Derivation of unique stable solutions to McKean-Vlasov SDEs

Alexander Kalinin, LMU Munich

Joint work with Thilo Meyer-Brandis and Frank Proske

Finance and Stochastics Seminar
Imperial College London, March 2023

Introduction

We seek to analyse the one-dimensional McKean-Vlasov SDE

$$dX_t = b(t, X_t, P_{X_t})\,dt + \sigma(t, X_t)\,dW_t \quad \text{for } t \geq 0. \tag{1}$$

Thereby,

- $b : \mathbb{R}_+ \times \mathbb{R} \times \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are the measurable drift and diffusion coefficients, respectively,
- $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions and
- W is a standard $(\mathscr{F}_t)_{t\geq 0}$-Brownian motion.

Further, $\mathscr{P}_1(\mathbb{R})$ stands for the Polish space of all Borel probability measures μ on $\mathbb{R}$ with finite first moment

$$\int_{\mathbb{R}} |x| \, \mu(dx),$$

equipped with the first Wasserstein metric given by

$$\vartheta_1(\mu, \nu) := \inf_{\theta \in \mathscr{P}(\mu,\nu)} \int_{\mathbb{R}\times\mathbb{R}} |x - y| \, \theta(dx, dy),$$

where $\mathscr{P}(\mu, \nu)$ is the convex space of all Borel probability measures θ on $\mathbb{R} \times \mathbb{R}$ with

$$\theta(B \times \mathbb{R}) = \mu(B) \quad \text{and} \quad \theta(\mathbb{R} \times B) = \nu(B)$$

for all $B \in \mathscr{B}(\mathbb{R})$.

We recall that a solution to the McKean-Vlasov SDE (1) is an adapted, continuous and integrable process X such that

$$\int_0^t |b(s, X_s, P_{X_s})| + |\sigma(s, X_s)|^2 \, ds < \infty$$

for all $t \geq 0$ and

$$X_t = X_0 + \int_0^t b(s, X_s, P_{X_s}) \, ds + \int_0^t \sigma(s, X_s) \, dW_s$$

for any $t \geq 0$ a.s. In our paper, we replace $\mathscr{P}_1(\mathbb{R})$ by a suitable metrisable topological space $\mathscr{P}$, to include SDEs.

For $p \geq 1$ and $l \in \mathbb{N}$ we let $\mathscr{L}^p_{loc}(\mathbb{R}^l)$ denote the linear space of all measurable maps $\eta : \mathbb{R}_+ \to \mathbb{R}^l$ such that

$$\int_0^t |\eta(s)|^p \, ds < \infty \quad \text{for all } t \geq 0$$

and we write $\mathscr{L}^p_{loc}(\mathbb{R}^l_+)$ for the convex cone of all $\eta \in \mathscr{L}^p_{loc}(\mathbb{R}^l)$ with

$$\eta_1 \geq 0, \ldots, \eta_l \geq 0.$$

In the same spirit we define $\mathscr{L}^\infty_{loc}(\mathbb{R}^l)$ and $\mathscr{L}^\infty_{loc}(\mathbb{R}^l_+)$.

Pathwise uniqueness

First, let us consider two requirements:

(C.1) (Local Hölder continuity condition)

For any $n \in \mathbb{N}$ there is $\hat{\eta}_n \in \mathscr{L}^2_{loc}(\mathbb{R}_+)$ such that

$$|\sigma(\cdot, x) - \sigma(\cdot, \tilde{x})| \leq \hat{\eta}_n |x - \tilde{x}|^{\frac{1}{2}}$$

for all $x, \tilde{x} \in [-n, n]$.

Example (sums of power functions)

For $l \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^l$ be measurable and $\alpha \in]0, \infty[^l$ be such that

$$\sigma(\cdot, x) = \kappa + \eta_1 |x|^{\alpha_1} + \cdots + \eta_l |x|^{\alpha_l}$$

for any $x \in \mathbb{R}$. Then (C.1) holds if

$$\eta \in \mathscr{L}^2_{loc}(\mathbb{R}^l) \quad \text{and} \quad \alpha_1 \geq \frac{1}{2}, \ldots, \alpha_l \geq \frac{1}{2}.$$

(C.2) (Partial Osgood continuity condition)

There exist $\eta, \lambda \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ and some increasing concave function $\rho \in C(\mathbb{R}_+)$ such that

$$\rho(0) = 0, \quad \rho > 0 \quad \text{on }]0, \infty[$$

and

$$\begin{aligned} \operatorname{sgn}(x - \tilde{x})\big(b(\cdot, x, \mu) - b(\cdot, \tilde{x}, \tilde{\mu})\big) \\ \leq \eta\rho(|x - \tilde{x}|) + \lambda\rho\big(\vartheta_1(\mu, \tilde{\mu})\big) \end{aligned}$$

for all $x, \tilde{x} \in \mathbb{R}$ and $\mu, \tilde{\mu} \in \mathscr{P}_1(\mathbb{R})$.

