The oriented derivative in stochastic calculus

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A unifying differentiability notion

The derivative of a function is a central object in calculus that carries intrinsic properties of the function itself.

Indeed, the analysis of a function by means of its derivative is an essential part of many textbooks on calculus and it goes back as far as the classical works of Fréchet [2] and Gâteaux [3].

In this talk, we will see a differential calculus that applies to the derivatives of Fréchet and Gâteaux and solely requires the function and its domain to be differentiable and open in the direction of the selected orientation, respectively.

In particular, in a Hilbert space X the oriented derivative can be identified with a gradient.

Namely, let S be a star convex set in X with the origin as center, which amounts to

$$[0, x] \subset S \quad \text{for each } x \in S.$$

Here, $[x, y]$ stands for the line segment $\{(1-t)x + ty \mid t \in [0,1]\}$ between two points x and y in X.

Further, let U be a non-empty S-open set in X in the sense that for any $x \in U$ there is $\delta > 0$ such that

$$x + B_\delta(0) \cap S \subset U,$$

where $B_r(y)$ is the open ball with center $y \in X$ and radius $r > 0$.

The oriented gradient (K., '23)

A function $\varphi : U \to \mathbb{R}$ is called S-differentiable if there is a map $L : U \to X$ satisfying

$$\varphi(x+h) - \varphi(x) = \langle L(x), h \rangle + o(|h|) \quad \text{as } h \to 0 \text{ on } S$$

for any $x \in U$, where $\langle \cdot, \cdot \rangle$ is the inner product on X that induces the complete norm $|\cdot|$.

In this case, there exists a unique map $\nabla_S \varphi$ on U taking all its values in the closure V of $\mathrm{span}(S)$ such that

$$\langle L, h \rangle = \langle \nabla_S \varphi, h \rangle \quad \text{for all } h \in V.$$

We shall call $\nabla_S \varphi$ the S-oriented gradient of φ.

Thus, the S-oriented gradient of such an S-differentiable function $\varphi : U \to \mathbb{R}$ is the unique V-valued map on U such that

$$\lim_{\substack{h \to 0, \\ h \in S}} \frac{\varphi(x+h) - \varphi(x) - \langle \nabla_S \varphi(x), h \rangle}{|h|} = 0$$

for any $x \in U$, by a simple uniqueness argument and an application of Riesz's representation theorem.

The resulting oriented differential calculus extends the mean value theorem, the chain rule and the Taylor formula, even in Banach spaces, as shown in the presented paper [4].

Let $C^1_S(U)$ stand for the linear space of all S-differentiable $\varphi \in C(U)$ for which $\nabla_S \varphi$ is continuous.

Several relevant applications and properties of the oriented gradient are as follows.

(i) While $\varphi : U \to \mathbb{R}$ is X-differentiable if and only if it is Fréchet differentiable, the introduced concept is redundant for $S = \{0\}$.

Thus, $C^1_X(U) = C^1(U)$ and $C^1_{\{0\}}(U) = C(U)$ and we recover the gradient and the zero operator,

$$\nabla_X = \nabla \quad \text{and} \quad \nabla_{\{0\}} = 0.$$

(ii) If $V = X$, then ∇_S determines ∇ in the following sense: Every differentiable extension $\tilde{\varphi}$ of φ to an open set $\tilde{U}$ in X with $U \subset \tilde{U}$ satisfies

$$\nabla \tilde{\varphi} = \nabla_S \varphi.$$

In the case that $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$ and S is the half-space $\mathbb{H}^d$ of all $x \in \mathbb{R}^d$ with $x_d \geq 0$, this fact is used to introduce differentiable manifolds with boundary.

In particular, for $d = 1$ the function φ is $\mathbb{R}_+$-differentiable if and only if its right-hand derivative $\varphi'_+(x)$ exists at any $x \in U$. In this case,

$$\nabla_{\mathbb{R}_+} \varphi = \varphi'_+.$$

(iii) Let S be balanced, which means that $-S \subset S$, and

$$V = X.$$

If U is convex and $\varphi \in C_S^1(U)$, then, without requiring that φ is differentiable in U°, the mean value theorem for the oriented derivative yields that

$$\varphi(x) - \varphi(y) = \int_0^1 \langle \nabla_S \varphi((1-t)x + ty), x - y \rangle \, \mathrm{d}t$$

for any $x, y \in U^\circ$. Moreover, if X is merely a Banach space, then the general definitions of

the S-derivative D_S and the linear space $C_S^1(U)$

in [4] apply, and to ensure that the above formula remains valid, we replace the integrand by $D_S\varphi((1-t)x+ty)(x-y)$, where $t \in [0,1]$.

(iv) The directional or Gâteaux derivative of φ at $x \in U$ in the positive direction of $h \in S$, defined as

$$D_h^+ \varphi(x) = \lim_{t \downarrow 0} \frac{\varphi(x + th) - \varphi(x)}{t},$$

exists if and only if φ is differentiable at x relative to the conic hull of $\{h\}$. In this case,

$$D_h^+ \varphi(x) = \langle \nabla_{\mathrm{coni}(h)} \varphi(x), h \rangle.$$

Hence, if U is convex, then $h = y - x$ with $y \in U$ is possible.

