A general stochastic calculus approach to insider trading

Francesca Biagini[1] Bernt Øksendal[2),3)]

March 11, 2008

[1] Department of Mathematics, University of Bologna,
Piazza di Porta S. Donato, 5
I-40127 Bologna, Italy
Email: biagini@dm.unibo.it

[2] Center of Mathematics for Applications (CMA)
Department of Mathematics , University of Oslo
Box 1053 Blindern , N-0316 Oslo, Norway ,
Email: oksendal@math.uio.no

[3] Norwegian School of Economics and Business Administration ,
Helleveien 30 , N-5045 Bergen, Norway

Abstract

The purpose of this paper is to present a general stochastic calculus approach to insider trading. We consider a market driven by a standard Brownian motion $B(t)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$ where the coefficients are adapted to a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$, with $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0,T]$, $T > 0$ being a fixed terminal time. By an *insider* in this market we mean a person who has access to a filtration (information) $\mathbb{H} = \{\mathcal{H}_t\}_{0\leq t\leq T}$ which is strictly bigger than the filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$. In this context an insider strategy is represented by an $\mathcal{H}_t$-adapted process $\phi(t)$ and we interpret all anticipating integrals as the forward integral defined in [23],[25].
We consider an optimal portfolio problem with general utility for an insider with access to a general information $\mathcal{H}_t \supset \mathcal{G}_t$ and show that if an optimal insider portfolio $\pi^*(t)$ of this problem exists, then $B(t)$ is an $\mathcal{H}_t$-semimartingale, i.e. the enlargement of filtration property holds. This is a converse of previously known results in this field. Moreover, if π^* exists we obtain an explicit expression in terms of π^* for the semimartingale decomposition of $B(t)$ with respect to $\mathcal{H}_t$. This is a generalization of results in [16], in [20] and in [2].

MSC 2000 subject classifications: Primary 60HXX; Secondary 91B28.
Key words and phrases: Forward integral, Skorohod integral, Wick product, insider trading, utility function.

Investigation supported by University of Bologna. Funds for selected research topics

1 Introduction

How do we model the hedging by an insider in finance? Let $\{B(t)\}_{t\geq 0} = \{B(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$. By an *insider* we mean a person who has access to a filtration $\mathbb{H} = \{\mathcal{H}_t\}_{0\leq t\leq T}$ which is strictly bigger than the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ of $B(t)$. Therefore the question is how to interpret integrals of the form

$$\int_0^T \phi(t,\omega)dB(t) \tag{1.1}$$

where ϕ is assumed to be adapted to $\mathcal{H}_t \supset \mathcal{F}_t$.
A natural, and the most common, approach to this question is to assume that $\mathcal{H}_t$ is such that $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$. In this case we can write

$$B(t) = \widehat{B}(t) + A(t), \quad 0 \leq t \leq T \tag{1.2}$$

where $\widehat{B}(t)$ is a $\mathcal{H}_t$-Brownian motion and $A(t)$ is a continuous $\mathcal{H}_t$-adapted finite variation process.
If $A(t)$ has the form

$$A(t) = \int_0^t \alpha(u)du \tag{1.3}$$

then the process $\alpha(\cdot)$ is called the *information drift*. In general, if a relation of the form (1.2) holds, then it is natural to define

$$\int_0^T \phi(t,\omega)dB(t) = \int_0^T \phi(t,\omega)d\widehat{B}(t) + \int_0^T \phi(t,\omega)dA(t) \tag{1.4}$$

because both terms of the right-hand side are well-defined.

Example 1.1 *Let $T_0 \geq T$ and*

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(B(T_0)); \quad 0 \leq t \leq T. \tag{1.5}$$

In other words $\mathcal{H}_t$ is the σ-algebra generated by $\mathcal{F}_t$ and the terminal value $B(T_0)$. Then it can be shown that (see e.g. [14])

$$\widehat{B}(t) := B(t) - \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds; \quad 0 \leq t \leq T \tag{1.6}$$

is an $\mathcal{H}_t$-Brownian motion, and in this case (1.2) holds with

$$A(t) := \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds; \quad 0 \leq t \leq T. \tag{1.7}$$

In general, there are several difficulties with this approach:

(i) How do we know if (1.2) is possible?
(ii) If (1.2) is possible, how do we find $A(t)$?
(iii) What do we do if (1.2) is not possible?
Partial answers to (i) and (ii) can be found in the contributions to the book of Jeulin and Yor ([14]). See also [12].
The purpose of this paper is to present a more general approach to insider trading which does not assume that (1.2) holds. One of our main results is in fact a kind of converse: we consider a market where the coefficients are adapted to a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0 \le t \le T}$ with $\mathcal{F}_t \subset \mathcal{G}_t$ for all $t \in [0, T]$. In this market we study an optimal portfolio problem with general utility for an insider with access to the information $\mathcal{H}_t \supset \mathcal{G}_t$. We show that, if an optimal insider portfolio $\pi^*(t)$ of this problem exists, then in fact (1.2) and (1.3) hold, with $\alpha(t)$ closely related to $\pi^*(t)$.

