Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs in arbitrary moments

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Introduction

Let us consider the McKean-Vlasov SDE

$$dX_t = b(t, X_t, P_{X_t})\, dt + \sigma(t, X_t, P_{X_t})\, dW_t \quad \text{for } t \geq 0. \tag{1}$$

Thereby, b and σ are the respective measurable drift and diffusion coefficients taking values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ defined on

$$\mathbb{R}_+ \times \mathbb{R}^m \times \mathscr{P}_p(\mathbb{R}^m), \quad \text{where } p \geq 2,$$

$(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions and W is a standard d-dimensional $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion.

Further, $\mathscr{P}_p(\mathbb{R}^m)$ denotes the Polish space of all Borel probability measures μ on $\mathbb{R}^m$ with finite p-th moment

$$\int_{\mathbb{R}^m} |x|^p \, \mu(dx),$$

equipped with the p-th Wasserstein metric given by

$$\vartheta_p(\mu, \nu) := \inf_{\theta \in \mathscr{P}(\mu,\nu)} \left(\int_{\mathbb{R}^m \times \mathbb{R}^m} |x - y|^p \, \theta(dx, dy) \right)^{\frac{1}{p}},$$

where $\mathscr{P}(\mu, \nu)$ is the convex space of all Borel probability measures θ on $\mathbb{R}^m \times \mathbb{R}^m$ with

$$\theta(B \times \mathbb{R}^m) = \mu(B) \quad \text{and} \quad \theta(\mathbb{R}^m \times B) = \nu(B)$$

for all $B \in \mathscr{B}(\mathbb{R}^m)$.

For $q \geq 1$ and $n \in \mathbb{N}$ we let $\mathscr{L}^q_{loc}(\mathbb{R}^n)$ denote the linear space of all measurable maps $\eta : \mathbb{R}_+ \to \mathbb{R}^n$ such that

$$\int_0^t |\eta(s)|^q \, ds < \infty \quad \text{for all } t \geq 0$$

and we write $\mathscr{L}^q_{loc}(\mathbb{R}^n_+)$ for the convex cone of all $\eta \in \mathscr{L}^q_{loc}(\mathbb{R}^n)$ with

$$\eta_1 \geq 0, \ldots, \eta_n \geq 0.$$

In the same spirit we define $\mathscr{L}^\infty_{loc}(\mathbb{R}^n)$ and $\mathscr{L}^\infty_{loc}(\mathbb{R}^n_+)$.

Stability

First, let us allow for negative partial Lipschitz coefficients.

(C.1) (Hölder continuity conditions)

There are $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}^2)$, $\hat{\eta} \in \mathscr{L}^2_{loc}(\mathbb{R}^2_+)$ and $\alpha, \beta \in (0, 1]$ such that

$$(x - \tilde{x})'\big(b(\cdot, x, \mu) - b(\cdot, \tilde{x}, \tilde{\mu})\big) \leq |x - \tilde{x}|\Big(\eta_1|x - \tilde{x}|^{\alpha} + \eta_2\vartheta_p(\mu, \tilde{\mu})^{\beta}\Big)$$

and

$$|\sigma(\cdot, x, \mu) - \sigma(\cdot, \tilde{x}, \tilde{\mu})| \leq \hat{\eta}_1|x - \tilde{x}|^{\alpha} + \hat{\eta}_2\vartheta_p(\mu, \tilde{\mu})^{\beta}$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathscr{P}_p(\mathbb{R}^m)$.

Under (C.1), we define $\gamma_p \in \mathscr{L}^1_{loc}(\mathbb{R})$ and $\delta_p \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ via

$$\gamma_p := (p-1+\alpha)\big(\eta_1^+ - \eta_1^- \mathbb{1}_{\{1\}}(\alpha)\big) + (p-1+\beta)\eta_2^+ + \frac{p-1}{2}\big(p-2(1-\alpha\vee\beta)\big)(\hat{\eta}_1+\hat{\eta}_2)^2$$

and

$$\delta_p := (1-\alpha)\eta_1^+ + (1-\beta)\eta_2^+ + \frac{p-1}{2}(1-\alpha\wedge\beta)(\hat{\eta}_1+\hat{\eta}_2)^2.$$

Further, let $\Theta_\beta : \mathbb{R}_+ \times \mathscr{P}_p(\mathbb{R}^m) \times \mathscr{P}_p(\mathbb{R}^m) \to \mathbb{R}_+$ be given by

$$\Theta_\beta(\cdot,\mu,\tilde{\mu}) := \eta_2^+ \vartheta_p(\mu,\tilde{\mu})^\beta + \hat{\eta}_2^2 \vartheta_p(\mu,\tilde{\mu})^{2\beta}.$$

Explicit L^p-comparison estimate (Meyer-Brandis, Proske and K., 2022)

