A probabilistic analysis of stochastic integral equations

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Randomised ordinary differential equations

For a clear overview that can be followed readily let us begin with the one-dimensional ordinary differential equation (ODE)

$$\dot{x}(t) = b(t, x(t)) \quad \text{for } t \geq 0 \tag{1}$$

with a product measurable function $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

As b is not required to be continuous, we may not expect solutions in the classical sense but an integral version of (1) can be used.

Namely, a mild solution to (1) is a continuous function $x : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\int_0^t |b(s, x(s))| \, ds < \infty \quad \text{and} \quad x(t) = x(0) + \int_0^t b(s, x(s)) \, ds$$

for any $t \geq 0$, two properties that entail its local absolute continuity.

We note that a mild solution x becomes a classical solution if and only if the measurable function

$$\mathbb{R}_+ \to \mathbb{R}, \quad s \mapsto b(s, x(s)),$$

which is its weak derivative $\dot{x}$, is continuous, by the Fundamental Theorem of Calculus.

To allow for randomness, we take a complete probability space $(\Omega, \mathcal{F}, P)$ on which there is a standard Brownian motion

$$W : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto W_t(\omega).$$

That means, W is a continuous process with independent increments such that $W_0 = 0$ a.s. and

$$W_t - W_s \sim \mathcal{N}(0, t - s)$$

for all $s, t \geq 0$ with $s < t$. In particular, W is a square-integrable martingale, and we let

$$\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$$

be another measurable function.

Thus, instead of analysing the ODE (1), we consider the stochastic differential equation (SDE)

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \quad \text{for } t \geq 0. \tag{2}$$

We recall that a solution to (2) is an adapted continuous process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

$$\int_0^t |b(s, X_s)| + \sigma(s, X_s)^2\,ds < \infty$$

for any $t \geq 0$ and

$$X_t = X_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s$$

for all $t \geq 0$ a.s.

Thereby, $\int_0^{\cdot} \sigma(s, X_s)\, dW_s$ is the stochastic integral of $\sigma(\cdot, X)$ with respect to W that is a local martingale with quadratic variation

$$\int_0^{\cdot} \sigma(s, X_s)^2\, ds.$$

In particular, if $\sigma = 0$ a.e., then any path of a solution to the SDE (2) is a mild solution to the ODE (1) and vice versa.

In general, however, we may only expect the paths of a solution X to (2) to be locally α-Hölder-continuous for any $\alpha \in]0, \frac{1}{2}[$, as

$$E[|W_s - W_t|^2] = |s - t|^{2 \cdot \frac{1}{2}}$$

for any $s, t \geq 0$ and W is a centred Gaussian process.

Stability and uniqueness

Let us consider a condition on the drift b that allows for negative partial Lipschitz coefficients:

(C.1) (Partial Hölder continuity condition)

There are $\alpha \in]0,1]$ and some measurable locally integrable function $\eta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\operatorname{sgn}(x - \tilde{x})\big(b(\cdot, x) - b(\cdot, \tilde{x})\big) \leq \eta |x - \tilde{x}|^{\alpha}$$

for any $x, \tilde{x} \in \mathbb{R}$.

Example (sums involving decreasing functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^m$ be measurable and $f : \mathbb{R} \to \mathbb{R}^m$ be increasing such that

$$b(\cdot, x) = \kappa - \eta_1 f_1(x) - \cdots - \eta_m f_m(x)$$

for any $x \in \mathbb{R}$. Then (C.1) holds for any $\alpha \in]0,1]$ if

$$\eta_1 \geq 0, \ldots, \eta_m \geq 0.$$

Thereby, b may fail to be continuous in $x \in \mathbb{R}$.

On the diffusion σ we impose a local continuity condition only:

(C.2) (Hölder continuity condition on compact sets)

For any $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|\sigma(\cdot, x) - \sigma(\cdot, \tilde{x})| \leq c_n |x - \tilde{x}|^{\frac{1}{2}}$$

for all $x, \tilde{x} \in [-n, n]$.

