{WASC}}(t, z, \Lambda_\Sigma, \Lambda_I)$ defined in Lemma 2.4, thus giving the following result.

Lemma 3.12. *Under the WASC model of Lemma 2.4 the price of a Double Digital Option on the first asset is given by:*

$$c_{\text{DDC}}(s_0^1, K, t) = e^{-rt} \int_{-\infty+i\gamma}^{+\infty+i\gamma} \hat{g}(z) \frac{1}{2\pi} \int_{-\infty+i\hat{\gamma}}^{+\infty+i\hat{\gamma}} e^{-i\hat{z}K_2} \frac{1}{i\hat{z}} G_{\text{WASC}} \left(t, ize_1, 0_n, \frac{i\hat{z}}{t}e_{11} \right) d\hat{z}dz,$$

with $\gamma = \Im(z) < 0$, $\hat{\gamma} = \Im(\hat{z}) < 0$.

4 Numerical Results

The four models were calibrated on same data so they produce approximately the same vanilla option values, they are extracted from Da Fonseca and Grasselli (2011) and reported in Table I. Naturally, models with more parameters lead to a smaller calibration error but even for the Heston model the pricing error is relatively small. For the values presented here the root mean square error for out-the-money option prices is 0.163% of the underlying forward price while for the BiHeston, WMSV and WASC models it is around 0.1%. The pricing of these exotic options with calibrated models allows us to put our results in the broader perspective of model risk, see Cont (2006) for related aspects, and raises some practical important problems.

[Insert Table I here]

The TVO prices are reported in Table II, the Corridor Variance Swap prices are reported in Table III while the double digital call prices are given, depending on the model considered, by Tables IV, V, VI and VII.

[Insert Tables II - VII here]

For the TVO prices the Heston and BiHeston models lead to similar prices. However, note that the percentage difference between the prices given by the two models can reach 10% (e.g., for the maturity 0.5 and strike 0.9) and on average around 5%. For the WASC and WMSV models (for both models we changed the Gindikin parameters so that they are greater than one) the average discrepancy between the TVO prices is 3% and decreases with the maturity.

For the Corridor Variance Swap the Heston and BiHeston models give prices that are close but the error increases with the maturity (0.8% for the maturity 0.5 and 5% for the one year maturity). For the WASC and WMSV, we observe the opposite, that is the average discrepancy decreases with the maturity, from 10% to 6%.

For the Double Digital Call the conclusions are similar. For example, comparing the Heston and BiHeston models illustrates the fact that models giving close vanilla prices can lead to substantial differences in derivative prices as a difference of more than 50% can easily be reached if we consider options whose payoff depends on the tail of the asset-volatility distribution. This problem is likely to be magnified by complex payoff structures. Similarly, the differences between the WASC and the WMSV can be substantial (30% for $T = 1$, $K_1 = 0.08$ and $K_1 = 1$). Also of interest is the fact that for the WASC and WMSV we changed slightly the Gindikin parameters, which gives us a rough idea of exotic option prices sensitivity to model parameter "uncertainty" and illustrates how prices produced by the pair Heston/BiHeston and the pair WMSV/WASC can diverge for small parameter perturbations. Because the DDC strongly depend on the tails of the stock-volatility distribution it constitutes a "worst" case example, the other products lead to similar, though less dramatic conclusions. A comparison between the Heston/BiHeston prices on one hand and the WMSV/WASC prices on the other hand shows a huge difference. Let us stress the fact that the magnitude of our values are in line with what happens in practice. It should be clear also that adding exotic options in the calibration objective function, so that all the models produce similar vanilla and exotic option prices, does not guarantee that exotic options not included in the calibration set will be similarly priced. Furthermore, this strategy supposes that these exotic option prices are given, that is to say are input values and implies that these products are liquid enough, which is often not the case.

As noted in Chiarella et al. (2014), in practice whenever a derivatives seller wants to gain some confidence in his pricing the standard procedure is to ask other market participants for their price. As some derivatives can be very exotic it might be difficult to obtain such information. To overcome this difficulty some companies provide a service that allows a market participant to know whether his price is close or within the range of prices proposed by the other participants (but without revealing the prices). Our equity-volatility option results confirm the issues raised with this practice and extend to this market the concerns developed in Chiarella et al. (2014) for the interest rate markets. It is unclear to us whether current market regulation rules address properly that problem.