Example (sums involving decreasing functions)

For $l \in \mathbb{N}$ let $\kappa, \lambda : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^l$ be measurable maps and $f : \mathbb{R} \to \mathbb{R}^l$ be increasing such that

$$b(\cdot, x, \mu) = \kappa - \eta_1 f_1(x) - \cdots - \eta_l f_l(x) + \lambda \int_{\mathbb{R}} y \, \mu(dy)$$

for any $x \in \mathbb{R}$ and $\mu \in \mathscr{P}_1(\mathbb{R})$. In this case, (C.2) holds for the choice

$$\rho(x) = x \quad \text{for all } x \geq 0$$

if $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}^l_+)$ and $\lambda \in \mathscr{L}^1_{loc}(\mathbb{R})$.

Pathwise uniqueness (Meyer-Brandis, Proske and K., 2021)

Let (C.1) and (C.2) hold such that $\int_0^1 \rho(x)^{-1}\,dx = \infty$ and define $\Theta : \mathbb{R}_+ \times \mathscr{P}_1(\mathbb{R}) \times \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}_+$ by

$$\Theta(\cdot, \mu, \tilde{\mu}) := \lambda\rho\big(\vartheta_1(\mu, \tilde{\mu})\big).$$

Then pathwise uniqueness for (1) relative to Θ holds. That is, any two solutions X and $\tilde{X}$ satisfying

$$X_0 = \tilde{X}_0 \quad \text{a.s.}$$

are indistinguishable if $\Theta(\cdot, P_X, P_{\tilde{X}}) \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$.

To allow for negative partial Lipschitz coefficients, we replace (C.2) by the following hypothesis:

(C.3) (Partial mixed Hölder continuity condition)

There are $l \in \mathbb{N}$,

$$\eta \in \mathscr{L}^1_{loc}(\mathbb{R}^l), \quad \lambda \in \mathscr{L}^1_{loc}(\mathbb{R}^l_+)$$

and $\alpha, \beta \in]0,1]^l$ such that

$$\operatorname{sgn}(x - \tilde{x})\big(b(\cdot, x, \mu) - b(\cdot, \tilde{x}, \tilde{\mu})\big) \leq \sum_{k=1}^{l} \eta_k |x - \tilde{x}|^{\alpha_k} + \lambda_k \vartheta_1(\mu, \tilde{\mu})^{\beta_k}$$

for any $x, \tilde{x} \in \mathbb{R}$ and $\mu, \tilde{\mu} \in \mathscr{P}_1(\mathbb{R})$.

Under (C.3), we define $\gamma_{\mathscr{P}_1} \in \mathscr{L}^1_{loc}(\mathbb{R})$ and $\delta_{\mathscr{P}_1} \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ via

$$\gamma_{\mathscr{P}_1} := \sum_{k=1}^{l} \alpha_k \big(\eta_k^+ - \eta_k^- \mathbb{1}_{\{1\}}(\alpha_k)\big) + \beta_k \lambda_k$$

and

$$\delta_{\mathscr{P}_1} := \sum_{k=1}^{l} (1-\alpha_k)\eta_k^+ + (1-\beta_k)\lambda_k.$$

Further, let $\Theta : \mathbb{R}_+ \times \mathscr{P}_1(\mathbb{R}) \times \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}_+$ be given by

$$\Theta(\cdot, \mu, \tilde{\mu}) := \sum_{k=1}^{l} \lambda_k \vartheta_1(\mu, \tilde{\mu})^{\beta_k}.$$

Explicit L^1-comparison estimate (Meyer-Brandis, Proske and K., 2021)

Let (C.1) and (C.3) hold and X and $\tilde{X}$ be two solutions such that $\Theta(\cdot, P_X, P_{\tilde{X}}) \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$. Then $Y := X - \tilde{X}$ satisfies

$$E\big[|Y_t|\big] \leq e^{\int_0^t \gamma_{\mathscr{P}_1}(s)\,ds} E\big[|Y_0|\big] + \int_0^t e^{\int_s^t \gamma_{\mathscr{P}_1}(\tilde{s})\,d\tilde{s}} \delta_{\mathscr{P}_1}(s)\,ds$$

for all $t \geq 0$. In particular, if $\gamma^+_{\mathscr{P}_1}$ and $\delta_{\mathscr{P}_1}$ are integrable, then

$$\sup_{t \geq 0} E\big[|Y_t|\big] < \infty.$$

If additionally $\int_0^\infty \gamma^-_{\mathscr{P}_1}(s)\,ds = \infty$, then $\lim_{t\uparrow\infty} E[|Y_t|] = 0$.

