Lyapunov stable solutions to ordinary and stochastic differential equations

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From ordinary to stochastic differential equations

For a product measurable function $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ let us first consider the ordinary differential equation (ODE)

$$\dot{x}(t) = b(t, x(t)) \quad \text{for } t \geq 0. \tag{1}$$

As b may fail to be continuous, we cannot expect to derive solutions in the classical sense but an integral version of (1) can be used.

Namely, a mild solution to (1) is a continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ such that $\int_0^t |b(s, x(s))| \, ds < \infty$ and

$$x(t) = x(0) + \int_0^t b(s, x(s)) \, ds \quad \text{for any } t \geq 0.$$

By the fundamental theorem of calculus, a mild solution x becomes a classical solution if and only if the function

$$\mathbb{R}_+ \to \mathbb{R}, \quad s \mapsto b(s, x(s)),$$

which is its weak derivative $\dot{x}$, is continuous.

To allow for randomness, we take a complete probability space $(\Omega, \mathcal{F}, P)$ on which there is a standard Brownian motion

$$W : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto W_t(\omega).$$

That means, W is a continuous process with independent increments such that $W_0 = 0$ and

$$W_t - W_s \sim \mathcal{N}(0, t - s)$$

for all $s, t \geq 0$ with $s < t$. In particular, W is a square-integrable martingale, and we let

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$$

be another measurable function.

Instead of analysing (1), let us focus on the stochastic differential equation (SDE)

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \quad \text{for } t \geq 0. \tag{2}$$

We recall that a solution to (2) is an adapted continuous process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

$$\int_0^{\cdot} |b(s, X_s)| + \sigma(s, X_s)^2\, ds < \infty$$

and

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s \quad \text{for all } t \geq 0 \text{ a.s.}$$

Thereby,

$$\int_0^\cdot \sigma(s, X_s)\, dW_s$$

is the stochastic integral of $\sigma(\cdot, X)$ with respect to W that is a local martingale with quadratic variation

$$\int_0^\cdot \sigma(s, X_s)^2\, ds.$$

In particular, if $\sigma = 0$, then for any $\omega \in \Omega$ the path

$$\mathbb{R}_+ \to \mathbb{R}, \quad t \mapsto X_t(\omega)$$

is a mild solution to the ODE (1). Conversely, every mild solution to (1) serves as path of a solution to (2).

Stability and uniqueness

Let us consider a regularity condition on the drift b that allows for negative partial Lipschitz coefficients:

(C.1) (Partial Hölder continuity condition)

There are $\alpha \in]0,1]$ and some measurable locally integrable function $\eta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\operatorname{sgn}(x - \tilde{x})\big(b(\cdot, x) - b(\cdot, \tilde{x})\big) \leq \eta |x - \tilde{x}|^{\alpha}$$

for any $x, \tilde{x} \in \mathbb{R}$.

Example (sums involving decreasing functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^m$ be measurable and $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ be increasing such that

$$b(\cdot, x) = \kappa - \eta_1 f_1(x) - \cdots - \eta_m f_m(x)$$

for any $x \in \mathbb{R}$. Then (C.1) holds if

$$\eta_1 \geq 0, \ldots, \eta_m \geq 0.$$

Thereby, b may fail to be continuous in $x \in \mathbb{R}$.

On the diffusion σ we impose a local regularity condition only:

(C.2) (Hölder continuity condition on compact sets)

For any $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|\sigma(\cdot, x) - \sigma(\cdot, \tilde{x})| \leq c_n |x - \tilde{x}|^{\frac{1}{2}}$$

for all $x, \tilde{x} \in [-n, n]$.

The exponent $\frac{1}{2}$ comes from the Yamada-Watanabe approach, since $\beta \in]0, 1]$ satisfies

$$\int_0^1 \frac{1}{x^{2\beta}}\, dx = \infty \quad \Leftrightarrow \quad \beta \geq \frac{1}{2}.$$

Example (sums of power functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^m$ be measurable and $\beta \in]0, \infty[^m$ be such that

$$\sigma(\cdot, x) = \kappa + \eta_1 |x|^{\beta_1} + \cdots + \eta_m |x|^{\beta_m}$$

for any $x \in \mathbb{R}$. Then (C.2) holds if η is bounded and

$$\beta_1 \geq \frac{1}{2}, \ldots, \beta_m \geq \frac{1}{2}.$$

Explicit L^1-comparison estimate (Meyer-Brandis, Proske and K., '21)

Let (C.1) and (C.2) hold and X and $\tilde{X}$ be two solutions to (2). Then $Y := X - \tilde{X}$ satisfies

$$E\big[|Y_t|\big] \leq e^{\alpha \int_0^t \eta_\alpha(s)\,ds} E\big[|Y_0|\big] + (1-\alpha)\int_0^t e^{\alpha \int_s^t \eta_\alpha(\tilde{s})\,d\tilde{s}} \eta^+(s)\,ds$$

for all $t \geq 0$ with $\eta_\alpha := \eta^+ - \eta^- \mathbb{1}_{\{1\}}(\alpha)$. In particular, if Y_0 and η^+ are integrable, then

$$\sup_{t\geq 0} E\big[|Y_t|\big] < \infty.$$

In this case, $\lim_{t\uparrow\infty} E[|Y_t|] = 0$ if $\alpha = 1$ and $\int_0^\infty \eta^-(s)\,ds = \infty$.

Proof ideas.

(i) The Yamada-Watanabe approach gives us a suitable increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C^2(\mathbb{R}_+)$ such that

$$\psi_n(0) = \psi_n'(0) = \psi_n''(0) = 0 \quad \text{for any } n \in \mathbb{N}$$

as well as $\sup_{n\in\mathbb{N}} \psi_n(x) = x$ and $\lim_{n\uparrow\infty} \psi_n'(x) = 1$ for each $x > 0$.

