# MEAN-VARIANCE HEDGING
# WITH RANDOM VOLATILITY JUMPS

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ABSTRACT. We introduce a general framework for stochastic volatility models, with the risky asset dynamics given by:

$$dX_t(\omega, \eta) = \mu_t(\eta) X_t(\omega, \eta) dt + \sigma_t(\eta) X_t(\omega, \eta) dW_t(\omega)$$

where $(\omega, \eta) \in (\Omega \times H, \mathcal{F}^\Omega \otimes \mathcal{F}^H, P^\Omega \otimes P^H)$.

In particular, we allow for random discontinuities in the volatility $\sigma$ and the drift $\mu$. First we characterize the set of equivalent martingale measures, then compute the *mean-variance optimal* measure $\tilde{P}$, using some results of Schweizer on the existence of an adjustment process $\beta$.

We show examples where the *risk premium* $\lambda = \frac{\mu - r}{\sigma}$ follows a discontinuous process, and make explicit calculations for $\tilde{P}$.

## 1. INTRODUCTION

The introduction of the mean-variance approach for pricing options under incomplete information is due to Föllmer and Sondermann [7], who first proposed the minimization of quadratic risk.

Their work, as well as that of Bouleau and Lamberton, focused on the case when the price of the underlying asset is a martingale. The more general semimartingale case was considered by: Duffie and Richardson [5], Schweizer ([19], [20], [21], [22], Monat and Stricker [14], Schäl [18],

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and a definitive solution was provided by Rheinländer and Schweizer [17], and Gourieroux, Laurent, and Pham [9], with different methods.

In the meantime, the random behavior of volatility turned out to be a major issue in applied option pricing, and Hull and White [11], Stein and Stein [23] and Heston [10] proposed different models with stochastic volatility. In fact, such models are special cases of incomplete information, and can be effectively embedded in the theoretical framework developed by mathematicians.

A particularly appealing feature of mean-variance hedging is that European options prices are calculated as the expectations of their respective payoff under a (possibly signed) martingale measure $\tilde{P}$, introduced by Schweizer [22]. The optimal strategy can also be found in terms of this measure, therefore it is not surprising that considerable effort has been devoted to its explicit calculation.

In this paper, we address the problem of calculating $\tilde{P}$ in presence of volatility jumps or, more generally, when the so-called *market price of risk* $\lambda = \frac{\mu - r}{\sigma}$ follows a possibly discontinuous process.

We consider the following market model, where each state of nature $(\omega, \eta)$ belongs to the product space $(\Omega \times H)$, endowed with the product measure $P^{\Omega} \otimes P^{H}$:

$$
(1) \quad \begin{cases} dS_t(\omega, \eta) = \mu_t(\eta)S_t(\omega, \eta)dt + \sigma_t(\eta)S_t(\omega, \eta)dW_t(\omega) \\ B_t = \exp\left(\int_0^t r_s ds\right) \end{cases}
$$

and $r$ is a constant.

We assume the existence on $H$ of a set of martingales with the representation property: this somewhat technical condition is in fact satisfied in most models present in the literature.

The results of [1] provide a characterization of the density of the *mean-variance optimal* martingale measure. When volatility follows a diffusion process (such as in the Heston or Hull and White models, only to mention two of them), this result was already obtained by Laurent and Pham [13] with stochastic control arguments. Here we illustrate in details calculations for sample models where volatility jumps are random both in size and in time of occurrence. For all of them, we calculate the density of the mean-variance optimal measure, and the law of the jumps under $\tilde{P}$. It turns out that the jump size distribution is a critical issue: in fact, finite distributions are easily handled by $n$ martingales, where $n$ is the cardinality of the jump size support. On the contrary, an infinite distribution for the jump size requires a more general approach. In this case, the density of the variance-optimal measure is characterized in terms of a compensated integer-valued random measure and we show some applications.

In the last section, we calculate the mean-variance hedging strategy for a call option, exploiting the *change of numéraire* technique of El

Karoui, Geman and Rochet [8], as well as the general formula in feedback form of Rheinländer and Schweizer [17].

## 2. The Market Model

We introduce here a simple model for a market with incomplete information.

We have two complete filtered probability spaces: $(\Omega, \mathcal{F}^\Omega, \mathcal{F}^\Omega_t, P^\Omega)$ and $(H, \mathcal{F}^H, \mathcal{F}^H_t, P^H)$. We denote by $W_t$ a standard Brownian Motion on $\Omega$, and assume that $\mathcal{F}^\Omega_t$ is the $P^\Omega \otimes P^H$-augmentation of the filtration generated by $W$. Our set of states of nature is given by the product space $(\Omega \times H, \mathcal{F}^\Omega \otimes \mathcal{F}^H, P^\Omega \otimes P^H)$.

We have a risk-free asset $B_t$ and a discounted risky asset $X_t = \dfrac{S_t}{B_t}$, with the following dynamics:

$$(2) \qquad \begin{cases} dX_t(\omega, \eta) = (\mu_t(\eta) - r_t)X_t(\omega, \eta)dt + \sigma_t(\eta)X_t(\omega, \eta)dW_t(\omega) \\ B_t = \exp(\int_0^t r_s ds) \end{cases}$$

where $r$ is a deterministic function of time. We assume that the equation for $X$ admits $P^H$-a.e. a unique strong solution with respect to the filtration $\mathcal{F}^W$ or, equivalently, that there exists a unique strong solution with respect to the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}^W_t \otimes \mathcal{F}^H$. This is satisfied under fairly weak assumptions: for example, it is sufficient that $\mu$ and $\sigma$ are $P^H$-a.e. bounded. Denoting by $\mathcal{F}^X$ the filtration generated by $X$, we also assume that $\mathcal{F}^X_t = \mathcal{F}^W_t \otimes \mathcal{F}^H_t$. In particular, at time $T$ all information is revealed through the observation of the process $X$.

The following proposition helps checking whether this condition is satisfied:

**Proposition 2.1.** *Let $X_t$ be defined as in (2). Then, if*

    *i) $\mu_t$ is $\mathcal{F}^X_t$-measurable,*

    *ii) $\mathcal{F}^H_t \subset \mathcal{F}^X_t$.*

*then $\mathcal{F}^X_t = \mathcal{F}^W_t \otimes \mathcal{F}^H_t$.*

*Proof.* By definition of $X_t$, we immediately have that $\mathcal{F}^X_t \subset \mathcal{F}^\Omega_t \otimes \mathcal{F}^H_t$. To see that the reverse inclusion holds, observe first that by (2):

$$(3) \qquad W_t = -\int_0^t \frac{\mu_s}{\sigma_s}ds + \int_0^t \frac{dX_s}{\sigma_s X_s}$$

By *i)*, the first term above is $\mathcal{F}^X_t$-measurable. For the second, note that

$$(4) \qquad \int_0^t \sigma(s, \eta)^2 X_s^2 ds = \lim_{\sup_i |t_{i+1} - t_i| \to 0} \sum_{i=1}^n |X_{t_{i+1}} - X_{t_i}|^2$$

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where the limit holds in probability, uniformly in $t$. This proves that $\mathcal{F}_t^\Omega \subset \mathcal{F}_t^X$, and by $ii)$ the proof is complete. $\qquad\square$

A simpler version of this model was introduced by Delbaen and Schachermayer [3], while it can be found in the above form in Pham, Rheinländer, and Schweizer [15].

