Mild to classical solutions for XVA equations under stochastic volatility

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German Probability and Statistics Days
Essen, 2023

A financial market model with default

We aim to evaluate a derivative contract between an investor $\mathscr{I}$ and a counterparty $\mathscr{C}$ in a financial market under

- default risk,
- collateralisation and
- funding costs and benefits.

To this end, we derive a valuation equation based on default-free information only and characterise its solutions.

Let $(\mathscr{F}_t)_{t\in[0,T]}$ and $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$ be two filtrations standing for the default-free and the whole available information, respectively.

The two $[0,T]\cup\{\infty\}$-valued random variables $\tau_{\mathscr{I}}$ and $\tau_{\mathscr{C}}$ model the respective default times of $\mathscr{I}$ and $\mathscr{C}$.

Then $\tau := \tau_{\mathscr{I}} \wedge \tau_{\mathscr{C}}$ stands for the time of a party to default first and we require that

$$\begin{aligned} \mathscr{F}_t \subset \tilde{\mathscr{F}}_t \subset \mathscr{F}_t \vee \sigma\big(\mathbb{1}_{\{\tau\leq s\}} : \tau \in \{\tau_{\mathscr{I}}, \tau_{\mathscr{C}}\}, s \in [0,t]\big), \\ P(\tau_{\mathscr{I}} = t) = P(\tau_{\mathscr{C}} = t) = P(\tau_{\mathscr{I}} = \tau_{\mathscr{C}}, \tau < \infty) = 0 \end{aligned} \tag{C}$$

for all $t \in [0,T]$.

Example (Hitting times involving a gamma distribution)

Let ξ_i be a gamma distributed random variable and $\lambda^{(i)}$ be a process, both with positive values, such that

$$\tau_i = \inf\left\{ t \in [0, T] \,\middle|\, \int_0^t \lambda_s^{(i)}\, ds \geq \xi_i \right\}$$

for $i \in \{\mathscr{I}, \mathscr{C}\}$. Then, under verifiable assumptions, the conditions in (C) on the distribution of $\tau_{\mathscr{I}}$ and $\tau_{\mathscr{C}}$ hold and

$$P(\tau \in B) = \int_{B \cap [0,T]} \varphi_\tau(s)\, ds + \left(1 - \int_0^T \varphi_\tau(s)\, ds\right) \delta_\infty(B)$$

for any Borel set B in $[0, T] \cup \{\infty\}$ and some explicitly determined measurable integrable function $\varphi_\tau : [0, T] \to [0, \infty]$.

Next, the measurable integrable function $r : [0, T] \to \mathbb{R}$ is the instantaneous risk-free interest rate and

$$D_{s,t}(r) := \exp\left(- \int_s^t r(\tilde{s})\, d\tilde{s} \right)$$

is the discount factor from time $s \in [0, T]$ to $t \in [s, T]$.

Let us assume that $\tilde{P}$ is an equivalent local martingale measure. That is,

$$P \sim \tilde{P}$$

and the discounted price process of the only traded risky asset is an $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-local martingale under $\tilde{P}$.

A stochastic volatility model

We suppose that $\hat{W}$ and $\tilde{W}$ are two $(\mathscr{F}_t)_{t\in[0,T]}$-Brownian motions with covariation

$$\langle \hat{W}, \tilde{W} \rangle = \int_0^{\cdot} \rho(s)\, ds \quad \text{a.s.}$$

and impose the following dynamics on the price process S of the only risky asset and its squared volatility process V:

$$\begin{aligned} dS_t &= b(t)S_t\, dt + \theta(t)\sqrt{V_t}S_t\, d\hat{W}_t \\ dV_t &= \big(k(t) - l_0(t)V_t + l(t)V_t^{\alpha}\big)\, dt + \lambda(t)V_t^{\beta}\, d\tilde{W}_t \end{aligned} \tag{1}$$

for $t \in [0, T]$ with initial condition $(S_0, V_0) = (s_0, v_0)$ a.s., where $\alpha \geq 1$ and $\beta \geq 1/2$.

From a pathwise uniqueness and a strong existence result in [2] and a positivity condition we draw the following conclusion.

Power diffusion as squared volatility (Brigo, Graceffa and K., 2021)

Let b, θ, k, l_0, l, λ be bounded, $l \leq 0$ and $\lambda^2/2 \leq k$. Then pathwise uniqueness for the SDE (1) holds and there is a unique strong solution (S, V) satisfying

$$S > 0, \quad V > 0 \quad \text{and} \quad (S_0, V_0) = (s_0, v_0) \quad \text{a.s.}$$

Example (Established models in the literature)

For $l = 0$ and $l_0 > 0$ we recover the dynamics

$$dV_t = \big(k(t) - l_0(t)V_t\big)\,dt + \lambda(t)V_t^{\beta}\,d\tilde{W}_t, \quad \text{for } t \in [0, T]$$

in time-dependent versions of the following option pricing models:

(i) The Heston model for $\beta = 1/2$. There, l_0 is the mean reversion speed, k/l_0 is the mean reversion level and the same positivity condition $\lambda \leq 2k$ applies.