First moment stability

We restrict (C.3) as follows:

(C.4) (Partial Lipschitz condition)
There are $\eta \in \mathscr{L}^1_{loc}(\mathbb{R})$ and $\lambda \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ such that

$$\operatorname{sgn}(x-\tilde{x})\big(b(\cdot,x,\mu)-b(\cdot,\tilde{x},\tilde{\mu})\big) \leq \eta|x-\tilde{x}| + \lambda\vartheta_1(\mu,\tilde{\mu})$$

for every $x,\tilde{x} \in \mathbb{R}$ and $\mu,\tilde{\mu} \in \mathscr{P}_1(\mathbb{R})$.

Under this requirement,

$$\delta_{\mathscr{P}_1} = 0, \quad \gamma_{\mathscr{P}_1} = \eta + \lambda \quad \text{and} \quad \Theta(\cdot,\mu,\tilde{\mu}) = \lambda\vartheta_1(\mu,\tilde{\mu})$$

for all $\mu,\tilde{\mu} \in \mathscr{P}_1(\mathbb{R})$.

Exponential first moment stability (Meyer-Brandis, Proske and K., 2021)

Let (C.1) and (C.4) hold. Further, let $\alpha > 0$ and $\hat{\lambda} < 0$ satisfy

$$\gamma_{\mathscr{P}_1}(s) \leq \hat{\lambda}\alpha s^{\alpha-1} \quad \text{for a.e. } s \geq 0.$$

Then (1) is α-exponentially stable in moment relative to Θ and $\hat{\lambda}$ is a Lyapunov exponent. That is, there is $c \geq 0$ such that

$$E\left[|X_t - \tilde{X}_t|\right] \leq c e^{\hat{\lambda} t^{\alpha}} E\left[|X_0 - \tilde{X}_0|\right]$$

for all $t \geq 0$ whenever X and $\tilde{X}$ are two solutions satisfying $\Theta(\cdot, P_X, P_{\tilde{X}}) \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$.

Derivation of strong solutions

We recall that a Borel measurable map $\mu : \mathbb{R}_+ \to \mathscr{P}_1(\mathbb{R})$ induces the SDE

$$dX_t = b_\mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \quad \text{for } t \geq 0 \tag{2}$$

with the measurable map $b_\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ given by

$$b_\mu(t, x) := b(t, x, \mu(t))$$

and let ξ denote an $\mathscr{F}_0$-measurable random variable.

Now we specify (C.1) as follows:

(C.5) (Local Hölder continuity condition and the origin as zero)

For every $n \in \mathbb{N}$ there is $\hat{\eta}_n \in \mathscr{L}^{\infty}_{loc}(\mathbb{R}_+)$ such that

$$|\sigma(\cdot, x) - \sigma(\cdot, \tilde{x})| \leq \hat{\eta}_n |x - \tilde{x}|^{\frac{1}{2}}$$

for any $x, \tilde{x} \in [-n, n]$ and

$$\sigma(\cdot, 0) = 0.$$

Example (sums of power functions)

The case in which there are $l \in \mathbb{N}$, $\eta \in \mathscr{L}^{\infty}_{loc}(\mathbb{R}^l)$ and $\alpha \in [1/2, \infty[^l$ such that

$$\sigma(\cdot, x) = \eta_1 |x|^{\alpha_1} + \cdots + \eta_l |x|^{\alpha_l}$$

for all $x \in \mathbb{R}$ is included, even though $\alpha_k > 1$ may hold for some $k = 1, \ldots, l$.

For local weak solutions to (2) we need the following condition:

(C.6) (Space continuity and boundedness on bounded sets)

b is continuous in $x \in \mathbb{R}$ and for each $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|b(s, x, \mu)| \leq c_n$$

for any $s \in [0, n]$, $x \in [-n, n]$ and $\mu \in \mathscr{P}_1(\mathbb{R})$ with

$$\int_{\mathbb{R}} |y| \, \mu(dy) \leq n.$$

Thereby, we recall that $\vartheta_1(\mu, \delta_0) = \int_{\mathbb{R}} |y| \, \mu(dy)$.