Let (C.1) hold and X and $\tilde{X}$ be two solutions such that $\Theta_\beta(\cdot, P_X, P_{\tilde{X}}) \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$. Then $Y := X - \tilde{X}$ satisfies

$$E\left[|Y_t|^p\right] \leq e^{\int_0^t \gamma_p(s)\,ds} E\left[|Y_0|^p\right] + \int_0^t e^{\int_s^t \gamma_p(\tilde{s})\,d\tilde{s}} \delta_p(s)\,ds$$

for all $t \geq 0$. In particular, if γ_p^+ and δ_p are integrable, then

$$\sup_{t \geq 0} E\left[|Y_t|^p\right] < \infty.$$

If in addition $\int_0^\infty \gamma_p^-(s)\,ds = \infty$, then $\lim_{t\uparrow\infty} E[|Y_t|^p] = 0$.

Derivation of strong solutions

Note that a Borel measurable map $\mu : \mathbb{R}_+ \to \mathscr{P}_p(\mathbb{R}^m)$ induces the SDE

$$dX_t = b_\mu(t, X_t)\, dt + \sigma_\mu(t, X_t)\, dW_t \quad \text{for } t \geq 0 \tag{2}$$

with the measurable maps b_μ and σ_μ given by

$$b_\mu(t, x) := b(t, x, \mu(t)) \quad \text{and} \quad \sigma_\mu(t, x) := \sigma(t, x, \mu(t))$$

for $t \geq 0$ and $x \in \mathbb{R}^m$. Additionally, assume that $\xi : \Omega \to \mathbb{R}^m$ is $\mathscr{F}_0$-measurable.

For local weak solutions to (2) we need the following condition:

(C.2) (Space continuity and boundedness on bounded sets)

b is continuous in $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|b(s,x,\mu)| \vee |\sigma(s,x,\mu)| \leq c_n$$

for any $s \in [0,n]$, $x \in \mathbb{R}^m$ and $\mu \in \mathscr{P}_p(\mathbb{R}^m)$ with

$$|x| \vee \left(\int_{\mathbb{R}^m} |y|^p \, \mu(dy) \right)^{\frac{1}{p}} \leq n.$$

For an explicit L^p-growth estimate we assume the following:

(C.3) (Affine growth conditions)
There are $\upsilon \in \mathscr{L}^1_{loc}(\mathbb{R}^3_+)$ and $\hat{\upsilon} \in \mathscr{L}^2_{loc}(\mathbb{R}^3_+)$ such that

$$x'b(\cdot, x, \mu) \leq |x|\left(\nu_1 + \nu_2|x| + \nu_3\left(\int_{\mathbb{R}^m} |y|\,\mu^p(dy)\right)^{\frac{1}{p}}\right),$$

$$|\sigma(\cdot, x, \mu)| \leq \hat{\nu}_1 + \hat{\nu}_2|x| + \hat{\nu}_3\left(\int_{\mathbb{R}^m} |y|\,\mu^p(dy)\right)^{\frac{1}{p}}$$

for each $x \in \mathbb{R}^m$ and $\mu \in \mathscr{P}_p(\mathbb{R}^m)$.

Existence of unique strong solutions (Meyer-Brandis, Proske and K., 2022)

Let (C.1)-(C.3) hold for $\alpha = \beta = 1$ and $E[|\xi|^p] < \infty$. Then the following two assertions hold:

(i) There is pathwise uniqueness for (1) relative to Θ_1. That is, any two solutions X and $\tilde{X}$ satisfying

$$X_0 = \tilde{X}_0 \quad \text{a.s.}$$

are indistinguishable if $\Theta_1(\cdot, P_X, P_{\tilde{X}}) \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$.

(ii) There is a unique strong solution X^ξ such that $X^\xi_0 = \xi$ a.s. and $E[|X^\xi|^p]$ is locally bounded.

Example (sums of power functions and integral maps)

Let $n \in \mathbb{N}$, $\hat{b} \in \mathscr{L}^{\infty}_{loc}(\mathbb{R}^n_+)$ and $\hat{c} \in \mathscr{L}^{\infty}_{loc}(\mathbb{R})$ and suppose that $a \in]0, \infty[^n$ and

$$f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$$

is Lipschitz continuous such that $f(0, \cdot) = 0$,

$$b(\cdot, x, \mu) = -x\Big(\hat{b}_1 |x|^{a_1 - 1} + \cdots + \hat{b}_n |x|^{a_n - 1}\Big) + \hat{c} \int_{\mathbb{R}^m} f(x, y)\, \mu(dy)$$

and $b(\cdot, 0, \mu) = 0$ for all $x \in \mathbb{R}^m \setminus \{0\}$ and $\mu \in \mathscr{P}_p(\mathbb{R}^m)$. Then (C.1)-(C.3) are valid for b with $\alpha = \beta = 1$.

References

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Thank you for your attention!