In this framework, we study the problem of an agent wishing to hedge a certain European option $H(X_T)$ expiring at a fixed time $T$. Hedging performance is defined as the $L^2$-norm of the difference, at expiration, between the liability and the hedging portfolio. More precisely, we look for a solution to the minimization problem:

$$(5) \qquad \min_{\substack{c \in \mathbb{R} \\ \theta \in \Theta}} E\left[\left(H(X_T) - c - G_T(\theta)\right)^2\right]$$

where

$$G_t(\theta) = \int_0^t \theta_s dX_s \qquad \text{and} \qquad \Theta = \left\{\theta \in L(X), \ G_t(\theta) \in \mathcal{S}^2(P)\right\}$$

Here $L(X)$ denotes the space of $X$-integrable predictable processes, and $\mathcal{S}^2$ the space of semimartingales $Y$ decomposable as $Y = Y_0 + M + A$, where $M$ is a square-integrable martingale, and $A$ is a process of square-integrable variation.

This problem is generally nontrivial, since the agent has not access to the filtration $\tilde{\mathcal{F}}$, but only to $\mathcal{F}^X$. Indeed, Reihnländer and Schweizer [17] and, independently, Gourieroux, Laurent and Pham [9], proved that problem (5) admits a unique solution for all $H \in L^2(P)$, under the standing hypothesis:

$$(\text{CL}) \qquad\qquad\qquad G_T(\Theta) \text{ is closed}$$

**Definition 2.2.** We define the sets of *signed martingale measures* $\mathcal{M}_s^2$, and of *equivalent martingale measures* $\mathcal{M}_e^2$:

$$(6) \quad \mathcal{M}_s^2 = \left\{Q \ll P : \frac{dQ}{dP} \in L^2(P), \ X_t \text{ is a } Q\text{-local martingale}\right\}$$

$$(7) \quad \mathcal{M}_e^2 = \left\{Q \in \mathcal{M}_s^2 : Q \sim P :\right\}$$

The option price, (i.e. the optimal value for $c$), and the mean-variance hedging strategy $\theta$ can be computed in terms of $\tilde{P}$, the *variance-optimal martingale measure*. If (5) has solution, in [22] it is shown that the optimal value for $c$ is given by $c = \tilde{E}[H]$. Moreover, by Theorem 6 in [17] one obtains the following characterization of the optimal strategy $\theta$.

**Proposition 2.3.** *If (CL) holds and $\mathcal{M}_s^2 \neq \emptyset$, for any $H \in L^2(P)$ the optimal strategy $\theta$ takes the form:*

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$$(8) \qquad \theta_t = \tilde{\xi}_t - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - c - \int_0^t \theta_s dX_s\right)$$

*where*

(1) $\tilde{V}_t = \tilde{E}\left[H|\mathcal{F}_t\right] = \tilde{V}_0 + \int_0^t \xi_s dX_s + \tilde{L}_t$ *and* $\tilde{L}_t$ *is a* $\tilde{P}$-*square integrable martingale orthogonal to* $X_t$

(2) $\tilde{Z}_t = \tilde{E}\left[\frac{d\tilde{P}}{dP}\Big|\mathcal{F}_t\right] = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_s dX_s$

**Definition 2.4.** The *variance optimal* martingale measure is the unique solution $\tilde{P}$ (if it exists) to the minimum problem:

$$(9) \qquad \min_{Q \in \mathcal{M}_s^2} E\left[\left(\frac{dQ}{dP}\right)^2\right]$$

If $\mathcal{M}_s^2$ is nonempty, then $\tilde{P}$ always exists, as it is the minimizer of the norm in a convex set: the problem is that it may not be positive definite, thereby leading to (possibly) negative option prices. However, if $X_t$ has continuous paths, and under the standard assumption

(NA) $\qquad\qquad\qquad\qquad \mathcal{M}_e^2 \neq \emptyset$

In [3] Delbaen and Schachermayer have shown that $\tilde{P} \in \mathcal{M}_e^2$. Since we are dealing with continuous processes, and we will always assume (NA), we need not worry about this issue.

In this paper, using a representation formula from [1], we compute explicitly $\tilde{P}$ for some sample models.

We denote by $\lambda_t = \dfrac{\mu_t - r_t}{\sigma_t}$ the so-called *market price of risk*. By Proposition 1.11 of [1], we obtain that, if there exists a $n$-dimensional martingale $M$ on $H$ such that:

i) $[M_i, M_j] \equiv 0$ for all $i \neq j$;

ii) $M$ has the representation property for $\mathcal{F}_t^H$;

then we have for every $Q \in \mathcal{M}_e^2$:

$$(10) \qquad \frac{dQ}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_t(\eta)dW_t\right)_T \mathcal{E}\left(\int_0^{\cdot} k_t(\omega, \eta)dM_t\right)_T$$

where $k_t$ is such that $\mathcal{E}\left(-\int_0^{\cdot}\lambda_t(\eta)dW_t\right)_t \mathcal{E}\left(\int_0^{\cdot} k_t(\omega,\eta)dM_t\right)_t$ is a square integrable martingale and $k_t \cdot \Delta M_t > -1$.

Consequently, a martingale measure is uniquely determined by the process $k$ which appears in its representation. In particular, $k = 0$ corresponds to the *minimal martingale measure* $\hat{P}$ introduced by Föllmer

and Schweizer in [6].

Also, in the same assumptions, from Theorem 1.16 in [1] it follows that:

$$(11) \qquad \frac{d\tilde{P}}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_t dW_t\right)_T \mathcal{E}\left(\int_0^{\cdot} \tilde{k}_t(\eta) dM_t\right)_T$$

where $\tilde{k}_t$ is a solution of the following equation

$$(12) \qquad \mathcal{E}\left(\int_0^{\cdot} \tilde{k}_t(\eta) dM_t\right)_T = \frac{\exp\left(-\int_0^T \lambda_t^2(\eta) dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta) dt\right)\right]}$$

such that $\mathcal{E}\left(-\int_0^{\cdot} \lambda_t dW_t\right)_t \mathcal{E}\left(\int_0^{\cdot} \tilde{k}_t(\eta) dM_t\right)_t$ is a square integrable martingale.