2 Some preliminaries

Here we recall the definition and some properties of the forward integral. For more information, see [23], [25], [26],[27].

Definition 2.1 *Let $\phi(t, \omega)$ be a measurable process. The forward integral of ϕ is defined by*

$$\int_0^\infty \phi(t, \omega) d^- B(t) = \lim_{\epsilon \to 0} \int_0^\infty \phi(t, \omega) \frac{B(t + \epsilon) - B(t)}{\epsilon} dt \tag{2.1}$$

if the limit exists in probability, in which case ϕ is called forward integrable. *If the limit exists also in $L^2(P)$, we write $\phi \in Dom_2 \delta^-$.*

Note that if ϕ is càglàd (i.e. left continuous with right limits) and forward integrable, then

$$\int_0^T \phi(t, \omega) d^- B(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) \cdot \Delta B(t_j) \tag{2.2}$$

To see this, we argue as follows. We may assume that $\phi(t, \omega) = \sum_{j=1}^n \phi(t_j, \omega) \chi_{(t_j, t_{j+1}]}(t)$. Then

$$\int_0^\infty \phi(t, \omega) d^- B(t) = \lim_{\epsilon \to 0} \int_0^\infty \phi(t, \omega) \frac{B(t + \epsilon) - B(t)}{\epsilon} dt =$$
$$\sum_{j=1}^n \phi(t_j) \lim_{\epsilon \to 0} \int_{t_j}^{t_{j+1}} \frac{B(t + \epsilon) - B(t)}{\epsilon} dt =$$

$$\sum_{j=1}^{n} \phi(t_j) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_j}^{t_{j+1}} (\int_{t}^{t+\epsilon} dB_u)dt =$$
$$\sum_{j=1}^{n} \phi(t_j) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_j}^{t_{j+1}} (\int_{u-\epsilon}^{u} dt)dB_u =$$
$$\sum_{j=1}^{n} \phi(t_j) \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_j}^{t_{j+1}} \epsilon \, dB_u =$$
$$\sum_{j=1}^{n} \phi(t_j)(B(t_{j+1}) - B(t_j)).$$

We now explain how the forward integral appears naturally in insider modeling.

Remark 2.2 *(i) First, consider a buy-and-hold strategy $\psi(t,\omega) = I_{\{\tau_1 < t \leq \tau_2\}}$, where τ_1, τ_2 are bounded random times. As an immediate consequence of (2.2) we obtain that*

$$\int_0^T \psi(t,\omega)d^- B(t) = \lim_{\Delta t_j \to 0} \sum_j \psi(t_j) \cdot \Delta B(t_j) = \int_{\tau_1}^{\tau_2} dB(t) = B(\tau_2) - B(\tau_1). \tag{2.3}$$

Hence, if $B(t)$ is interpreted as the price process, then the forward integral of ψ gives exactly the money gained by this strategy, as it should. We also remark that this property holds even if τ_1, τ_2 are not necessarily stopping times.
(ii) Second, let $\mathcal{H}_t \supset \mathcal{F}_t$ as in Section 1 and assume that $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$, so that (1.2) holds, i.e.

$$B(t) = \widehat{B}(t) + A(t); \qquad 0 \leq t \leq T$$

where $\widehat{B}(t)$ is a $\mathcal{H}_t$-adapted Brownian motion, $A(t)$ is a $\mathcal{H}_t$-adapted finite variation continuous process. Let $\phi(s,\omega)$ be forward integrable and càglàd. Then $\int_0^T \phi(t)dB(t)$ exists as a semimartingale integral and

$$\int_0^T \phi(t)dB(t) = \int_0^T \phi(t)d^- B(t) \tag{2.4}$$

Proof. By equation (1.2) we get

$$\int_0^T \phi(t)d\widehat{B}(t) + \int_0^T \phi(t)dA(t) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) \cdot (\Delta\widehat{B}(t_j) + \Delta A(t_j)) =$$
$$\lim_{\Delta t_j \to 0} \sum_j \phi(t_j) \cdot \Delta B(t_j) = \int_0^T \phi(t)d^- B(t)$$

□

In view of (2.4) we see that if (1.2) holds, then it is natural to interpret "$\int_0^T \phi(t,\omega)dB(t)$" as $\int_0^T \phi(t,\omega)d^- B(t)$ in insider trading models.

