Stochastic Volterra equations with random functional coefficients in Banach spaces

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The probabilistic framework

In what follows, let I be a non-degenerate interval in $\mathbb{R}_+$ with $0 \in I$ and E be a separable Banach space.

Further, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t \in I}$ be a filtration of $\mathcal{F}$ such that the usual conditions hold.

We shall consider the separable Hilbert space ℓ^2 of all sequences $a = (a_i)_{i \in \mathbb{N}}$ of real numbers with $\sum_{i=1}^{\infty} a_i^2 < \infty$, and we recall from the Riesz-Fischer Theorem that any separable Hilbert space of infinite dimension is isometrically isomorphic to ℓ^2.

Then there is a norm $|\cdot|_2$ that turns the linear space $\mathcal{L}_2(\ell^2, E)$ of all E-valued linear, continuous and radonifying maps on ℓ^2 into a separable Banach space.

Under the hypothesis that E is 2-smooth, for any $\mathbb{F}$-progressively measurable process $U : I \times \Omega \to \mathcal{L}_2(\ell^2, E)$ satisfying

$$\int_0^{\cdot} |U_s|_2^2 \,\mathrm{d}s < \infty \quad \text{a.s.}$$

and a sequence $W = (W^{(i)})_{i \in \mathbb{N}}$ of independent standard Brownian motions with respect to $\mathbb{F}$, the stochastic integral

$$\int_0^{\cdot} U_s \,\mathrm{d}W_s$$

can be constructed as continuous $\mathbb{F}$-local martingale.

For a sequence $(Y_i)_{i\in\mathbb{N}}$ of independent $\mathcal{N}(0,1)$-distributed random variables a linear continuous map $L:\ell^2\to E$ is called radonifying if the series $\sum_{i=1}^{\infty} Y_i L(e_i)$ converges in second moment.

This property is independent of the choice of $(Y_i)_{i\in\mathbb{N}}$, and if it holds, then the distribution of the series is uniquely determined.

It follows that the resulting linear space $\mathcal{L}_2(\ell^2,E)$, equipped with the norm

$$|L|_2 := \mathbb{E}\left[\left|\sum_{i=1}^{\infty} Y_i L(e_i)\right|^2\right]^{\frac{1}{2}},$$

is indeed a separable Banach space and we have $|L|\le|L|_2$ for all $L\in\mathcal{L}_2(\ell^2,E)$.

Example (Hilbert-Schmidt norm)

For $d, m \in \mathbb{N}$ let $E = \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times d}$. If $L : \ell^2 \to \mathbb{R}^m$ is given by

$$L(a) := A \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix},$$

then $L \in \mathcal{L}_2(\ell^2, \mathbb{R}^m)$ and

$$|L|_2^2 = \mathbb{E}\big[|AY|^2\big] = \mathrm{tr}(A^\top A) = |A|^2$$

for the random vector $Y := (Y_1, \dots, Y_d)^\top$ and the Hilbert-Schmidt norm $|\cdot|$ on $\mathbb{R}^{m \times d}$.

We recall that E is 2-smooth if and only if there is a norm $\|\cdot\|$ that is equivalent to $|\cdot|$ such that

$$\|x+y\|^2 + \|x-y\|^2 \leq 2\|x\|^2 + \hat{c}\|y\|^2$$

for all $x, y \in E$ and some $\hat{c} \geq 2$. In this case, E is uniformly smooth and reflexive.

If equality holds for $\hat{c} = 2$, then we recover the parallelogram identity and $\|\cdot\|$ is induced by an inner product.

In general, an equivalent norm to $|\cdot|$ that turns E into a Hilbert space does not need to exist, since

$$L^p(S, \mathcal{B}, \mu)$$

is a 2-smooth Banach space whenever $(S, \mathcal{B}, \mu)$ is a σ-finite measure space and $p \geq 2$.

Within this framework, we consider the stochastic Volterra equation

$$X_t = \xi_t + \int_0^t \mathrm{B}_{t,s}(X_s)\,\mathrm{d}s + \int_0^t \Sigma_{t,s}(X_s)\,\mathrm{d}W_s \quad \text{a.s.} \tag{1}$$

for $t \in I$ coupled with a value condition. Here,

ξ is an E-valued $\mathbb{F}$-progressively measurable process,

$\mathcal{D}$ is a non-empty set of E-valued random vectors such that if $Y \in \mathcal{D}$ and Z is an E-valued random vector with

$$Y = Z \quad \text{a.s.}, \quad \text{then} \quad Z \in \mathcal{D},$$

and the random coefficients

$$\mathrm{B} : I \times I \times \Omega \times \mathcal{D} \to E \quad \text{and} \quad \Sigma : I \times I \times \Omega \times \mathcal{D} \to \mathcal{L}_2(\ell^2, E)$$

are admissible in a suitable sense.

As Banach-valued functionals,

$$\mathrm{B}_{t,s}(\cdot)(\omega) \quad \text{and} \quad \Sigma_{t,s}(\cdot)(\omega)$$

depend on the random vector X_s of any solution X, and their domain $\mathcal{D}$ could be the linear space

$$\mathcal{L}^p(\Omega, E)$$

of all E-valued p-fold integrable random vectors, where $s, t \in I$ with $s \leq t$, $\omega \in \Omega$ and $p \geq 1$.

Hence, (1) represents a novel type of stochastic Volterra equation for which the presented paper derives unique solutions.

In particular, for controlled and distribution-dependent coefficients we obtain strong solutions.