The subsequent assumption leads to an *explicit L^1-growth estimate*:

(C.7) (Partial affine growth condition)
There are $\kappa, \chi \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ and $\upsilon \in \mathscr{L}^1_{loc}(\mathbb{R})$ such that

$$\operatorname{sgn}(x) b(\cdot, x, \mu) \leq \kappa + \upsilon |x| + \chi \int_{\mathbb{R}} |y| \, \mu(dy)$$

for each $x \in \mathbb{R}$ and $\mu \in \mathscr{P}_1(\mathbb{R})$.

Example (sums involving polynomials and integral functions)

For $l \in \mathbb{N}$ let $\kappa, \eta_0, \lambda \in \mathscr{L}^\infty_{loc}(\mathbb{R})$ and

$$\eta \in \mathscr{L}^\infty_{loc}(\mathbb{R}^l_+)$$

as well as $n \in \mathbb{N}^l$ be such that

$$b(\cdot, x, \mu) = \kappa + \eta_0 x - \eta_1 x^{n_1} - \cdots - \eta_l x^{n_l} + \lambda \int_{\mathbb{R}} |y| \, \mu(dy)$$

for all $x \in \mathbb{R}$ and $\mu \in \mathscr{P}_1(\mathbb{R})$. Then (C.4), (C.6) and (C.7) are valid if the coordinates of n are odd.

An explicit L^1-growth estimate (Meyer-Brandis, Proske and K., 2021)

Let (C.5) and (C.7) be valid and X be a solution to (1) such that $\chi E[|X|] \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$. Then

$$E\big[|X_t|\big] \leq e^{\int_0^t (\upsilon+\chi)(s)\,ds} E\big[|X_0|\big] + \int_0^t e^{\int_s^t (\upsilon+\chi)(\tilde{s})\,d\tilde{s}} \kappa(s)\,ds$$

for all $t \geq 0$. In particular, if $(\upsilon + \chi)^+$ and κ are integrable, then $\sup_{t\geq 0} E\big[|X_t|\big] < \infty$. In this case,

$$\lim_{t\uparrow\infty} E[|X_t|] = 0$$

follows from $\int_0^\infty (\upsilon + \chi)^-(s)\,ds = \infty$.

Existence of unique strong solutions (Meyer-Brandis, Proske and K., 2021)

Let (C.4)-(C.7) be satisfied and $E[|\xi|] < \infty$. Moreover, define $\Theta : \mathbb{R}_+ \times \mathscr{P}_1(\mathbb{R}) \times \mathscr{P}_1(\mathbb{R}) \to \mathbb{R}_+$ by

$$\Theta(\cdot, \mu, \tilde{\mu}) := \lambda \vartheta_1(\mu, \tilde{\mu}).$$

Then pathwise uniqueness for (1) relative to Θ holds and there exists a unique strong solution X^ξ such that

$$X_0^\xi = \xi \quad \text{a.s.}$$

and $E[|X^\xi|]$ is locally bounded.

Proof ideas.

For any Borel measurable map $\mu : \mathbb{R}_+ \to \mathscr{P}_1(\mathbb{R})$ we show that the SDE (2) admits a unique strong solution $X^{\xi,\mu}$ such that

$$X_0^{\xi,\mu} = \xi \quad \text{a.s.}$$

and $E[|X^{\xi,\mu}|]$ is locally bounded as soon as the function $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto \int_{\mathbb{R}} |x| \, \mu(t)(dx)$ is locally bounded.

Then we prove that the sequence $(\mu_n)_{n\in\mathbb{N}}$ of $\mathscr{P}_1(\mathbb{R})$-valued Borel measurable maps on $\mathbb{R}_+$ recursively given by

$$\mu_n := P_{X^{\xi,\mu_{n-1}}} \quad \text{with} \quad \mu_0 := \mu$$

converges locally uniformly to the law of the strong solution to (1).

$\square$

An Application in Mathematical Finance

Joint work with Damiano Brigo and Federico Graceffa

A stochastic volatility model

Let us consider a financial market model with time horizon $T > 0$ consisting of only one riskless and one risky asset that are traded.

In this setting, the measurable integrable function $r : [0, T] \to \mathbb{R}$ is the instantaneous risk-free interest rate and

$$D_t(r) := \exp\left(- \int_0^t r(s)\, ds \right)$$

is the discount factor from the initial time to $t \in [0, T]$.

Similarly as before, let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \in [0,T]}, P)$ be a filtered probability space satisfying the usual conditions.