In view of the above, from now on we adopt the forward integral as our mathematical model in insider trading in general, without assuming that (1.2) holds. Thus in (1.1) we put

$$\int_0^T \phi(t,\omega)dB(t) := \int_0^T \phi(t)d^-B(t) \tag{2.5}$$

for all processes $\phi(t,\omega) \in Dom_2\delta^-$.

3 Optimal portfolio of an insider with general utility

Let $B(t)$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Let $\{\mathcal{G}_t\}_{t\geq 0}$, $\{\mathcal{H}_t\}_{t\geq 0}$ be filtrations such that

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t \subset \mathcal{F} \qquad \forall t \in [0,T] \tag{3.1}$$

where $T > 0$ is a fixed terminal time. Consider the following financial market, with two investment possibilities:

1. A risk-free investment, with price

$$dS_0(t) = r(t,\omega)S_0(t)dt; \quad S_0(0) = 1 \tag{3.2}$$

2. A risky investment, with price

$$dS_1(t) = S_1(t)[\mu(t,\omega)dt + \sigma(t,\omega)d^-B(t)]; \quad S_1(0) = x > 0. \tag{3.3}$$

We assume that the coefficients $r(t) = r(t,\omega)$, $\mu(t) = \mu(t,\omega)$ and $\sigma(t) = \sigma(t,\omega)$ satisfy the following conditions:

$$r(t),\ \mu(t) \text{ and } \sigma(t) \text{ are } \mathcal{G}_t\text{-adapted} \tag{3.4}$$

$$E\left[\int_0^T \left\{|r(t)| + |\mu(t)| + \sigma(t)^2\right\} dt\right] < \infty \tag{3.5}$$

$$\sigma(t) \text{ is càglàd and forward integrable} \tag{3.6}$$

The corresponding anticipating integral on the right hand side of (3.3) is interpreted as a **forward** integral. This models a market which is influenced by large investors with insider information, i.e., with access to the information $\mathcal{G}_t$ or, more generally, a market possibly influenced by other random events than those described by $\mathcal{F}_t$.
In this insider market we consider an agent with access to a filtration $\mathcal{H}_t \supset \mathcal{G}_t$.
Let $\pi(t)$ be a portfolio denoting the fraction of the wealth invested in the stock at time t by an insider. Thus $\pi(t)$ is a $\mathcal{H}_t$*-adapted* stochastic process. If $\sigma(t)\pi(t)$ is càglàd, forward integrable and

$$\int_0^T (|\mu(s) - r(s)||\pi(s)| + \sigma^2(s)\pi^2(s))ds < \infty \quad \text{a.s.}$$

holds, then the corresponding wealth $X(t) = X^{(\pi)}(t)$ of the insider at time t will satisfy the stochastic forward equation

$$dX(t) = X(t)\left[\{r(t) + (\mu(t) - r(t))\pi(t)\}\, dt + \sigma(t)\pi(t)d^{-}B(t)\right]; \quad X(0) = x_0. \tag{3.7}$$

Fix a generalised utility function

$$U : [0, \infty) \longrightarrow [-\infty, \infty) \tag{3.8}$$

assumed to be continuously differentiable on $(0, \infty)$. If the function U is also concave and non-decreasing, it is an utility function in the regular sense, but these assumptions are not needed in our argument.
We now introduce the set $\mathcal{A}$ of admissible strategies for the insider trader.

Definition 3.1 *An $\mathcal{H}_t$-adapted stochastic process $\pi(t) = \pi(t, \omega)$ is called* admissible *if*

1. *$\pi(t)$ is càglàd.*
2. *$\sigma(t)\pi(t)$ is forward integrable.*
3. *$\int_0^T (|\mu(s) - r(s)||\pi(s)| + \sigma^2(s)\pi^2(s))ds < \infty$ a.s.*
4. *$U'(X^{(\pi)}(T)) > 0$ a.s. and $E\left[U'(X^{(\pi)}(T))X^{(\pi)}(T)\right] < \infty$, where $U'(x) = \frac{d}{dx}U(x)$.*

We denote by $\mathcal{A}$ the set of all admissible portfolios π.