*Remark* 2.5. We show now how the change of measure works on $\Omega$ and $H$. In fact, provided that $\hat{P}$ exists, we can write:

$$\frac{d\tilde{P}}{dP} = \frac{d\tilde{P}}{d\hat{P}} \cdot \frac{d\hat{P}}{dP}$$

where

$$\frac{d\hat{P}}{dP} = \mathcal{E}\left(-\int_0^T \lambda_t dW_t\right) \qquad \text{and} \qquad \frac{d\tilde{P}}{d\hat{P}} = \frac{\exp\left(-\int_0^T \lambda_t^2(\eta) dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2(\eta) dt\right)\right]}$$

Since in our model $\dfrac{d\tilde{P}}{d\hat{P}}$ does not depend on $\omega$, we have:

$$(13) \quad \frac{d\tilde{P}_H}{dP_H} = E\left[\frac{d\tilde{P}}{dP}\bigg| \mathcal{F}_H\right] =$$

$$= E\left[\frac{d\tilde{P}}{d\hat{P}} \cdot \frac{d\hat{P}}{dP}\bigg| \mathcal{F}_H\right] = \frac{d\tilde{P}}{d\hat{P}} E\left[\frac{d\hat{P}}{dP}\bigg| \mathcal{F}_H\right] = \frac{d\tilde{P}}{d\hat{P}}$$

This provides a rule of thumb for changing measure from $P$ to $\tilde{P}$ via $\hat{P}$. First change $P$ to $\hat{P}$ by a direct use of Girsanov theorem: this amounts to replacing $\mu$ with $r$ in (2), and is the key of risk-neutral valuation. $P_H$ is not affected by this step.

In principle, one could repeat the same argument from $\hat{P}$ to $\tilde{P}$, but this involves calculating the $\tilde{k}_t(\eta)$. As we show with an example in the last section, this task may prove hard even in simple cases.

A more viable alternative is calculating $\tilde{P}_H$ with the above formula. This avoids dealing with $\tilde{k}$ directly, although its existence is still needed.

A sufficient condition for the existence of $\hat{P}$ is the Novikov condition, namely:

$$(14) \qquad E\left[\exp\left(\frac{1}{2}\int_0^T \lambda_t^2 dt\right)\right] < \infty$$

and is satisfied by all the examples in the last section.

## 3. VOLATILITY JUMPS AND RANDOM MEASURES

Although most markets models considered in the literature can be embedded in a framework consistent with the described one, there are some remarkable exceptions. For example, continuously distributed jumps in volatility can generate filtrations where no finite set of martingales has the representation property (see the examples in the next section).

In these cases, we can still represent martingales in terms of integrals with respect to a compensated random measure $\nu - \nu^p$, thereby obtaining an analogous of Theorem 1.16 and Proposition 1.11 of [1].

Note that the following results are complementary to those in the previous section, but do not directly generalize them: in fact any model with volatility following a diffusion process is covered in the previous section, and not in the present one.

**Theorem 3.1.** *If there exists a compensated, integer-valued, random measure $\nu - \nu^p$ on $E \times \mathbb{R}_+ \times \mathbb{R}$ such that:*

*i) $\mathcal{F}^H$ coincides with the smallest filtration under which $\nu$ is optional;*

*ii) $\nu - \nu^p$ has the representation property on $(H, \mathcal{F}^H, P^H)$.*

*Then we have for every $Q \in \mathcal{M}_e^2$:*

$$(15) \qquad \frac{dQ}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_t dW_t\right)_T \mathcal{E}\left(k * (\nu - \nu^p).\right)_T$$

*where $k_t$ is such that $\Delta(k*\nu)_t > -1$ and $\mathcal{E}\left(-\int_0^{\cdot} \lambda_t dW_t\right)_t \mathcal{E}\left(k * (\nu - \nu^p).\right)_t$ is a square integrable martingale.*

As in the previous section, we need these lemmas:

**Lemma 3.2.** *In the same assumptions as Theorem 3.1, every square integrable martingale $M_t$ on the space $(\Omega \times E, \mathcal{F}^\Omega \otimes \mathcal{F}^H, P^\Omega \otimes P^H)$ with respect to $\mathcal{F}_t$ can be written as*

$$(16) \qquad M_t = M_0 + \int_0^t h_s dW_s + k * (\nu - \nu^p)_t$$

*Proof.* Let us denote the set of martingales for which the thesis holds by $\mathfrak{M}$. We want to show that $\mathfrak{M} = M^2(\Omega \times H)$.

By representation property, every square integrable martingale $M_t(\omega)$ on $\Omega \times H$ depending only on $\omega$ belongs to $\mathfrak{M}$, since it can be written as $M_t = M_0 + \int_0^t h_s dW_s$. Analogously, every $N_t(\eta)$ belongs to $\mathfrak{M}$, since it is of the form $N_t = N_0 + k * (\nu - \nu^p)_t$, where $k$ is a $\tilde{\mathcal{P}}$-measurable process and $|k| * \nu_t$ is locally integrable.

Denoting $P_t(\omega, \eta) = N_t(\eta)M_t(\omega)$, we have that $P_t$ is a square integrable martingale on $\Omega \times H$. Setting $P_t = P_t^d + P_t^c$, where $P_t^c$ and $P_t^d$ are the continuous and purely discontinuous parts of $P$, we have that $\Delta P_t = \Delta P_t^d = M_t \Delta N_t$. By Definition II.1.27 in [12], it follows that $P_t^d = P_0^d + (Mk) * (\nu - \nu^p)_t$. Also, by Itô's formula, $P_t^c = P_0^c + \int_0^t N_s h_s dW_s$. This shows that any linear combination $\sum_i M_i(\omega)N_i(\eta)$ belongs to $\mathfrak{M}$ and, by a monotone class argument, it is easy to see that $\mathfrak{M}$ is dense in $\mathbf{M}^2$. Hence, for every $Z_t \in M^2$ there exist a sequence $X_t^n$ of square-integrable martingales such that $X_t^n = X_0 + \int_0^t h_s^n dW_s + k^n * (\nu - \nu^p)_t$. By the identity:

$$(17) \qquad E[X_T^n] = E\left[\int_0^T (h_s^n)^2 ds\right] + E\left[(k_s^n)^2 * \nu_s^p\right]$$

it follows that $h^n$ and $k^n$ are Cauchy sequences respectively in

$$\{h_t(\eta) \text{ predictable}: E\left[\int_0^T h_s^2 ds\right] < \infty\} \qquad \text{and}$$

$$\{k_t(\eta, x) \text{ predictable}: E\left[k_s^2 * \nu_s^p\right] < \infty\}$$

Since these spaces are complete, the proof is finished.

$\square$

**Lemma 3.3.** *Let $Z_T$ be a strictly positive, square-integrable random variable and denote $Z_t = E[Z_T | \mathcal{F}_t]$. Then, if $Z_t = Z_0 + H * (\nu - \nu^p)_t$, we have:*

$$Z_t = Z_0 \mathcal{E}\left(\frac{H}{Z_-} * (\nu - \nu^p)_t\right).$$

*Proof.* If there exists a martingale $M_t = M_0 + K * (\nu - \nu^P)_t$ such that $Z_t = Z_0 \mathcal{E}(M_t)$, then it is unique. In fact, if $N_t = N_0 + H * (\nu - \nu^P)_t$ and $Z_t = Z_0 \mathcal{E}(N_t)$, we immediately have $\Delta M_t = \Delta N_t$. Since $M_t$ and $N_t$ are purely discontinuous martingales by Definition II.1.27 in [12], they must coincide up to evanescent sets.