We suppose that $\hat{W}$ and $\tilde{W}$ are two $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motions with covariation

$$\langle \hat{W}, \tilde{W} \rangle = \int_0^{\cdot} \rho(s)\, ds \quad \text{a.s.}$$

and impose the following dynamics on the price process S of the only risky asset and its squared volatility process V:

$$\begin{aligned} dS_t &= b(t)S_t\, dt + \theta(t)\sqrt{V_t}S_t\, d\hat{W}_t \\ dV_t &= \big(k(t) - l_0(t)V_t + l(t)V_t^{\alpha}\big)\, dt + \lambda(t)V_t^{\beta}\, d\tilde{W}_t \end{aligned} \tag{3}$$

for $t \in [0, T]$ with $\alpha \geq 1$ and $\beta \geq 1/2$.

From the pathwise uniqueness and strong existence results in [1] and a positivity condition we draw the following conclusion.

Power diffusion as squared volatility (Brigo, Graceffa and K., 2021)

Let b, θ, k, l_0, l, λ be bounded, $l \leq 0$ and $\lambda^2/2 \leq k$. Then pathwise uniqueness for (3) holds and there is a unique strong solution (S, V) satisfying

$$S > 0, \quad V > 0 \quad \text{and} \quad (S_0, V_0) = (s_0, v_0) \quad \text{a.s.},$$

where $s_0, v_0 > 0$. Furthermore, $\sup_{t\in[0,T]} |\log(S_t)|$ and V are integrable.

Example (Established models in the literature)

For $l = 0$ and $l_0 > 0$ we recover the dynamics

$$dV_t = \big(k(t) - l_0(t)V_t\big)\,dt + \lambda(t)V_t^{\beta}\,d\tilde{W}_t, \quad \text{for } t \in [0, T]$$

in time-dependent versions of the following option pricing models:

(i) The Heston model for $\beta = 1/2$. There, l_0 is the mean reversion speed, k/l_0 is the mean reversion level and the same positivity condition $\lambda \leq 2k$ applies.

(ii) The Garch diffusion model for $\beta = 1$. Similarly, l_0 is the mean reversion speed and k/l_0 the mean reversion level.

Market prices of risk

The model allows for an equivalent local martingale measure.

That is, there is a probability measure $\tilde{P}$ on $(\Omega, \mathscr{F})$ such that $\tilde{P} \sim P$ and the discounted price process

$$[0, T] \times \Omega \to]0, \infty[, \quad (t, \omega) \mapsto D_t(r) S_t(\omega)$$

is a local martingale under $\tilde{P}$.

Indeed, let us define a continuous local martingale Z via

$$Z = \exp\left(- \int_0^{\cdot} \kappa_s \, dW_s - \int_0^{\cdot} \tilde{\kappa}_s \, d\tilde{W}_s - \frac{1}{2} \int_0^{\cdot} \kappa_s^2 + \tilde{\kappa}_s^2 \, ds \right) \quad \text{a.s.},$$

where W is an $(\mathscr{F}_t)_{t \in [0,T]}$-Brownian motion independent of $\tilde{W}$.

If $E[Z_T] = 1$, then Z induces an equivalent local martingale measure via Girsanov's theorem if and only if

$$(b - r)(t) = \theta(t)\sqrt{V_t}\big(\kappa_t\sqrt{1 - \rho(t)^2} + \tilde{\kappa}_t\rho(t)\big)$$

for a.e. $t \in [0, T]$ a.s. If in addition $\theta > 0$, then we propose to take the market prices of risk

$$\tilde{\kappa}_t = \gamma\theta(t)\sqrt{V_t} \quad \text{and} \quad \kappa_t = \left(\frac{(b-r)(t)}{\theta(t)\sqrt{V_t}} - \tilde{\kappa}_t\rho(t)\right)\frac{1}{\sqrt{1-\rho(t)^2}}$$

for all $t \in [0, T]$ and fixed $\gamma \geq 0$.

In this case, Novikov's condition implies that $E[Z_T] = 1$ as soon as

$$E\left[\exp\left(\frac{\gamma^2}{2}\int_0^T \theta(t)^2 V_t\,dt\right)\right] < \infty.$$

In particular, the choice $\gamma = 0$ is feasible and V satisfies the same SDE

$$dV_t = \big(k(t) - l_0(t)V_t + l(t)V_t^{\alpha}\big)\,dt + \lambda(t)V_t^{\beta}\,d\tilde{W}_t$$

under the resulting risk-neutral measure.

By also considering the dynamics of $\log(S)$, we can turn to the evaluation of contingent claims in a subsequent analysis.

References

[1] A. Kalinin, T. Meyer-Brandis, and F. Proske.
Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs: a multidimensional Yamada-Watanabe approach.
arXiv:2107.07838 , 2021.

[2] D. Brigo, F. Graceffa, and A. Kalinin.
Mild to classical solutions for XVA equations under stochastic volatility.
arXiv:2112.11808, 2021.

Thank you for your attention!