For $\pi \in \mathcal{A}$ define

$$M_\pi(t) = \int_0^t \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s)\right\} ds + \int_0^t \sigma(s)d^{-}B(s). \tag{3.9}$$

We assume that the following holds:
5. For all $\pi, \theta \in \mathcal{A}$ with θ bounded there exists $\delta > 0$ such that the family

$$\left\{U'(X^{\pi+\epsilon\theta}(T))X^{\pi+\epsilon\theta}(T)|M_{\pi+\epsilon\theta}(T)|\right\}_{0\le\epsilon\le\delta} \tag{3.10}$$

is uniformly integrable, and
6. for all $t \in [0, T]$ the process $\pi(s) := \chi_{(t,t+h]}(s)\theta_0(\omega)$, with $h > 0$ and $\theta_0(\omega)$ a bounded $\mathcal{H}_t$-measurable random variable, belongs to $\mathcal{A}$.

Remark 3.2 *It is clear that many choices of such spaces $\mathcal{A}$ are possible. For example, let $\mathcal{A} = \mathcal{A}_0$ be the set of all step processes of the form*

$$\phi(\omega, t) = \sum_{k=0}^{N-1} \psi_k(\omega)\chi_{(t_k,t_{k+1}]}(t)$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of $[0, T]$ and $\psi_k(\omega)$ is $\mathcal{H}_{t_k}$-measurable and bounded. Then the space $\mathcal{A}_0$ satisfies 1.,2.,3., 6. and also 4. and 5. for many choices of utility functions U (e.g. $U(x) = \ln x$, $U(x) = \frac{1}{\gamma}x^\gamma$ for $\gamma \in (-\infty, 1)\setminus\{0\}$ constant or $U(x) = -e^{-\beta x}$ with $\beta > 0$ constant). Or let $\bar{\mathcal{A}}_0$ be the closure of $\mathcal{A}_0$ in the norm

$$\|\phi\|^2 := E\left[\int_0^T (|\mu(t)-r(t)||\pi(t)| + \phi(t)^2\sigma(t)^2)dt + (\int_0^T \phi(t)d^-B(t))^2\right]$$

and choose $\mathcal{A}$ to be the set of càglàd processes $\phi(t)$ in $\bar{\mathcal{A}}_0$ which satisfy 2., 4. and 5.

Consider the following *insider optimal portfolio problem*:

Problem 3.3 : *Find $V_T^{\mathbb{G},\mathbb{H}} \in \mathbb{R}$ and $\pi^* \in \mathcal{A}$ such that*

$$V_T^{\mathbb{G},\mathbb{H}} := \sup_{\pi\in\mathcal{A}} E\left[U(X^{(\pi)}(T))\right] = E\left[U(X^{(\pi^*)}(T))\right] \tag{3.11}$$

We call $V_T^{\mathbb{G},\mathbb{H}} \leq \infty$ the value of the optimal portfolio problem and $\pi^* \in \mathcal{A}$ the optimal portfolio (if it exists).

This problem was first studied by Pikovsky and Karatzas ([16]) in the special case when $\mathcal{F}_t = \mathcal{G}_t$. They assume that

$$U(x) = \log x \tag{3.12}$$

and that $\mathcal{H}_t$ has the form

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(L); \quad 0 \leq t \leq T \tag{3.13}$$

for some fixed random variable L. They also assume that there exists a $\mathcal{H}_t$-adapted process $\alpha(t)$ such that

$$\widehat{B}(t) = B(t) - \int_0^t \alpha(s)ds \tag{3.14}$$

is a $\mathcal{H}_t$-Brownian motion.
Subsequently, this problem has been studied by many authors, but to the best of our knowledge they all assume that (3.13) and (3.14) hold. See for example Leon, Navarro and Nualart [20] and Imkeller [12] and the references therein. The recent paper Corcuera et al. [4] has a different, but related assumption. The purpose of our paper is to study Problem 3.3 for general filtrations $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$ and for a general utility function U, without assuming (3.12), (3.13) or (3.14).

We first recall the following result ([27]):

Theorem 3.4 *Let $\xi(t)$ and $\eta(t)$ be $\mathcal{G}_t$-adapted processes such that $\int_0^t (|\xi(s)| + \eta^2(s))\, ds < \infty$ a.s. for all $t > 0$ and η is forward integrable. Then the equation*

$$dX(t) = X(t)[\xi(t)dt + \eta(t)d^-B(t)]; \quad X(0) = x_0 \tag{3.15}$$

has the unique solution

$$X(t) = x_0 \exp\left(\int_0^t \left\{\xi(s) - \frac{1}{2}\eta^2(s)\right\} ds + \int_0^t \eta(s)d^-B(s)\right); \quad t \geq 0 \tag{3.16}$$

We now return to Problem 3.3.