In particular, we have that $\Delta M_t = \frac{\Delta Z_t}{Z_{t-}}$. For all $t > 0$, $\frac{\Delta Z_t}{Z_{t-}}$ coincides with the jumps of the purely discontinuous martingale $\frac{H}{Z_-} * (\nu - \nu^P)_t$, therefore $M_t$ exists and is given by $M_t = \log(Z_0) + \frac{H}{Z_-} * (\nu - \nu^P)_t$. $\quad\square$

**Proposition 3.4.** *In the same assumptions as Theorem 3.1, we have:*

$$(18) \qquad \frac{d\tilde{P}}{dP} = \mathcal{E}\left( -\int_0^{\cdot} \lambda_t dW_t \right)_T \mathcal{E}\left( \tilde{k} * (\nu - \nu^p). \right)_T$$

*where $\tilde{k}_t$ is a solution of the following equation*

$$(19) \qquad \mathcal{E}\left( \tilde{k} * (\nu - \nu^p). \right)_T = \frac{\exp\left( -\int_0^T \lambda_t^2(\eta) dt \right)}{E\left[ \exp\left( -\int_0^T \lambda_t^2(\eta) dt \right) \right]}$$

*such that $\mathcal{E}\left( -\int_0^{\cdot} \lambda_t dW_t \right)_t \mathcal{E}\left( \tilde{k} * (\nu - \nu^p). \right)_t$ is a square integrable martingale.*

*Proof.* The proof is formally analogous to that of Theorem 1.16 of [1], by the previous results and the representation property of the compensated random measure $\nu - \nu^p$. $\quad\square$

## 4. EXAMPLES

We now show how the results in the previous sections provide convenient tools for calculating $\tilde{P}$ (and thus pricing options) in models where volatility jumps. We start with a simple model where jumps occur at fixed times, and can take only two values. We then discuss the more general cases of jumps occurring at stopping times, and with arbitrary distributions.

4.1. **Deterministic Volatility Jumps.** In discrete-time fashion, the following model was introduced in [24] as an improvement of the standard lognormal model for calculating Value at Risk. We set $H = \{0,1\}^n$ and denote $\eta = \{a_1, \ldots, a_n\}$. $a_1. \ldots, a_n$ are Bernoulli IID random variable, so that $H$ is endowed with the product measure from $\{0,1\}$. $\mathcal{F}_t^H$ contains all information on jumps up to time $t$, therefore it is equal to the parts of $\{a_i\}_{t_i \leq t}$. Setting $t_i = i\frac{T}{n+1}$, the dynamics of $\mu$ and $\sigma$ is given by:

$$(20) \qquad \begin{cases} \mu_t = 1_{\{0 \leq t < t_1\}}\mu + \sum_{i=1}^n 1_{\{t_i \leq t < t_{i+1}\}}\mu_{a_i} \\ \sigma_t = 1_{\{0 \leq t < t_1\}}\sigma + \sum_{i=1}^n 1_{\{t_i \leq t < t_{i+1}\}}\sigma_{a_i} \end{cases}$$

In fact, all we need for mean-variance hedging is the dynamics for $\lambda$:

$$\lambda_t^2 = \lambda^2 + \sum_{i=1}^n 1_{\{t_i \leq t < t_{i+1}\}}\lambda_{a_i}^2$$

It is easy to check that a martingale with the representation property on $H$ is given by: $M_t = \sum_{t_i \leq t}(1_{\{a_i = 0\}} - p)$, where $p = P(a_1 = 0)$.

We are now ready to see how the change of measure works: in fact, by remark 2.5, we have that:

$$(21) \qquad \frac{d\tilde{P}}{d\hat{P}} = \frac{d\tilde{P}_H}{dP_H} = \frac{\exp\left(-\frac{\lambda_1^2 T}{n}\right)^{a_1 + \cdots + a_n} \exp\left(-\frac{\lambda_0^2 T}{n}\right)^{n - (a_1 + \cdots + a_n)}}{\left(p \exp\left(-\frac{\lambda_0^2 T}{n}\right) + (1 - p) \exp\left(-\frac{\lambda_1^2 T}{n}\right)\right)^n}$$

Since the density above can be written as:

$$(22) \qquad \frac{d\tilde{P}_H}{dP_H} = \prod_{i=1}^{n} \frac{\exp(-a_i \frac{\lambda_1^2 T}{n} - (1 - a_i)\frac{\lambda_0^2 T}{n})}{\left(p \exp\left(-\frac{\lambda_0^2 T}{n}\right) + (1 - p) \exp\left(-\frac{\lambda_1^2 T}{n}\right)\right)^n}$$

it follows that under $\tilde{P}$ the variables $a_1, \ldots, a_n$ are still independent, and $p$ is replaced by:

$$(23) \qquad \tilde{p} = p \frac{\exp(-\frac{\lambda_1^2 T}{n})}{p \exp\left(-\frac{\lambda_0^2 T}{n}\right) + (1 - p) \exp\left(-\frac{\lambda_1^2 T}{n}\right)}$$

4.2. **Random Volatility Jumps.** Consider the following model, where $\mu$ and $\sigma$ are constant, until some unexpected event occurs. In other words:

$$(24) \qquad \begin{cases} \mu_t &= \mu_1 1_{\{t < \tau\}} + \mu_2 1_{\{t \geq \tau\}} \\ \sigma_t &= \sigma_1 1_{\{t < \tau\}} + \sigma_2 1_{\{t \geq \tau\}} \end{cases}$$

In fact, all we need is the dynamics for $\lambda$:

$$\lambda_t^2 = \lambda_1^2 + \alpha 1_{\{t \geq \tau\}}$$

where $\alpha = \lambda_2^2 - \lambda_1^2$. The event $\tau$ which triggers the jump is a *totally inaccessible* stopping time. That is to say, any attempt to predict it by means of previous information is deemed to failure. $\alpha$ represents the jump size, and it may be deterministic or random. We now solve the problem in three cases: $\alpha$ deterministic, $\alpha$ Bernoulli, and $\alpha$ continuously distributed.

4.2.1. *$\alpha$ Deterministic.* Since our goal is to find the variance-optimal martingale measure, we start exhibiting a martingale with the representation property for $\mathcal{F}_H$, which in this case is the filtration generated by $\tau$.

**Proposition 4.1.** *Let $\tau$ be a stopping time with a diffuse law, and $\tilde{A}$ the compensator of $1_{\{\tau \leq t\}}$. Then the martingale $M_t = 1_{\{\tau \leq t\}} - \tilde{A}_t$ has the representation property.*

*Proof.* Let $Q$ be a martingale measure for $M$, and $\tilde{A}^Q$ the compensator of $1_{\{\tau \leq t\}}$ in $Q$. Both $M_t$ and $1_{\{\tau \leq t\}} - \tilde{A}^Q$ are $Q$-martingales, therefore their difference $\tilde{A}^Q - \tilde{A}$ is also a martingale. However, it is also a finite variation process, therefore it must be identically zero.