5 Remarks and Open Problems

The previous results might suggest that any closed-form results for the Heston model can be easily extended to the BiHeston, WMSV or WASC models. We already mentioned that this statement is not correct. In addition to the problems underlined in the corridor variance swap section, the pricing of option on the discretely sampled variance once more illustrates the difficulty the handle multivariate models. It was performed in the Heston model in Lian et al. (2014) and it is simple to check that adapting to the WMSV or WASC models their results is a non-trivial task. Let us further illustrate with another equity-stochastic volatility product the difficulties related to the dimension. The timer option is a recent product whose payoff is given by

$$\mathbb{E}\left[e^{-r\tau_t}(s_{\tau_t} - K)_+\right]$$

with $\tau_t = \inf\{u; \int_0^u v_s ds = t\}$ and v_u is the volatility of the stock. As for the products considered in this work the timer option payoff depends on the stock and its volatility (path). For this product a closed-form solution is available for the Heston model, see Li (2013), but the extension to the WMSV or WASC model leads to important difficulties that we were not able to solve. Even the BiHeston, which does not involve any matrix in its characteristic function, brings some to tedious numerical difficulties.

Lastly, we only consider continuous time diffusion processes while the univariate stochastic volatility model can have different kind of jumps (on the stock and/or the volatility). An alternative to the Wishart-based models was proposed in the series of papers Barndorff-Nielsen and Stelzer (2013) and Muhle-Karbe et al. (2012) who develop a multi-asset matrix jump process model, the pricing of exotic derivatives within that framework is an open question[8].

6 Conclusion

In this paper we exploited a powerful technique in order to price some equity-volatility products that have been recently introduced in the market. Our approach combines conditioning with respect to the subfiltration generated by the volatility path with simple Fourier techniques. Our methodology allows one to price in closed form Target Volatility options, Corridor Variance Swaps and Double Digital calls regardless of the dimension of the stochastic process used to describe the volatility process. We investigated the Affine Class with a special emphasis on the recent Wishart based specifications introduced by Da Fonseca, Grasselli, and Tebaldi (2007) and Da Fonseca, Grasselli, and Tebaldi (2008), for which closed form solutions are available for the moment generating function.

A numerical exercise for the TVO and Corridor Variance Swap shows that the Heston and BiHeston models lead to similar prices for short maturities; however the discrepancy between the prices increases with the maturity and can reach 10% but is on average around 5%. For the WASC and WMSV models we observe the opposite, that is the average discrepancy between the prices is 3% and decreases with the maturity. These results suggest that models giving close vanilla prices can lead to substantial differences in exotic derivative prices. Furthermore, a small perturbation of data parameters can produce huge differences for exotic derivative prices and raises the question of how to define a robust pricing for these derivatives.

Our results clearly illustrate the remarkable flexibility of Wishart-based models as they enable to increase the dimension of the state variables, either of the volatility or the number of assets, and yet remain highly tractable. Although we showed how to overcome some difficulties we pointed out some open and challenging questions that we leave for future research.

[8]There are only very few multiasset stochastic volatility models, apart from those mentioned above that have the feature of employing matrix diffusion processes let us also mention Yoon et al. (2011).

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Appendix

Tables

Table I: Model Parameter Values

Heston	Value	BiHeston	Value	WASC	Value	WSMV	Value
v_t	0.0414	v_t^0	0.0187	Σ_t^{11}	0.0446	Σ_t^{11}	0.0298
κ	1.4078	κ_0	1.3080	Σ_t^{12}	0.0366	Σ_t^{12}	0.0119
θ	0.0838	θ_0	0.0281	Σ_t^{22}	0.0424	Σ_t^{22}	0.0108
σ	0.9319	σ_1	1.1202	β	1.7332	β	1.5776
ρ	-0.5409	ρ_0	-0.3884	M_{11}	-0.7820	M_{11}	-1.2479
		v_t^1	0.0229	M_{12}	-0.3772	M_{12}	-0.8985
		κ_1	1.4134	M_{21}	-0.0539	M_{21}	-0.0820
		θ_1	0.0485	M_{22}	-1.2497	M_{22}	-1.1433
		σ_1	0.4822	Q_{11}	0.3898	Q_{11}	0.3417
		ρ_1	-0.8395	Q_{12}	0.3573	Q_{12}	0.3493
				Q_{21}	0.2809	Q_{21}	0.1848
				Q_{22}	0.3362	Q_{22}	0.3090
				ρ_1	-0.6407	R_{11}	-0.2243
				ρ_2	-0.1105	R_{12}	-0.1244
						R_{21}	-0.2545
						R_{22}	-0.7230

These parameter values are those of Da Fonseca and Grasselli (2011) and were obtained by performing a calibration on the DAX vanilla options on the day August, 20 2008. For the WASC model the calibration was performed on the options for the pair EuroStoxx50/DAX. Without loss of generality we will take $s_0 = 1$ and the risk free rate $r = 0$. We changed the Gindikin parameter values β so that $\beta > 1$.

