Support characterization for regular path-dependent stochastic Volterra integral equations

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Support of the law of a diffusion

Let S be a metrizable topological space and μ be an inner regular measure on $\mathscr{B}(S)$.

Support of a measure

The support of μ, denoted by $\operatorname{supp}(\mu)$, is the smallest closed set in S whose complement has μ-measure zero.

By using a metric that generates the topology of S, it follows readily that

$$\operatorname{supp}(\mu) = \{x \in S \mid \mu(B_\varepsilon(x)) > 0 \text{ for all } \varepsilon > 0\}.$$

Let $C_r^\alpha([0,T],\mathbb{R}^m)$ be the Banach space of all $x \in C([0,T],\mathbb{R}^m)$ that are α-Hölder-continuous on $[r,T]$, equipped with the delayed α-Hölder norm

$$\|x\|_{\alpha,r} := \|x^r\|_\infty + \sup_{s,t\in[r,T]\colon s\neq t} \frac{|x(s)-x(t)|}{|s-t|^\alpha}.$$

Moreover, let $X : [0,T] \times \Omega \to \mathbb{R}^m$ be a stochastic process on some probability space $(\Omega, \mathscr{F}, P)$ satisfying

$$X \in C_r^\alpha([0,T],\mathbb{R}^m) \quad \text{a.s.}$$

By viewing X as a random variable and assuming inner regularity, we may consider the support of $P \circ X^{-1}$.

Namely, a path $x \in C_r^\alpha([0, T], \mathbb{R}^m)$ belongs to $\mathrm{supp}(P \circ X^{-1})$ if and only if

$$P(\|X - x\|_{\alpha,r} < \varepsilon) > 0 \quad \text{for any } \varepsilon > 0.$$

We note that $P \circ X^{-1}$ is inner regular as soon as

$$X \in C_r^\beta([0, T], \mathbb{R}^m) \quad \text{a.s.}$$

for some $\beta \in (\alpha, 1]$.

Let $W_r^{1,p}([0,T],\mathbb{R}^m)$ be the Banach space of all $x \in C([0,T],\mathbb{R}^m)$ that are absolutely continuous on $[r,T]$ such that

$$\int_r^T |\dot{x}(s)|^p \, ds < \infty,$$

endowed with the delayed Sobolev L^p-norm

$$\|x\|_{1,p,r} := \|x^r\|_\infty + \left(\int_r^T |\dot{x}(s)|^p \, ds \right)^{\frac{1}{p}}.$$

Then $W_r^{1,p}([0,T],\mathbb{R}^m) \subsetneq C_r^{1/q}([0,T],\mathbb{R}^m)$ whenever $p > 1$ and q is its dual exponent.

Stochastic Volterra integral equations

For two non-anticipative product measurable maps b and σ on

$$[r, T]^2 \times C([0, T], \mathbb{R}^m)$$

with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, we consider the SVIE

$$X_t = X_r + \int_r^t b(t, s, X)\, ds + \int_r^t \sigma(t, s, X)\, dW_s \tag{1}$$

for $t \in [r, T]$ with initial condition $X_q = \hat{x}(q)$ for all $q \in [0, r]$ a.s.

We introduce the map $\rho : [r, T]^2 \times C([0, T], \mathbb{R}^m) \to \mathbb{R}^m$ by

$$\rho_k(t, s, x) := \sum_{l=1}^{d} \underbrace{\partial_x \sigma_{k,l}(t, s, x)}_{\in \mathbb{R}^{1\times m}} \underbrace{\sigma(s, s, x)}_{\in \mathbb{R}^{m\times d}} \underbrace{e_l}_{\in \mathbb{R}^d},$$

if $s < t$, and $\rho_k(t, s, x) := 0$, otherwise.

For any $p \geq 1$ and $h \in W_r^{1,p}([0, T], \mathbb{R}^d)$, we study the VIE

$$\begin{aligned} x_h(t) = x_h(r) + \int_r^t (b - (1/2)\rho)(t, s, x_h)\, ds \\ + \int_r^t \sigma(t, s, x_h)\, dh(s) \end{aligned} \tag{2}$$

for $t \in [r, T]$ with initial condition $x_h(q) = \hat{x}(q)$ for any $q \in [0, r]$.

Strong solutions as semimartingales (K., 2019)

Pathwise uniqueness for (1) holds and there is a unique strong solution X. Further, X is a semimartingale and

$$E[\|X\|_{\alpha,r}^{p}] < \infty$$

for any $\alpha \in [0, 1/2)$ and $p \geq 1$.

Related work

Volterra equations driven by semimartingales, Protter ('85).

Mild solutions and the flow map (K., 2019)

For any $p \geq 1$ and $h \in W_r^{1,p}([0,T],\mathbb{R}^d)$, there is a unique solution x_h to (2). Moreover, the flow map

$$W_r^{1,p}([0,T],\mathbb{R}^d) \to W_r^{1,p}([0,T],\mathbb{R}^m), \quad h \mapsto x_h$$

is Lipschitz continuous on bounded sets.

Support characterization for SVIEs (K., 2019)

Under suitable conditions, it holds that

$$\mathrm{supp}(P \circ X^{-1}) = \overline{\{x_h \mid h \in W_r^{1,p}([0,T], \mathbb{R}^d)\}}$$

in $C_r^{\alpha}([0,T], \mathbb{R}^m)$ for every $\alpha \in [0, 1/2)$ and $p \geq 2$.

Works on support theorems for SDEs

- Stroock and Varadhan ('72), Gyöngy and Pröhle ('90), Aida ('90),
- Ben Arous, Grădinaru and Ledoux ('94), Millet and Sanz-Solé ('94), Cont and Kalinin ('20).

Example (diffusion coefficients with regular kernels)

Let $d = m = 1$, $k \in C([r,T]^2, \mathbb{R})$ and $F : [r,T] \times C([0,T], \mathbb{R}) \to \mathbb{R}$ be of class $\mathbb{C}^{1,2}$ such that

$$\sigma(t,s,x) = k(t,s)F(s,x)$$

for every $s, t \in [r,T]$ and $x \in C([0,T], \mathbb{R})$. Then the correction map ρ satisfies

$$\rho(t,s,x) = k(t,s)k(s,s)\partial_x F(s,x)F(s,x)$$

for any $s, t \in [r,T]$ with $s < t$ and $x \in C([0,T], \mathbb{R})$.

Thank you for your attention!

Reference

Support characterization for regular path-dependent stochastic Volterra integral equations, A. Kalinin, *Electronic Journal of Probability*, 2021.