Since $\tilde{A}^Q = \tilde{A}$, by Proposition 6.9, the c.d.f.'s of $\tau$ under $P$ and $Q$ are equal. This implies that $Q = P$, $\mathcal{F}^\tau$-a.e. Theorem IV.37 in [16] concludes the proof. □

We now compute $P_H$, that is the law of $\tau$ under $\tilde{P}$. For simplicity, assume that $\tau$ has a density, and denote it by $f_t$. We have:

$$(25) \quad \tilde{f}_t = \frac{d\tilde{P}}{d\hat{P}} f_t =$$

$$= \frac{\exp\left(-\int_0^T \lambda_s^2(\eta) ds\right)}{c} f_t = \begin{cases} \frac{1}{c} \exp\left(-\lambda_1 T + \alpha t\right) f_t & \text{if } t < T \\ \frac{1}{c} \exp\left(-\lambda_1 T\right) f_T & \text{if } t \geq T \end{cases}$$

where $c = E\left[\exp\left(-\int_0^T \lambda_s^2(\eta) ds\right)\right]$. In this simple example we also calculate $\tilde{k}_t$ explicitly, although the computational effort required suggests that in more complex situations it may not be a good idea to do so.

First, we see how stochastic integrals with respect to $M$ look like. Recall that, by Proposition 6.9 $\tilde{A}_t = a(t \wedge \tau)$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$.

**Lemma 4.2.** *Let $k_t$ be a $\mathcal{F}^\tau$-measurable process. Then we have:*

$$(26) \qquad \int_0^T k_t dM_t = k_\tau(\tau) 1_{\{\tau \leq T\}} - \int_0^{T \wedge \tau} k_t(t) da_t$$

*Proof.* By Corollary 6.3, we have that any $\mathcal{F}^\tau$-measurable process can be written as $k_t(t \wedge \tau)$. Hence:

$$\int_0^T k_t(t \wedge \tau) dM_t = k_\tau(\tau) 1_{\{\tau \leq T\}} - \int_0^T k_t(t \wedge \tau) da_{t \wedge \tau} =$$

$$= k_\tau(\tau) 1_{\{\tau \leq T\}} - \int_0^{T \wedge \tau} k_t(t \wedge \tau) da_t =$$

$$= k_\tau(\tau) 1_{\{\tau \leq T\}} - \int_0^{T \wedge \tau} k_t(t) da_t$$

□

The above lemma shows that $k_t(s)$ needs only be defined for $s = t$, so from now on we shall unambiguously write $k_t$ instead of $k_t(t)$.

Now we can compute $\tilde{k}_t$:

**Proposition 4.3.** $\tilde{k}_t$ *is the unique solution of the following ODE:*

(27)
$$\begin{cases} k_t' = & \alpha + (a_t' + \alpha)k_t + a_t' k_t^2 \\ k_0 = & \frac{\exp(-\lambda_2^2 T)}{c} - 1 \end{cases}$$

*where*

(28) $\quad c = \exp\left(-\lambda_2^2 T\right) \int_0^T \exp\left(\alpha t\right) dF_t + \exp\left(-\lambda_1^2 T\right)(1 - F_T)$

*and* $F_t = P(\tau \leq t)$.

*Proof.* By lemma 4.2, and the generalization of Itô's formula for processes with jumps, we have:

(29) $\quad \mathcal{E}\left(\int_0^T k_t(\eta)dM_t\right) = \mathcal{E}\left(k_\tau 1_{\{\tau \leq T\}} - \int_0^{T \wedge \tau} k_t da_t\right) =$

$$= \left(1 + k_\tau 1_{\{\tau \leq T\}}\right) \exp\left(-\int_0^{\tau \wedge T} k_t da_t\right)$$

Hence, by section 2 of [1] we have:

(30)
$$\left(1 + k_\tau 1_{\{\tau \leq T\}}\right) \exp\left(-\int_0^{\tau \wedge T} k_t da_t\right) = \frac{\exp\left(-T\lambda_1^2 - (T - \tau \wedge T)\alpha\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2 dt\right)\right]}$$

Taking logarithms of both sides, and setting $c = E\left[\exp\left(-\int_0^T \lambda_t^2 dt\right)\right]$, we get:

$$\ln\left(1 + k_\tau 1_{\{\tau \leq T\}}\right) - \int_0^{\tau \wedge T} k_t da_t = -T\lambda_1^2 - (T - \tau \wedge T)\alpha - \ln c$$

Differentiating with respect to $\tau$, for $\tau \leq T$ we obtain equation (27). $\qquad \square$

*Remark* 4.4. Equation (27) is a Riccati ODE, and can be solved in terms of the function $a_t$. Depending on the form of $a_t$, explicit solutions may or may not be available.

4.2.2. $\alpha$ *Bernoulli.* In this case, $\alpha$ is a Bernoulli random variable, independent of $\tau$, with values $\{\alpha_0, \alpha_1\}$. We also set $A = \{\alpha = \alpha_0\}$, $B = \{\alpha = \alpha_1\}$, and $p = P(B)$. Since the support of $\alpha$ is no longer a single point, a martingale will not be sufficient for representation purposes. In fact, two martingales do the job, as we prove in the following:

**Proposition 4.5.** *Let* $N_t = 1_{\{\tau \leq t\}}(1_B - p)$. *Then the set of two martingales* $\{M, N\}$ *has the representation property.*

*Proof.* First we check that $M$ and $N$ are orthogonal. This is easily seen, since $MN = Na_\tau$. We now prove that the martingale measure is unique.

Let $Q$ be a martingale measure for $\{M, N\}$. As shown in the proof of Proposition 4.1, the distribution of $\tau$ under $Q$ must be the same as under $P$. However, we also need that $Q(B) = P(B)$, otherwise $N$ would not be a martingale. $\qquad\square$

The change from $P$ to $\tilde{P}$ is a change in the joint law of $(\tau, \alpha)$. Under $P$ this is a product measure, since $\tau$ and $\alpha$ are independent. However, we cannot expect that the same holds under $\tilde{P}$. For $t \leq T$ we have:

$$(31)\quad \frac{d\tilde{P}_H}{dP_H} =$$

$$= \frac{1_A \exp\left(-\lambda_1^2 T - (T-t)\alpha_0\right) + 1_B \exp\left(-\lambda_1^2 T - (T-t)\alpha_1\right)}{c}$$

Therefore the law of $\tau$ under $\tilde{P}$ is given by:

$$(32)\quad \tilde{f}_t =$$

$$= \frac{1}{c}\left(p \exp\left(-\lambda_1^2 T - (T-t)\alpha_1\right) + (1-p) \exp\left(-\lambda_1^2 T - (T-t)\alpha_0\right)\right) f_t$$

And the conditional law of $\alpha$ with respect to $\tau$ is given by:

$$(33)\quad \tilde{P}(B|\tau \in dt) =$$

$$= \frac{p \exp\left(-\lambda_1^2 T - (T-t)\alpha_1\right)}{\left(p \exp\left(-\lambda_1^2 T - (T-t)\alpha_1\right) + (1-p) \exp\left(-\lambda_1^2 T - (T-t)\alpha_0\right)\right)}$$

In particular, it is immediately seen that $\alpha$ is independent of $\tau$ if and only if it degenerates in the previous case.

When $\alpha$ is Bernoullian, calculating $k$ involves solving a system of two Riccati ODEs, which is somewhat cumbersome. More generally, if the support of $\alpha$ is made of $n$ points, it is reasonable that $n$ martingales are required for representation purposes. As a result, the values of $k$ would be the solutions of a system of $n$ ODEs.