(ii) We may apply Itô's formula to $\psi_n(|Y|)$ for all $n \in \mathbb{N}$, since $\psi_n(|\cdot|) \in C^2(\mathbb{R})$. Further, we take a locally absolutely continuous function

$$u : \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{with} \quad u(0) = 1$$

and deduce the dynamics of $u \cdot \psi_n(|Y|)$ from Itô's product rule, which is the novel part of our work.

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(iii) Next, we show that Y is integrable and the function $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto E[|Y_t|]$ is locally bounded, provided $E[|Y_0|] < \infty$. Then

$$u(t)E\left[|Y_t|\right] = \lim_{n\uparrow\infty} u(t)E\left[\psi_n(|Y_t|)\right]$$
$$\leq E\left[|Y_0|\right] + \int_0^t E\left[\dot{u}(s)|Y_s| + u(s)\eta(s)|Y_s|^{\alpha}\right] ds$$

for any $t \geq 0$, by monotone convergence.

(iv) Hence, Young's inequality and the choice

$$u(t) = \exp\left(-\alpha \int_0^t \eta_\alpha(s)\, ds\right)$$

for all $t \geq 0$ yield the asserted estimate. □

As corollaries we obtain stability results in the sense of Lyapunov.

Exponential first moment stability (Meyer-Brandis, Proske and K., '21)

Let (C.1) and (C.2) hold for $\alpha = 1$. Further, let $\beta > 0$ and $\lambda < 0$ satisfy

$$\eta(s) \leq \lambda \beta s^{\beta - 1} \quad \text{for a.e. } s \geq 0.$$

Then (2) is β-exponentially stable in moment and λ is a Lyapunov exponent. That is, there is $c \geq 0$ such that

$$E\big[|X_t - \tilde{X}_t|\big] \leq c e^{\lambda t^\beta} E\big[|X_0 - \tilde{X}_0|\big]$$

for all $t \geq 0$ whenever X and $\tilde{X}$ are two solutions to (2).

Pathwise exponential stability (Meyer-Brandis, Proske and K., '21)

In addition to the preceding assumptions, let

$$\sup_{n \in \mathbb{N}} c_n < \infty.$$

Then (2) is pathwise β-exponentially stable with Lyapunov exponent $\lambda/2$. So, for any two solutions X and $\tilde{X}$ we have

$$\limsup_{t \uparrow \infty} \frac{1}{t^\beta} \log\left(|X_t - \tilde{X}_t|\right) \leq \frac{\lambda}{2} \quad \text{a.s.}$$

Derivation of strong solutions

For weak solutions to (2) we rely on the subsequent requirements:

(C.3) b is continuous in $x \in \mathbb{R}$ and locally bounded.

(C.4) (Partial affine growth condition)

There are measurable locally bounded functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ and $\upsilon : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\mathrm{sgn}(x) b(\cdot, x) \leq \kappa + \upsilon |x|$$

for every $x \in \mathbb{R}$.

Example (sums involving decreasing functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+^m$ be measurable and locally bounded and $n \in \mathbb{N}^m$ be such that

$$b(\cdot, x) = \kappa - \eta_1 x^{n_1} - \cdots - \eta_m x^{n_m}$$

for any $x \in \mathbb{R}$. Then (C.1), (C.3) and (C.4) are satisfied for $\alpha = 1$ if

the coordinates of n are odd.

However, b does not need to be of affine growth in $x \in \mathbb{R}$.

Existence of unique strong solutions (Meyer-Brandis, Proske and K., '21)

Let (C.1)-(C.4) hold for $\alpha = 1$ and $\sigma(\cdot, 0) = 0$. Then we have pathwise uniqueness for (2) and for any $\mathcal{F}_0$-measurable integrable random variable ξ there is a unique strong solution X^ξ such that

$$X_0^\xi = \xi \quad \text{a.s.}$$

Moreover, X^ξ is integrable and its first absolute moment function $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto E[|X_t^\xi|]$ is locally bounded.

As we show in our paper [1], all these methods can be applied to the McKean-Vlasov SDE

$$dX_t = b\big(t, X_t, \mathcal{L}(X_t)\big)\,dt + \sigma(t, X_t)\,dW_t \quad \text{for } t \geq 0,$$

where the product measurable drift b is defined on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R})$ instead of $\mathbb{R}_+ \times \mathbb{R}$.

In such a setting, $\mathcal{P}_p(\mathbb{R})$ is the Polish space of all Borel probability measures μ on $\mathbb{R}$ with finite p-th absolute moment

$$\int_{\mathbb{R}} |x|^p \,\mu(dx),$$

equipped with the p-th Wasserstein metric for $p \geq 1$.

If the diffusion σ depends on the law of the solution, we provide methods in another work [2] to handle the McKean-Vlasov SDE

$$dX_t = b\big(t, X_t, \mathcal{L}(X_t)\big)\,dt + \sigma\big(t, X_t, \mathcal{L}(X_t)\big)\,dW_t \quad \text{for } t \geq 0,$$

where the product measurable drift b and diffusion σ are defined on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_p(\mathbb{R})$ for $p \geq 2$ instead of $\mathbb{R}_+ \times \mathbb{R}$.

References

[1] A. Kalinin, T. Meyer-Brandis, and F. Proske.
Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs: a multidimensional Yamada-Watanabe approach.
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[2] A. Kalinin, T. Meyer-Brandis, and F. Proske.
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Thank you for your attention!