Theorem 3.5 *(i) Suppose that there exists an optimal portfolio $\pi \in \mathcal{A}$ for the problem*

$$V_T^{\mathbb{G},\mathbb{H}} := \sup_{\pi \in \mathcal{A}} E\left[U(X^{(\pi)}(T))\right]. \tag{3.17}$$

Then the process $M_\pi(t)$ of (3.9) is an $(\mathcal{H}, Q_\pi)$-martingale, where

$$dQ_\pi(\omega) = F_\pi(T)dP(\omega) \quad \text{on } \mathcal{H}_T \tag{3.18}$$

with

$$F_\pi(T) = E\left[U'(X^{(\pi)}(T))X^{(\pi)}(T)\right]^{-1} U'(X^{(\pi)}(T))X^{(\pi)}(T) \tag{3.19}$$

(ii) Conversely, suppose that there exists $\pi \in \mathcal{A}$ such that the process $M_\pi(t)$ of (3.9) is an $(\mathcal{H}, Q_\pi)$-martingale. Then π is an optimal portfolio for problem (3.17).

Proof. In the following we may assume $x_0 = 1$ without loss of generality. (i) By Theorem 3.4 the solution $X^{(\pi)}(t)$ of the wealth equation (3.7) is

$$X^{(\pi)}(t) = \exp\left(\int_0^t \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\} ds + \int_0^t \sigma(s)\pi(s)d^-B(s)\right) \tag{3.20}$$

where $t \geq 0$. Hence

$$E\left[U(X(T))\right] = H(\pi) :=$$
$$E\left[U(\exp\left(\int_0^T \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\} ds + \int_0^T \sigma(s)\pi(s)d^-B(s)\right))\right] \tag{3.21}$$

for $\pi \in \mathcal{A}$. Now suppose π maximizes $H(\pi)$ over $\mathcal{A}$. Then if $y \in \mathbb{R}$ and $\theta(t)$ is another ($\mathcal{H}_t$-adapted) process in $\mathcal{A}$ we have that the function

$$y \longrightarrow H(\pi + y\theta)$$

is maximal for $y = 0$. Therefore

$$0 = \frac{d}{dy}H(\pi + y\theta)|_{y=0} =$$
$$E\left[U'(X^{(\pi)}(T))X^{(\pi)}(T)(\int_0^T \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s)\right\}\theta(s)ds + \int_0^T \sigma(s)\theta(s)d^-B(s))\right]. \tag{3.22}$$

Now fix $t \in [0, T)$ and apply (3.22) to the process

$$\theta(s) = \chi_{[t,t+h)}(s)\theta_0(\omega); \quad 0 \leq s \leq T$$

where $h > 0$ is a constant such that $t + h \leq T$ and $\theta_0(\omega)$ is a bounded $\mathcal{H}_t$-measurable random variable. Then (3.22) and (3.10) give

$$0 = E\left[F_\pi(T)\left(\int_t^{t+h}\left\{\mu(s) - r(s) - \sigma^2(s)\pi(s)\right\}ds + \int_t^{t+h}\sigma(s)d^-B(s)\right)\theta_0(t)\right] \quad (3.23)$$

Since this holds for all bounded $\mathcal{H}_t$-measurable $\theta_0(\omega)$ we conclude that

$$E\left[F_\pi(T)\left(\int_t^{t+h}\left\{\mu(s) - r(s) - \sigma^2(s)\pi(s)\right\}ds + \int_t^{t+h}\sigma(s)d^-B(s)\right)\bigg|\,\mathcal{H}_t\right] = 0. \quad (3.24)$$

Then, with $M_\pi(t)$ as in (3.9) and Q_π as in (3.18)-(3.19) we get, by Bayes' Theorem,

$$\begin{aligned} &E_{Q_\pi}\left[M_\pi(t+h) - M_\pi(t)|\,\mathcal{H}_t\right] = \\ &E\left[F_\pi(T)|\,\mathcal{H}_t\right]^{-1} E\left[F_\pi(T)(M_\pi(t+h) - M_\pi(t))|\,\mathcal{H}_t\right] = 0 \end{aligned} \quad (3.25)$$