4.2.3. *$\alpha$ Continuously Distributed.* In this case the support of $\alpha$ is an infinite set, therefore Theorem 1.16 of [1] is no longer applicable. In fact we need its random measure analogous, given by Theorem 3.1. If the filtration $\mathcal{F}^\lambda$ generated by $\lambda_t$ coincides with the one generated by $\mu_t$ and $\sigma_t$, we can assume $\mathcal{F}^H = \mathcal{F}^\lambda$. By Proposition II.1.16 in [12], there exists a random measure $\nu$ associated to $\lambda$, and given by:

$$\nu(\eta; dt, dx) = \epsilon_{\{\tau, \alpha(\eta)\}}(dt, dx)$$

Since this is a multivariate point process, and $\mathcal{F}^H$ coincides with the smallest filtration under which $\nu$ is optional, by Theorem III.4.37 in

[12] the compensated measure $\nu - \nu^p$ has the representation property on $H$. Also, $k * (\nu - \nu^p)$ is a purely discontinuous martingale for all $\tilde{\mathcal{P}}$-measurable processes $k$, therefore $[W, k * (\nu - \nu^p)] = 0$. This means that the assumptions of Theorem 3.1 are satisfied, and $\tilde{P}$ is given by Proposition 3.4.

Suppose that $\alpha$ has a density, say $g(x)$. We have:

$$(34) \qquad \frac{d\tilde{P}_H}{dP_H} = h(t, x) = \frac{\exp\left(-\lambda_1^2 T - x(T - t)\right)}{c}$$

Denoting by $j(t, x)$ and $\tilde{j}(t, x)$ the joint densities of $(\tau, \alpha)$ under $P$ and $\tilde{P}$ respectively, we have:

$$(35) \qquad \tilde{j}(t, x) = h(t, x)j(t, x) = h(t, x)f(t)g(x)$$

$$(36) \qquad \tilde{f}(t) = f(t)\int h(t, x)g(x)dx$$

$$(37) \qquad \tilde{g}(x|t) = \frac{j(t, x)}{\tilde{f}(t)} = \frac{h(t, x)g(x)}{\int h(t, x)g(x)dx}$$

If $\alpha \sim \mathcal{N}(\delta, v)$, the density of $\tau$ under $\tilde{P}$ is given by:

$$(38) \quad \tilde{f}_t = f_t \int_{-\infty}^{+\infty} \frac{1}{c} \exp\left(-\lambda_1^2 T - \alpha(T - t)\right) n\left(\frac{\alpha - \delta}{\sqrt{v}}\right) d\alpha =$$

$$= \frac{1}{c} \exp\left(-\lambda_1^2 T - \delta(T - t) + \frac{1}{2}(T - t)^2 v\right) f_t$$

where $n(x)$ is the standard normal density function. It is easy to check that the conditional density of $\alpha$ given $\tau$ is of the form:

$$(39) \qquad \tilde{g}(x|t) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2}\frac{(x - (\alpha + (T - t)v))^2}{v}\right)$$

Therefore $\alpha$ is conditionally normal under $\tilde{P}$, with distribution $\mathcal{N}(\delta + (T - t)v, v)$.

*Remark* 4.6. In the specific case of $\lambda_t$ being normally distributed, it can be shown that $G_T(\Theta)$ is not closed. However, in [1] is shown that in this example $G_T(\Theta)$ is closed if and only if the support of $\alpha$ is bounded from above.

4.2.4. *Multiple Random Jumps.* Leaving $\alpha$ deterministic for simplicity, we now study the following model:

$$\lambda_t^2 = \lambda_0^2 + \sum_{i=1}^{n} 1_{\{t \geq \tau_i\}}\alpha_i$$

We assume that $\tau_{i+1} - \tau_i$ are IID random variables, with common density $f(x)$. Denote by $H$ the space $[0, T]^n$, endowed with the image measure of the mapping $(\tau_1, \ldots, \tau_n) \mapsto (\tau_1 \wedge T, \ldots, \tau_n \wedge T)$, and with the

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natural filtration $\mathcal{F}_t$ generated by $\{\tau_1 \wedge t, \ldots, \tau_n \wedge t\}$. A martingale with the predictable representation property is given by $M_t = \sum_{i=1}^n 1_{\{t \geq \tau_i\}}$. In this case, the density of $\tilde{P}$ is given by:

$$(40) \qquad \frac{d\tilde{P}_H}{dP_H} = \frac{\exp\left(-\lambda_0^2 T - \sum_{i=1}^n \alpha_i(T - \tau_i) \wedge 0\right)}{c}$$

Since this density cannot be factored into a product of densities each one involving at most a $\tau_i - \tau_{i-1}$, it follows that under $\tilde{P}$ the increments of the stopping times are no longer independent.

For example, consider the following case, with $n = 2$ and the stopping times exponentially distributed with parameter $b$. In other words:

$$(41) \qquad \begin{cases} P(x \in dt) = & be^{-bt} \quad \text{for } t < T \\ P(x = T) = & e^{-bT} \end{cases}$$

for $x = \tau_1, \tau_2 - \tau_1$. We obtain that:

$$(42) \qquad \begin{cases} \tilde{P}(\tau_1 \in dt) = & \frac{\exp\left(-(\lambda_0^2 + \alpha)T\right)}{c} e^{-(b - \alpha)t} \\ \tilde{P}(\tau_1 = T) = & \frac{\exp\left(-(\lambda_0^2 + b)T\right)}{c} \end{cases}$$

The conditional law of $\tau_2 - \tau_1$ turns out to be of the same form of (42) where $T$ is replaced by $T - \tau_1$. This shows that under $\tilde{P}$ the law of $\tau_2$ is not independent of $\tau_1$.

## 5. The optimal strategy for a call option

In this section, we adapt the technique of *change of numéraire* presented in [8] to write explicitly the optimal strategy for a call option. We shall make the following assumptions:

   i) The space $G_T(\Theta)$ is closed in $L^2(P)$.
   ii) $\mu_t$ and $\sigma_t$ depend only on $\eta$.