Further, positive homogeneity of S and S-differentiability of φ at x imply that

$$D_h^+ \varphi(x) = \langle \nabla_S \varphi(x), h \rangle \quad \text{for all } h \in S.$$

(v) Let S be balanced and for $n \in \mathbb{N}$ let $S_1, \dots, S_n$ be pairwise orthogonal balanced sets in X such that

$$S = S_1 \oplus \cdots \oplus S_n.$$

Then $C_S^1(U) = C_{S_1}^1(U) \cap \cdots \cap C_{S_n}^1(U)$ and we obtain the orthogonal decomposition

$$\nabla_S \varphi = \nabla_{S_1} \varphi + \cdots + \nabla_{S_n} \varphi \quad \text{for any } \varphi \in C_S^1(U),$$

as the explicit description of $C_S^1(U)$ in [4] shows.

Applications in stochastic calculus

For $T > 0$, $d \in \mathbb{N}$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, two kinds of extendible applications can be indicated:

(i) Let us focus on the separable Banach space $C([0,T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued continuous maps on $[0,T]$, equipped with the supremum norm $\|\cdot\|_\infty$.

For a continuous process $X : [0,T] \times \Omega \to \mathbb{R}^d$ and a continuous functional $F : C([0,T], \mathbb{R}^d) \to \mathbb{R}$, we seek to rewrite the difference

$$F(X^t) - F(X^s),$$

where $s, t \in [0,T]$ satisfy $s \leq t$ and X^t is the process X stopped at time t.

So, let $\mathcal{V}$ denote a linear subspace of $C([0,T],\mathbb{R}^d)$ containing all piecewise affine maps, F be $\mathcal{V}$-differentiable and the $\mathcal{V}$-oriented derivative $D_{\mathcal{V}}F$ be continuous. Then

$$F(X^t) - F(X^s) = D_{\mathcal{V}}F(X^s)(X^t - X^s) + R(X^s, X^t - X^s),$$

where the first-order reminder term is given by

$$R(x,h) := \int_0^1 \big(D_{\mathcal{V}}F(x+\lambda h) - D_{\mathcal{V}}F(x)\big)(h)\,\mathrm{d}\lambda$$

for all $x, h \in C([0,T],\mathbb{R}^d)$ and satisfies $R(x,h) = o(\|h\|_\infty)$ as $h \to 0$. In particular, for $\mathcal{V}$ we may take the Cameron-Martin space

$$H^1([0,T],\mathbb{R}^d)$$

of all absolutely continuous maps with a square-integrable weak derivative that plays a major role in [1].

(ii) For the Lebesgue measure λ on $[0, T]$ let us call two measurable processes $u, v : [0, T] \times \Omega \to \mathbb{R}^d$ equivalent if

$$u = v \quad \lambda \otimes \mathbb{P}\text{-a.e.}$$

Further, let $\mathbb{U}_+$ be the convex space of all equivalence classes of progressively measurable processes $u : [0, T] \times \Omega \to \mathbb{R}^d_+$ satisfying

$$\mathbb{E}\left[\int_0^T |u_t|^2 \, \mathrm{d}t \right] < \infty, \tag{1}$$

where $|\cdot|$ is the Euclidean norm induced by the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$.

For a measurable bounded function $f : [0, T] \times \mathbb{R}^d_+ \to \mathbb{R}$ the oriented derivative can be useful to minimise the functional

$$J(u) := \mathbb{E}\left[\int_0^T f(t, u_t)\, \mathrm{d}t \right]$$

over all control processes $u \in \mathbb{U}_+$.

Namely, the linear space $\mathbb{U}$ of all equivalence classes of progressively measurable processes $u : [0, T] \times \Omega \to \mathbb{R}^d$ satisfying (1), endowed with the inner product

$$\langle u, v \rangle_{\mathbb{U}} := \mathbb{E}\left[\int_0^T \langle u_t, v_t \rangle\, \mathrm{d}t \right]$$

is a Hilbert space.

Hence, by oriented differential calculus, if $u \in \mathbb{U}_+$ minimises J and the directional derivative $D^+_{v-u}J(u)$ exists for $v \in \mathbb{U}_+$, then

$$D^+_{v-u}J(u) \geq 0.$$

In particular, if J happens to be differentiable at u with respect to the 0-star convex set

$$\mathbb{S}_u := \{\lambda(v-u) \mid \lambda \in [0,1],\, v \in \mathbb{U}_+\},$$

then, as u is an $\mathbb{S}_u$-interior point of $\mathbb{U}_+$, it follows that

$$\langle \nabla_{\mathbb{S}_u} J(u), v-u \rangle_{\mathbb{U}} = D^+_{v-u}J(u) \geq 0 \quad \text{for all } v \in \mathbb{U}_+.$$

See [4, Lemma 2.28] for details.

Thus, if there is $L \geq 0$ such that $f(s, \cdot)$ is continuously differentiable on $]0, \infty[^d$, $\nabla_x f(s, \cdot)$ extends continuously to $\mathbb{R}^d_+$ and

$$|\nabla_x f(s, x) - \nabla_x f(s, y)| \leq L|x - y|$$

for all $s \in [0, T]$ and $x, y \in \mathbb{R}^d_+$, then it follows that

$$\langle \nabla_{\mathbb{S}_u} J(u), v - u \rangle_{\mathbb{U}} = \mathbb{E}\left[\int_0^T \langle \nabla_x f(t, u_t), v_t - u_t \rangle \, \mathrm{d}t \right] \geq 0$$

for each $v \in \mathbb{U}_+$. This leads to necessary first-order conditions to be satisfied by a minimiser u.

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Thank you for your attention!