Since $M_\pi(t)$ is $\mathcal{H}_t$-adapted, this gives

$$E_{Q_\pi}\left[M_\pi(t+h)|\,\mathcal{H}_t\right] = M_\pi(t). \quad (3.26)$$

Hence $M_\pi(t)$ is an $(\mathcal{H}_t, Q_\pi)$-martingale, as claimed.
(ii) Conversely, if $\pi \in \mathcal{A}$ is such that (3.26) holds, then (3.25) follows and hence (3.24) and (3.23) also. By linearity it follows that (3.22) holds for all $\theta \in \mathcal{A}$ of the form

$$\theta(s) = \sum_{j=1}^{N} \theta_{t_j}(\omega)\chi_{(t_j,t_{j+1}]}(s); \quad 0 \le s \le T \quad (3.27)$$

where θ_{t_j} is $\mathcal{H}_{t_j}$-measurable and bounded. Let $\mathcal{A}_0$ denote the set of such θ. Then we can conclude that the directional derivative of $H(\cdot)$ at π is 0 in all the directions $\theta \in \mathcal{A}_0$. Since $H(\cdot)$ is concave, the result follows from this by a density argument. □

We proceed to prove the main result of this paper:

Theorem 3.6 *(i) A process $\pi \in \mathcal{A}$ is optimal for the problem (3.17) if and only if the process*

$$\hat{M}_\pi(t) := M_\pi(t) - \int_0^t \frac{d[M_\pi, Z](s)}{Z(s)} \quad (3.28)$$

is an $(\mathcal{H}, P)$-martingale, where

$$Z(t) = E\left[F_\pi(T)|\,\mathcal{H}_t\right]^{-1}, \quad (3.29)$$

with $M_\pi(t)$ and $F_\pi(T)$ given by (3.9) and (3.19) respectively.
(ii) In particular, If an optimal $\pi \in \mathcal{A}$ exists, then the process

$$N(t) = \int_0^t \sigma(s)d^-B(s) \quad (3.30)$$

is an $(\mathcal{H}, P)$-semimartingale.
(iii) If an optimal $\pi \in \mathcal{A}$ exists and

$$\sigma(s) \neq 0 \quad \text{for a.a. } (s,\omega) \in [0,T] \times \Omega \quad (3.31)$$

then

$$B(t) \text{ is an } (\mathcal{H}, P)\text{-semimartingale.} \quad (3.32)$$

PROOF. By Theorem 3.5 we know that $M_\pi(t)$ of (3.9) is an $(\mathcal{H}, Q_\pi)$-martingale, with the notation of (3.19) and (3.19). Hence

$$dP(\omega) = G_\pi(T) dQ_\pi(\omega) \quad \text{on } \mathcal{H}_T,$$

where

$$G_\pi(T) = F_\pi(T)^{-1}.$$

Let $Z(t)$ be the $(\mathcal{H}, Q_\pi)$-martingale defined by

$$Z(t) = E_{Q_\pi}\left[G_\pi(T) | \mathcal{H}_t \right] = E\left[F_\pi(T) | \mathcal{H}_t \right]^{-1} E\left[F_\pi(T) G_\pi(T) | \mathcal{H}_t \right] = E\left[F_\pi(T) | \mathcal{H}_t \right]^{-1}.$$

By the Girsanov Theorem we get that the process $\hat{M}_\pi(t)$ of (3.28) is an $(\mathcal{H}, P)$-martingale, as claimed. The argument goes both ways.
(ii) is a direct consequence of (i).
(iii) By (ii) we know that

$$N(t) = \int_0^t \sigma(s) d^- B(s)$$

is an $(\mathcal{H}, P)$-semimartingale. Then if (3.31) holds, we get that

$$\int_0^t \sigma^{-1}(s) dN(s) = \int_0^t \sigma^{-1}(s) \sigma(s) d^- B(s) = B(t)$$

is an $(\mathcal{H}, P)$-semimartingale also. □

Remark 3.7 *This result is related to Theorem 7.2 on page 504 of [6], where it is proved that if there is no arbitrage for an insider using simple integrands, then the price process is a semimartingale. However, our result is not a consequence of the result in [6], since we are considering a generalised utility function which may not be even concave and non-decreasing (see 3.8).*
Theorem 3.6 is also related to a result of [19], where it is proved that the existence of an optimal strategy for some insider having strictly monotonic continuous and convex preferences implies the absence of free lunches. This again implies the semimartingale property for the asset prices [7], [21].