Condition $i$) guarantees the existence of an optimal strategy for any option $H$ such that $\frac{H}{B_T} \in L^2(P)$, as showed in [17]. In the examples contained in the previous section, closedness of $G_T(\Theta)$ turns out to be equivalent to restrictions on the distribution of the volatility jumps. Condition $ii$) allows to write $\tilde{P}$ as:

$$(43) \qquad \frac{d\tilde{P}}{dP} = \mathcal{E}\left(-\int_0^{\cdot} \lambda_s dW_s\right)_T \frac{\exp\left(-\int_0^T \lambda_t^2 dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2 dt\right)\right]}$$

Consider now a call option $H = (S_T - K)^+$ on the asset $S_t$ with strike price $K$. Recall that $X_t$ is the discounted price of $S_t$. Under the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{E}$ the model is complete, hence $H$ is attainable. As a result, the discounted value at time $t$ of the unique

replicating portfolio can be obtained via the usual Black-Scholes formula:

$$X_t N(d_1(t, \eta, X_t)) - \frac{K}{B_T} N(d_2(t, \eta, X_t))$$

where $N(\cdot)$ is the distribution function of the standard normal variable and

(44)
$$d_{1,2}(t, \eta, x) = \frac{\ln\left(\frac{x}{KB(t,T)}\right) \pm \int_t^T \sigma^2(s, \eta) ds}{\left(\int_t^T \sigma^2(s, \eta) ds\right)^{\frac{1}{2}}}$$

with $B(t, T) = \frac{B_t}{B_T}$. For all $t$, the filtration $\tilde{\mathcal{F}}_t$ contains the information on volatility up to $T$: more precisely, the random variable $\int_t^T \sigma^2(s, \eta) ds$ is $\tilde{\mathcal{F}}_t$-measurable. It is easy to see that the probability $\tilde{P}$ is an equivalent martingale measure with respect to $\tilde{\mathcal{F}}_t$. The change of numéraire technique applies since

(45)
$$\frac{d\tilde{P}^X}{d\tilde{P}} = \frac{X_T}{E[X_T]}$$

as proved in [9] and by the same argument as in [8], we can write the replicating portfolio as

(46)
$$\tilde{E}\left[\left(X_T - \frac{K}{B_T}\right)^+ \bigg| \tilde{\mathcal{F}}_t\right] = X_t \tilde{E}^X\left[1_A \bigg| \tilde{\mathcal{F}}_t\right] - \frac{K}{B_T} \tilde{E}\left[1_A \bigg| \tilde{\mathcal{F}}_t\right]$$

where $A = \{S_T > K\}$ and $\tilde{E}^X$ denotes the expectation under the probability $\tilde{P}^X$. We are going to use the above calculations to write the optimal strategy with respect to the filtration $\mathcal{F}_t$.

Let now $\xi_t^1$ and $\xi_t^2$ the predictable projections of $\tilde{\xi}_t^1 = \tilde{E}^X\left[1_A \bigg| \tilde{\mathcal{F}}_t\right]$ and $\tilde{\xi}_t^2 = \tilde{E}\left[1_A \bigg| \tilde{\mathcal{F}}_t\right]$ with respect to the filtration $\tilde{\mathcal{F}}_t$ and the probability $\tilde{P}$.

*Remark* 5.1. By the same argument used in Proposition 5.1 of [2], it follows that $\xi_t^1$ coincides with the predictable projection of the process $Y(t, \omega) = 1_A(\omega)$ with respect to the probability $\tilde{P}^X$ and the filtration $\mathcal{F}_t$. More precisely, for all predictable stopping times $\tau$ we have:

$$\xi_\tau^1 = \tilde{E}^X[1_A | \mathcal{F}_{\tau-}]$$

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Moreover, since the left-continuous versions of the stochastic processes $\tilde{E}^X\left[1_A|\mathcal{F}_{t-}\right]$ and $\tilde{E}\left[1_A|\mathcal{F}_{t-}\right]$ always exist, we have that they coincide with the predictable projections $\xi_t^1$ and $\xi_t^2$.

We can finally state the following

**Proposition 5.2.** *If $S_T$ is square-integrable with respect to $\tilde{P}$, the optimal strategy $\theta_t$ is given in the following feedback form:*

$$(47)\quad \theta_t = \xi_t^1 - \frac{\lambda_t}{\sigma_t X_t}\left(\xi_t^1 X_t - \frac{K}{B_T}\xi_t^2 - \tilde{E}\left[\frac{(S_T-K)^+}{B_T}\right] - \int_0^t \theta_s dX_s\right)$$

*Proof.* The expression for $\theta$ is given by Proposition 2.3:

$$\theta_t = \tilde{\xi}_t - \frac{\tilde{\zeta}_t}{\tilde{Z}_t}\left(\tilde{V}_{t-} - c - \int_0^t \theta_s dX_s\right)$$

We need only to evaluate the terms $\tilde{\xi}_t$, $\dfrac{\tilde{\zeta}_t}{\tilde{Z}_t}$, $V_{t-}$ and $c$.

By [22], it follows immediately that $c = \tilde{E}\left[\dfrac{(S_T-K)^+}{B_T}\right]$ and we obtain

$\dfrac{\tilde{\zeta}_t}{\tilde{Z}_t} = \dfrac{\lambda_s}{\sigma_s X_s}$ from the equality:

$$\frac{d\tilde{P}}{dP} = \frac{\mathcal{E}\left(-\int_0^T \frac{\lambda_s}{\sigma_s X_s}dX_s\right)}{E\left[\mathcal{E}\left(-\int_0^T \frac{\lambda_s}{\sigma_s X_s}dX_s\right)\right]}$$

Moreover, by Bayes' formula and equation (46) we get

$$\tilde{V}_t = \tilde{E}\left[\frac{(S_T-K)^+}{B_T}\bigg|\mathcal{F}_t\right] = X_t\tilde{E}^X\left[1_A|\mathcal{F}_t\right] - \frac{K}{B_T}\tilde{E}\left[1_A|\mathcal{F}_t\right]$$

As a result of Remark 5.1, $\tilde{V}_{t-} = X_t\xi_t^1 - \dfrac{K}{B_T}\xi_t^2$.

Finally, in order to compute $\tilde{\xi}_t$ we need a suitable decomposition of

$$\tilde{V}_t = \tilde{E}\left[\frac{(S_T-K)^+}{B_T}\bigg|\mathcal{F}_t\right]$$

with respect to $X_t$ under $\tilde{P}$. From the calculations preceeding Remark 5.1, we have

$$\frac{H}{B_T} = \frac{(S_T-K)^+}{B_T} = \tilde{E}\left[\frac{H}{B_T}\bigg|\tilde{\mathcal{F}}_0\right] + \int_0^T \tilde{\xi}_s^1 dX_s$$

because $\tilde{\mathcal{F}}_T \equiv \mathcal{F}_T$. Then, by Theorem 2.5 in [20], we obtain:

$$\frac{H}{B_T} = E\left[\frac{H}{B_T}\right] + \int_0^T \xi_s^1 dX_s + \tilde{L}_t$$

where $\xi_t^1$ is the $\mathcal{F}_t$-predictable projection of $\tilde{\xi}_t^1$ (calculated under $\tilde{P}$) and $\tilde{L}_t$ is a $\tilde{P}$-square-integrable martingale, orthogonal to $X$. Since the decomposition is unique, $\tilde{\xi}_t$ coincides with $\xi_t^1$ and this concludes the proof.