A solution to (1) is an E-valued $\mathbb{F}$-progressively measurable process X such that

$$X_s \in \mathcal{D} \quad \text{for a.e. } s \in I$$

and the set of all $t \in I$ for which

$$\begin{aligned} &\int_0^t |\mathrm{B}_{t,s}(X_s)| + |\Sigma_{t,s}(X_s)|_2^2 \,\mathrm{d}s < \infty \quad \text{and} \\ &X_t = \xi_t + \int_0^t \mathrm{B}_{t,s}(X_s)\,\mathrm{d}s + \int_0^t \Sigma_{t,s}(X_s)\,\mathrm{d}W_s \quad \text{a.s.} \end{aligned} \tag{2}$$

fails has Lebesgue measure zero. Moreover, if in addition $X_t \in \mathcal{D}$ and (2) holds for every $t \in I$, then X is called regular.

Controlled distribution-dependent coefficients

For $p \geq 1$ let $\mathcal{D} = \mathcal{L}^p(\Omega, E)$ and

$$\mathrm{B}_{t,s}(X_s) = f(t-s) b\big(X_s, \alpha_s, \mathcal{L}(X_s, \alpha_s)\big),$$

$$\Sigma_{t,s}(X_s) = g(t-s)\sigma\big(X_s, \alpha_s, \mathcal{L}(X_s, \alpha_s)\big)$$

for all $s, t \in I$ with $s < t$ and $X_s \in \mathcal{D}$, where $f, g : I \to \mathbb{R}$ are measurable,

A is a separable Banach space, α is an A-valued $\mathbb{F}$-progressively measurable control process and the maps

$$b : E \times A \times \mathcal{P}_p(E \times A) \to E, \quad \sigma : E \times A \times \mathcal{P}_p(E \times A) \to \mathcal{L}_2(\ell^2, E)$$

are Borel measurable.

For a separable Banach space S we recall that $\mathcal{P}_p(S)$ is the Polish space of all Borel probability measures on S satisfying

$$\int_S |x|^p \, \mu(\mathrm{d}x) < \infty,$$

equipped with the pth Wasserstein metric given by

$$\vartheta_p(\mu, \nu) := \inf_{\theta \in \mathcal{P}(\mu,\nu)} \left(\int_{S \times S} |x - y|^p \, \mathrm{d}\theta(x, y) \right)^{\frac{1}{p}},$$

where $\mathcal{P}(\mu, \nu)$ is the convex space of all Borel probability measures θ on $S \times S$ that admit

$$\mu \quad \text{and} \quad \nu$$

as first and second marginal distributions, respectively.

Moreover, note that strong solutions to (1) can be defined by using the natural filtration of ξ, α and $W^{(i)}$, where $i \in \mathbb{N}$.

Unique strong solutions (K., 2026)

Let $p \geq 2$, f and g^2 be locally integrable and b and σ be Lipschitz continuous. Then the following assertions hold:

(i) If $\operatorname{ess\,sup}_{s\in[0,t]} \mathbb{E}[|\xi_s|^p] + \mathbb{E}[|\alpha_s|^p] < \infty$ for all $t \in I$, then there is a unique strong solution X^ξ to (1) such that

$$\operatorname*{ess\,sup}_{s\in[0,t]} \mathbb{E}\big[|X_s^\xi|^p\big] < \infty \quad \text{for all } t \in I.$$

(ii) If just $\int_0^t \mathbb{E}[|\xi_s|^p] + \mathbb{E}[|\alpha_s|^p]\,\mathrm{d}s < \infty$ for any $t \in I$, then we obtain a unique strong solution X^ξ satisfying

$$\int_0^t \mathbb{E}\big[|X_s^\xi|^p\big]\,\mathrm{d}s < \infty \quad \text{for any } t \in I.$$

Unique strong solutions (cont.)

(iii) Let $\xi = \xi_0$ a.s. and $\mathbb{E}[|\xi_0|^q] + \operatorname{ess\,sup}_{s\in[0,t]} \mathbb{E}[|\alpha_s|^q] < \infty$ for all $q \geq 2$ and assume that

$$f(u) = u^{\gamma-1} \quad \text{and} \quad g(u) = u^{\delta-\frac{1}{2}}$$

for all $u \in I \setminus \{0\}$ with $\gamma \in]0,1]$ and $\delta \in]0,\frac{1}{2}]$. Then (1) admits a unique, strong and regular solution X^{ξ_0} such that

$$\sup_{s\in[0,t]} \mathbb{E}\left[|X_s^{\xi_0}|^q\right] < \infty$$

for all $t \in I$ and $q \geq 2$ and whose paths are locally β-Hölder continuous for any $\beta \in]0, \gamma \wedge \delta[$.

For establishing the existence of solutions, the progressive σ-field $\mathcal{A}$ is used as follows.

Progressively measurable weak modifications of stochastic Volterra integrals

Let $U : I \times I \times \Omega \to \mathcal{L}_2(\ell^2, E)$ be a $\mathcal{B}(I) \otimes \mathcal{A}$-measurable map such that

$$\int_0^t |U_{t,s}|_2^2 \, \mathrm{d}s < \infty \quad \text{a.s.} \quad \text{for any } t \in I.$$

Then there is an E-valued $\mathbb{F}$-progressively measurable process X such that

$$X_t = \int_0^t U_{t,s} \, \mathrm{d}W_s \quad \text{a.s.} \quad \text{for a.e. } t \in I.$$

The fact that solutions can be chosen to be strong relies on the following general result.

Measurability in the pth Wasserstein space

Let J be a non-empty set and S be a separable Banach space. Then for any product measurable process $X : J \times \Omega \to S$ such that

$$\mathbb{E}\big[|X_t|^p\big] < \infty$$

for all $t \in J$, the distribution map

$$J \to \mathcal{P}_p(S), \quad t \mapsto \mathcal{L}(X_t)$$

is Borel measurable.

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Thank you for your attention!