Theorem 3.8 *Suppose $\sigma(t) \neq 0$ for a.a. (t, ω). Suppose that there exists an optimal portfolio $\pi^*(t)$ for Problem 3.3. Then $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$ and P, i.e. there exists an $\mathcal{H}_t$-adapted finite variation process $A(t)$ such that*

$$\hat{B}(t) := B(t) - A(t) \quad \text{is an } \mathcal{H}_t\text{-Brownian motion.} \tag{3.33}$$

Moreover, we have the following explicit relation between A and the optimal portfolio π^*

$$\pi^*(t) := \sigma^{-2}(t) \left[\mu(t) - r(t) - Z^{-1}(t) \frac{d}{dt} [M_\pi^*, Z]^{ac}(t) + \sigma(t) \frac{d}{dt} A^{ac}(t) \right] \tag{3.34}$$

where $A^{ac}(t)$ denotes the absolutely continuous part of $A(t)$, given by the Lebesgue decomposition theorem, and similarly with $[M_\pi^*, Z]^{ac}(t)$.

PROOF. The statement (3.33) follows from Theorem 3.6. Then (3.34) is obtained by combining (3.33) with (3.28), (3.3) and (3.9) and solving for π^*. □

Note that this result gives an explicit expression for the semimartingale decomposition of B with respect to $\mathcal{H}$ in terms of the optimal portfolio π^*.

Remark 3.9 *If the filtration $\mathcal{H}_t$ is generated by $\hat{B}(t)$, then by Girsanov Theorem and by the representation property of Brownian martingales it follows that there exists an $\mathcal{H}_t$-adapted integrable process $\alpha(t)$ such that*

$$\hat{B}(t) := B(t) - \int_0^t \alpha(s)ds \quad \text{is an } \mathcal{H}_t\text{-Brownian motion.} \tag{3.35}$$

In this case (3.34) provides an explicit representation of the optimal portfolio $\pi^*(t)$ in terms of the information drift $\alpha(t)$.

We can apply Theorem 3.6 to the particular case of logarithmic utility and get

Corollary 3.10 (Logarithmic utility case) *Suppose that there exists an optimal portfolio $\pi^*(t)$ for Problem 3.3 when $U(x) = \log x$.*

1. *Define α^* by*

$$\pi^*(s) = \frac{\mu(s) - r(s)}{\sigma^2(s)} + \frac{\alpha^*(s)}{\sigma(s)} \tag{3.36}$$

and put

$$\beta(s) = \frac{\mu(s) - r(s)}{\sigma(s)}. \tag{3.37}$$

Then

$$\hat{B}(t) := B(t) - \int_0^t \alpha^*(s)ds \quad \text{is an } (\mathcal{H}, P)\text{-Brownian motion}$$

and

$$V_T^{\mathbb{G},\mathbb{H}} = \log x_0 + E\left[\int_0^T \left\{r(s) + \frac{1}{2}(\beta(s) + \alpha^*(s))^2\right\} ds\right] = V_T^{\mathbb{F},\mathbb{F}} + E\left[\int_0^T (\beta(s)\alpha^*(s) + \frac{1}{2}(\alpha^*(s))^2)ds\right]. \tag{3.38}$$

2. *Suppose in addition that $\beta(s)$ is $\mathcal{F}_s$-measurable, $0 \le s \le T$. Then*

$$E\left[\int_0^T \beta(s)\alpha^*(s)ds\right] = 0 \tag{3.39}$$

and the corresponding value is

$$V_T^{\mathbb{G},\mathbb{H}} = \log x_0 + E\left[\int_0^T \left\{r(s) + \frac{1}{2}[\beta(s)^2 + (\alpha^*(s))^2]\right\} ds\right] = V_T^{\mathbb{F},\mathbb{F}} + \frac{1}{2}E\left[\int_0^T (\alpha^*(s))^2 ds\right] \tag{3.40}$$

PROOF. First we note that since π^* is admissible, then the corresponding optimal value function in (3.38) is finite.
It only remains to prove (3.39). If $\beta(\cdot)$ is $\mathcal{F}$-adapted, then by (3.33)

$$E\left[\int_0^T \beta(s)\alpha^*(s)ds\right] = E\left[\int_0^T \beta(s)(dB(s) - d\hat{B}(s))\right] =$$
$$E\left[\int_0^T \beta(s)dB(s)\right] - E\left[\int_0^T \beta(s)d\hat{B}(s)\right] = 0.$$