$\square$

*Remark* 5.3. The application of Proposition 5.2 in concrete cases involves the calculation of the terms:

$$(48) \qquad \tilde{E}^X\left[1_A | \mathcal{F}_{t-}\right] = \tilde{E}\left[\tilde{E}^X\left[1_A | \mathcal{F}_t\right] \Big| \tilde{\mathcal{F}}_{t-}\right]$$

$$(49) \qquad \tilde{E}\left[1_A | \mathcal{F}_{t-}\right] = \tilde{E}\left[\tilde{E}\left[1_A | \mathcal{F}_t\right] \Big| \tilde{\mathcal{F}}_{t-}\right]$$

we know that:

$$(50) \qquad \tilde{E}^X\left[1_A | \mathcal{F}_t\right] = N(d_1(t, \eta, X_t))$$

$$(51) \qquad \tilde{E}\left[1_A | \mathcal{F}_t\right] = N(d_2(t, \eta, X_t))$$

Denoting by $P_E$ and $\tilde{P}_E$ respectively the projections of $P$ and $\tilde{P}$ on $E$, we have:

$$\frac{d\tilde{P}_E}{dP_E} = \frac{\exp\left(-\int_0^T \lambda_t^2 dt\right)}{E\left[\exp\left(-\int_0^T \lambda_t^2 dt\right)\right]}$$

Recalling that $X_t$ is $\mathcal{F}_{t-}$-measurable, we obtain, for $i \in \{1, 2\}$:

$$(52) \qquad \tilde{E}\left[N(d_i(t, \eta, X_t)) | \mathcal{F}_{t-}\right] = F_i(t, \eta, X_t)$$

here:

$$(53) \qquad F_i(t, \eta, x) = \tilde{E}\left[N(d_i(t, \eta, x)) | \mathcal{F}_{t-}\right] = \tilde{E}\left[N(d_i(t, \eta, x)) | \mathcal{E}_{t-}\right]$$

For instance, consider Example 4.1 with $n = 1$. In this case, $\eta$ is a Bernoulli random variable under $\tilde{P}$, and we denote $\tilde{p} = \tilde{P}(\eta = 0)$. The strategy is given by:

$$(54) \qquad \xi_t^1 = (\tilde{p}\xi_t^1(0) + (1-\tilde{p})\xi_t^1(1))1_{\{t<t_1\}} + \xi_t^1(\eta)1_{\{t \geq t_1\}}$$

In a similar fashion, the optimal strategy can be calculated in more complex examples, the computational effort becoming correspondingly higher.

## 6. Appendix

We refer to [4] for all standard definitions on stochastic processes and to [12] for a complete treatment of random measures theory.

6.1. **Stopping Times.** We recall here some definitions and properties of stopping times. We assume that all random variables are defined on some probability space $(\Omega, \mathcal{F}, P)$.

**Proposition 6.1.** *Let $\tau$ be a real-valued, Borel random variable, and $\mathcal{F}^\tau$ the smallest filtration under which $\tau$ is a stopping time. We have that:*

$$(55) \qquad \mathcal{F}_t^\tau = \tau^{-1}(\mathcal{B}([0,t])) \vee \tau^{-1}(t, \infty)$$

*where $\mathcal{B}(A)$ is the family of Borel subsets of $A$.*

*Proof.* By definition, $\mathcal{F}_t^\tau = \sigma(\{\tau \le s\}, s \le t)$. Since $\mathcal{F}_t^\tau$ is a $\sigma$-field and contains all sublevels of $\tau$ within $[0, t]$, it necessarily contains the inverse images of all Borel sets of $[0, t]$. The set $\tau^{-1}(t, \infty)$ is the complement of $\tau^{-1}[0, t]$, and belongs to the $\sigma$-field. The reverse inclusion is trivial. Finally, the right-hand side in (55) is easily seen to be a $\sigma$-field. $\qquad \square$

*Remark* 6.2. An immediate consequence of Proposition 6.1 is the right-continuity of $\mathcal{F}_t^\tau$. Moreover, $\mathcal{F}_{t-}^\tau = \tau^{-1}(\mathcal{B}([0, t))) \vee \tau^{-1}[t, \infty)$. This means that the augmentation of $\mathcal{F}_t^\tau$ is continuous if and only if the law of $\tau$ is diffuse.

**Corollary 6.3.** *The filtration generated by the random variable $\tau \wedge t$ coincides with $\mathcal{F}^\tau$ if and only if $\tau$ is an optional time, that is, if $\{\tau < t\} \in \mathcal{F}_t$ for all $t$.*

**Definition 6.4.** A stopping time $\tau$ is *totally inaccessible* if it is strictly positive and for every increasing sequence of stopping times $\tau_1, \ldots, \tau_n$, such that $\tau_n < \tau$ for all $n$, $P(\lim_{n \to \infty} \tau_n = \tau, \ \tau < \infty) = 0$.

Totally inaccessible stopping times can be characterized as follows:

**Proposition 6.5.** *Let $\tau$ a strictly positive stopping time. The following properties are equivalent:*

   *i) $\tau$ is totally inaccessible;*

   *ii) There exists a uniformly integrable martingale $M_t$, continuous outside the graph of $\tau$ such that $M_0 = 0$ and $\Delta M_\tau = 1$ on $\{\tau < \infty\}$.*

*Proof.* See [4], VI.78. $\qquad\square$

**Proposition 6.6.** *If the law of a stopping time is diffuse, then it is totally inaccessible with respect to $\mathcal{F}^\tau$.*

*Proof.* See [4], IV.107. $\qquad\square$

**Definition 6.7.** Let $A$ be an adapted process, with $A_0 = 0$ and locally integrable variation. The *compensator* of $A$ is defined as the unique predictable process $\tilde{A}$ such that $A - \tilde{A}$ is a local martingale.

*Remark* 6.8. In particular, the compensator of an increasing process is itself an increasing predictable process.

**Proposition 6.9.** *Let $\tau$ be a stopping time with a diffuse law. Then the compensator of the process $1_{\{\tau \leq t\}}$ with respect to $\mathcal{F}^\tau$ is given by*

$$(56) \qquad\qquad \tilde{A}_t = -\log(1 - F_{t\wedge\tau})$$

*where $F_x = P(\tau \leq x)$.*

*Proof.* By assumption, $M_t = 1_{\{\tau \leq t\}} - \tilde{A}_t$ is a local martingale, and by Remark 6.8, $\tilde{A}_t$ is an increasing process. Therefore we have:

$$\sup_{s \leq t} M_s \leq 1$$

and $M_t$ is in fact a martingale (see for instance [16], page 35, Theorem 47). We look for a compensator of the form $\tilde{A}_t = a_{t\wedge\tau}$, with $a : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Hence:

$$E\left[1_{\{\tau \leq s\}} - 1_{\{\tau \leq t\}}\,\middle|\,\mathcal{F}_t\right] = E\left[a_{s\wedge\tau} - a_{t\wedge\tau}\,\middle|\,\mathcal{F}_t\right] \quad \forall s \geq t$$

It follows that:

$$(57) \qquad \frac{P\left(t < \tau \leq s\right)}{P\left(t < \tau\right)} = \frac{E\left[1_{\{t < \tau \leq s\}}(a_\tau - a_t) + 1_{\{t < \tau\}}(a_s - a_t)\right]}{P\left(t < \tau\right)}$$

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This equality can be rewritten as an integral equation:

$$(58) \qquad F_s - F_t = \int_t^s (a_x - a_t)dF_x + \int_s^\infty (a_s - a_t)dF_x$$

It is easy to check by substitution that the unique solution to this integral equation is given by $a_x = -\log(1 - F_x)$. By the uniqueness, this is the only compensator of $1_{\{\tau \leq t\}}$. $\qquad \square$

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