(ii) The Garch diffusion model for $\beta = 1$. Similarly, l_0 is the mean reversion speed and k/l_0 the mean reversion level.

We derive an equation for the value process, denoted by $\tilde{\mathscr{V}} \in \tilde{\mathscr{S}}$, of a trading strategy that hedges the contract under $\tilde{P}$.

In the end, we seek a default-free valuation, and the equation for $\tilde{\mathscr{V}}$ includes quantities that merely depend on its pre-default part.

So, let $G(\tau)$ be an $(\mathscr{F}_t)_{t\in[0,T]}$-survival process of τ under $\tilde{P}$, which is an $[0,1]$-valued supermartingale such that

$$\tilde{P}(\tau > t|\mathscr{F}_t) = G_t(\tau) \quad \text{a.s. for all } t \in [0, T].$$

Further, a process $\tilde{X}$ is called integrable up to time τ if $\tilde{X}\mathbb{1}_{\{\tau>\cdot\}}$ is integrable.

We refine a classical result as follows.

Pre-default versions

A process $\tilde{X} \in \tilde{\mathscr{S}}$ is integrable up to time τ if and only if there is $X \in \mathscr{S}$ such that $XG(\tau)$ is integrable and $X_s = \tilde{X}_s$ a.s. on $\{\tau > s\}$ for all $s \in [0, T]$. In this case,

$$X_s G_s(\tau) = \tilde{E}[\tilde{X}_s \mathbb{1}_{\{\tau > s\}} | \mathscr{F}_s] \quad \text{a.s.}$$

for all $s \in [0, T]$. If in addition $G_s(\tau) > 0$ a.s. for all $s \in [0, T]$, then X is unique up to a modification.

We shall call X a pre-default version of $\tilde{X}$.

The discounted cash flows

For simplicity of the talk, let us focus on a part of the considered financial quantities:

1. **The contractual cash flows** depend on a payoff function and the risky asset that is influenced by its squared volatility:

$$ {}_{\mathrm{con}}\mathrm{CF}_s := D_{s,T}(r)\phi(S_T, V_T)\mathbb{1}_{\{\tau > T\}}. $$

2. **The cash flows arising on the default** of $\mathscr{I}$ or $\mathscr{C}$ can be computed with the residual value of the claim:

$$ {}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V}) := D_{s,\tau}(r)\varepsilon_\tau(\mathscr{V}) $$

on $\{s < \tau < T\}$ and ${}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V}) := 0$ on the complement of this set.

Under mild conditions, we require that $\tilde{\mathscr{V}}$ satisfies the valuation equation

$$\tilde{\mathscr{V}}_s = \tilde{E}\left[{}_{\mathrm{con}}\mathrm{CF}_s + {}_{\mathrm{def}}\mathrm{CF}_s(\mathscr{V})\middle|\tilde{\mathscr{F}}_s\right] \tag{2}$$

a.s. for all $s \in [0, T]$. Then (2) is satisfied if and only if

$$\begin{aligned}\mathscr{V}_s G_s(\tau) = \tilde{E}\left[D_{s,T}(r)\phi(S_T, V_T)G_T(\tau)\,\middle|\,\mathscr{F}_s\right]\\ - \tilde{E}\left[\int_s^T D_{s,t}(r)\varepsilon_t(\mathscr{V})\,dG_t(\tau)\,\middle|\,\mathscr{F}_s\right] \quad \text{a.s.}\end{aligned}$$

Characterisation of pre-default value semimartingales (Brigo, Graceffa and K., 2021)

Under weak conditions, a continuous $(\mathscr{F}_t)_{t\in[0,T]}$-semimartingale $\mathscr{V}$ is a pre-default value process if and only if $\tilde{E}[|\mathscr{V}_0|] < \infty$ and

$$\mathscr{V}_s = \phi(S_T, V_T) + \int_s^T -r(t)\mathscr{V}_t \, dt - \int_s^T \frac{\varepsilon_t(\mathscr{V}) - \mathscr{V}_t}{G_t(\tau)} \, dG_t(\tau) - \int_s^T \frac{D_{0,t}(-r)}{G_t(\tau)} \, dM_t$$

for all $s \in [0, T]$ a.s. and some continuous $(\mathscr{F}_t)_{t\in[0,T]}$-martingale M.

The pre-default valuation PDE

Next, we explicitly construct a local martingale measure $\tilde{P}_V$ via Girsanvov's theorem, by proposing suitable market prices of risk.

Then we deduce the dynamics of $(\log(S), V)$ under $\tilde{P}_V$ and derive a parabolic semilinear PDE with terminal condition.

Finally, under certain conditions, we prove that for any mild solution u to this PDE the process $\mathscr{V} \in \mathscr{S}$ defined via

$$\mathscr{V}_t := u(t, \log(S_t), V_t)$$

is a pre-default value process under $\tilde{P}_V$.

References

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[2] A. Kalinin, T. Meyer-Brandis, and F. Proske.
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Thank you for your attention!