□

Here $V_T^{\mathbb{F},\mathbb{F}}$ represents the value of the honest trader when $\mathbb{G} = \mathbb{F}$ and

$$\frac{1}{2}E\left[\int_0^T (\alpha^*(s))^2 ds\right]$$

is the *additional value (utility)* obtained by the insider. Theorem 3.10 represents a converse of Theorem 2.1 in [12].
Under certain assumptions, including the one that $B(t)$ is a $\mathcal{H}_t$-semimartingale, the optimal portfolio Problem 3.3 has been studied by many authors, also for other utility functions than the logarithm. See e.g. [9] and the references therein. We note in particular that in [9] the following general approach is used. Suppose that there exists an $\mathcal{H}$-adapted process $\alpha(s)$ such that the process

$$\hat{B}(t) := B(t) - \int_0^t \alpha(s)ds \tag{3.41}$$

is an $(\mathcal{H}, P)$-Brownian motion. Then we have

$$\int_0^t \phi(s)d^-B(s) = \int_0^t \phi(s)d\hat{B}(t) + \int_0^t \phi(s)\alpha(s)ds \tag{3.42}$$

for all the forward integrable processes ϕ. Hence the Problem 3.3 reduces to a classical optimal portfolio problem with the process $\hat{B}(t)$ as the driving Brownian motion and with

$$\hat{\mu}(t) := \mu(t) + \sigma(t)\alpha(t) \tag{3.43}$$

as the new mean rate of return in the stock price model (3.3), i.e.

$$dS_1(t) = S_1(t)[(\mu(t) + \sigma(t)\alpha(t))dt + \sigma(t)d^-\hat{B}(t)] \tag{3.44}$$

Under certain conditions one can now apply the classical martingale method to solve optimal consumption and portfolio problems in the new setting (3.44). We refer to [9] and [18] for details.
Combining this approach with Theorem 3.6 we obtain explicit formulas for the optimal insider portfolio without the assumption (3.41), but with the only assumption that an optimal portfolio exists. We illustrate this by giving the solution in the power utility case.

Corollary 3.11 (Power utility case) *Suppose*

$$U(x) = \frac{1}{\gamma}x^\gamma; x \in [0, \infty) \tag{3.45}$$

where $\gamma \in (-\infty, 1) - \{0\}$ *is a constant. Suppose* $\sigma \neq 0$ *for a.a.* (t, ω)*. Let* $\mathcal{G}_t \subset \mathcal{H}_t$ *be general filtrations as before. Suppose that there exists an optimal portfolio* $\pi^* \in \mathcal{A}$ *for the problem*

$$\Phi_{\mathcal{H}} := \sup_{\pi \in \mathcal{A}} E\left[\frac{1}{\gamma}(X^{(\pi)}(T))^{\gamma}\right]. \tag{3.46}$$

Then there exists an $\mathcal{H}$*-adapted process* $\alpha(s)$ *such that (3.33) holds and hence the optimal insider portfolio can be found as follows.*
Define the (modified) market price of risk, $\theta(t)$*, by*

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)} + \alpha(t).$$

Assume that

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta^2(t)dt\right)\right] < \infty$$

and define

$$H_0(t) = \exp\left(-\int_0^t \theta(s)d\hat{B}(s) - \frac{1}{2}\int_0^t (\theta^2(s) + r(s))ds\right)$$

and

$$X^*(t) = H_0^{-1}(t)E\left[H_0^{\frac{\gamma}{\gamma-1}}(T)\right]^{-1} E\left[H_0^{\frac{\gamma}{\gamma-1}}(T)\middle| \mathcal{H}_t\right]. \tag{3.47}$$

Let $\psi(t)$ *be the unique* $\mathcal{H}_t$*-adapted process such that*

$$\int_0^T \psi^2(t)dt < \infty \quad a.s.$$

and

$$E\left[H_0^{\frac{\gamma}{\gamma-1}}(T)\right]^{-1} H_0^{\frac{\gamma}{\gamma-1}}(T) = 1 + \int_0^T \psi(t)d\hat{B}(t).$$

Then

$$\pi^*(t) = \sigma^{-1}(t)\left[\frac{\psi(t)}{H_0(t)} + X^*(t)\theta(t)\right] \tag{3.48}$$

is the optimal insider portfolio and $X^*(t) = X^{(\pi^*)}(t)$ *is the corresponding optimal insider wealth process.*

Acknowledgments

We wish to thank Bruno Bouchard, Nils Christian Framstad, Axel Grorud, Peter Imkeller, Thilo Meyer-Brandis, Huyen Pham, Monique Pontier, Martin Schweizer, Agnès Sulem, Fabio Trojani and anonymous referees for helpful comments.

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