Table II: TVO Prices

		Heston		BiHeston		WASC		WMSV	
Maturity		0.5	1.0	0.5	1.0	0.5	1.0	0.5	1.0
	0.9	0.93084	0.98910	0.84322	0.92055	0.58416	0.66302	0.62240	0.68281
K	1	0.56497	0.63592	0.51799	0.60051	0.38502	0.48436	0.40299	0.48864
	1.1	0.30230	0.37550	0.28313	0.36242	0.23660	0.34307	0.24106	0.33748

We report the TVO price c_{TVO} for the maturity $t \in \{0.5, 1\}$, the strike $K \in \{0.9, 1, 1.1\}$ and model parameter values given in Table I.

Table III: Corridor Variance Swap Prices

		Heston		BiHeston		WASC		WMSV	
	Maturity	0.25	0.75	0.25	0.75	0.25	0.75	0.25	0.75
K	[0.95 1.05]	0.014354	0.010273	0.014462	0.009730	0.025362	0.018026	0.028327	0.019274
	[0.9 1.1]	0.025931	0.019664	0.026170	0.018824	0.047200	0.035517	0.053169	0.038036

We report the quantities $I_{t,h} - I_{t,l}$ for $t \in \{0.5, 1\}$, $[L, H]$ equals to [0.95 1.05] or [0.9 1.1] for the four models and parameter values given in Table I.

Table IV: Double Digital Call Prices - Heston

		$T = 0.5$			$T = 1$		
	K_1	0.9	1	1.1	0.9	1	1.1
	0.03	0.19901	0.13377	0.06070	0.18938	0.13806	0.08370
K_2	0.05	0.10889	0.06956	0.03737	0.10493	0.07440	0.04845
	0.08	0.03186	0.01788	0.01475	0.03066	0.02049	0.01655

We report the double digital call prices for the maturity $T \in \{0.5, 1\}$, $K_1 \in \{0.9, 1, 1.1\}$ and $K_2 \in \{0.03, 0.05, 0.08\}$ for the Heston model with parameter values given in Table I.

Table V: Double Digital Call Prices - BiHeston

		$T = 0.5$			$T = 1$		
	K_1	0.9	1	1.1	0.9	1	1.1
	0.03	0.21303	0.13979	0.05990	0.20186	0.14483	0.08550
K_2	0.05	0.11407	0.06842	0.03136	0.10982	0.07388	0.04326
	0.08	0.02044	0.00755	0.00688	0.02049	0.01001	0.00673

We report the double digital call prices for the maturity $T \in \{0.5, 1\}$, $K_1 \in \{0.9, 1, 1.1\}$ and $K_2 \in \{0.03, 0.05, 0.08\}$ for the BiHeston model with parameter values given in Table I.

Table VI: Double Digital Call Prices - WMSV

		$T = 0.5$			$T = 1$		
	K_1	0.9	1	1.1	0.9	1	1.1
	0.03	0.32062	0.23711	0.14847	0.29719	0.23862	0.18013
K_2	0.05	0.26166	0.18894	0.11706	0.26909	0.21324	0.15919
	0.08	0.15828	0.10818	0.06565	0.20056	0.15521	0.11421

We report the double digital call prices for the maturity $T \in \{0.5, 1\}$, $K_1 \in \{0.9, 1, 1.1\}$ and $K_2 \in \{0.03, 0.05, 0.08\}$ for the WMSV model with parameter values given in Table I.

Table VII: Double Digital Call Prices - WASC

		$T = 0.5$			$T = 1$		
	K_1	0.9	1	1.1	0.9	1	1.1
	0.03	0.29818	0.21497	0.12874	0.29105	0.22963	0.16940
K_2	0.05	0.22936	0.16139	0.09694	0.24292	0.18912	0.13861
	0.08	0.12577	0.08412	0.05112	0.15466	0.11719	0.08500

We report the double digital call prices for the maturity $T \in \{0.5, 1\}$, $K_1 \in \{0.9, 1, 1.1\}$ and $K_2 \in \{0.03, 0.05, 0.08\}$ for the WASC model with parameter values given in Table I.