The exponent $\frac{1}{2}$ comes from the Yamada-Watanabe approach, since $\beta \in]0, 1]$ satisfies

$$\int_0^1 \frac{1}{x^{2\beta}}\, dx = \infty \quad \Leftrightarrow \quad \beta \geq \frac{1}{2}.$$

Example (sums of power functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}^m$ be measurable and $\beta \in]0, \infty[^m$ be such that

$$\sigma(\cdot, x) = \kappa + \eta_1 |x|^{\beta_1} + \cdots + \eta_m |x|^{\beta_m}$$

for any $x \in \mathbb{R}$. Then (C.2) holds if η is bounded and

$$\beta_1 \geq \frac{1}{2}, \ldots, \beta_m \geq \frac{1}{2}.$$

Explicit L^1-comparison estimate (Meyer-Brandis, Proske and K., '21)

Let (C.1) and (C.2) hold and X and $\tilde{X}$ be two solutions to (2). Then $Y := X - \tilde{X}$ satisfies

$$E\big[|Y_t|\big] \leq e^{\alpha \int_0^t \eta_\alpha(s)\,ds} E\big[|Y_0|\big] + (1-\alpha)\int_0^t e^{\alpha \int_s^t \eta_\alpha(\tilde{s})\,d\tilde{s}} \eta^+(s)\,ds$$

for all $t \geq 0$ with $\eta_\alpha := \eta^+ - \eta^- \mathbb{1}_{\{1\}}(\alpha)$. In particular, if Y_0 and η^+ are integrable, then

$$\sup_{t\geq 0} E\big[|Y_t|\big] < \infty.$$

In this case, $\lim_{t\uparrow\infty} E[|Y_t|] = 0$ if $\alpha = 1$ and $\int_0^\infty \eta^-(s)\,ds = \infty$.

Proof ideas.

(i) The Yamada-Watanabe approach gives us a suitable increasing sequence $(\psi_n)_{n\in\mathbb{N}}$ in $C^2(\mathbb{R}_+)$ such that

$$\psi_n(0) = \psi_n'(0) = \psi_n''(0) = 0 \quad \text{for any } n \in \mathbb{N}$$

as well as $\sup_{n\in\mathbb{N}} \psi_n(x) = x$ and $\lim_{n\uparrow\infty} \psi_n'(x) = 1$ for each $x > 0$.

(ii) We may apply Itô's formula to $\psi_n(|Y|)$ for all $n \in \mathbb{N}$, since $\psi_n(|\cdot|) \in C^2(\mathbb{R})$. Further, we take a locally absolutely continuous function

$$u : \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{with} \quad u(0) = 1$$

and deduce the dynamics of $u \cdot \psi_n(|Y|)$ from Itô's product rule, which is the novel part of our work.

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(iii) Next, we show that Y is integrable and the function $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto E[|Y_t|]$ is locally bounded provided $E[|Y_0|] < \infty$. Then

$$
\begin{aligned}
u(t)E\left[|Y_t|\right] &= \lim_{n\uparrow\infty} u(t)E\left[\psi_n(|Y_t|)\right] \\
&\leq E\left[|Y_0|\right] + \int_0^t E\left[\dot{u}(s)|Y_s| + u(s)\eta(s)|Y_s|^{\alpha}\right] ds
\end{aligned}
$$

for any $t \geq 0$, by monotone convergence.

(iv) Hence, Young's inequality and the choice

$$
u(t) = \exp\left(- \alpha \int_0^t \eta_\alpha(s)\, ds\right)
$$

for all $t \geq 0$ yield the asserted estimate. $\square$

As a corollary we obtain a stability result in the sense of Lyapunov.

Exponential first moment stability (Meyer-Brandis, Proske and K., '21)

Let (C.1) and (C.2) hold for $\alpha = 1$. Further, let $\beta > 0$ and $\lambda < 0$ satisfy

$$\eta(s) \leq \lambda\beta s^{\beta-1} \quad \text{for a.e. } s \geq 0.$$

Then (2) is β-exponentially stable in moment and λ is a Lyapunov exponent. That is, there is $c \geq 0$ such that

$$E\big[|X_t - \tilde{X}_t|\big] \leq c e^{\lambda t^\beta} E\big[|X_0 - \tilde{X}_0|\big]$$

for all $t \geq 0$ whenever X and $\tilde{X}$ are two solutions to (2).

Derivation of strong solutions

For weak solutions to (2) we rely on the subsequent requirements:

(C.3) b is continuous in $x \in \mathbb{R}$ and locally bounded.

(C.4) (Partial affine growth condition)

There are measurable locally bounded functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ and $\upsilon : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\mathrm{sgn}(x) b(\cdot, x) \leq \kappa + \upsilon |x|$$

for every $x \in \mathbb{R}$.

Example (sums involving decreasing functions)

For $m \in \mathbb{N}$ let $\kappa : \mathbb{R}_+ \to \mathbb{R}$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+^m$ be measurable and locally bounded and $n \in \mathbb{N}^m$ be such that

$$b(\cdot, x) = \kappa - \eta_1 x^{n_1} - \cdots - \eta_m x^{n_m}$$

for any $x \in \mathbb{R}$. Then (C.1), (C.3) and (C.4) are satisfied for any $\alpha \in]0,1]$ if

the coordinates of n are odd.

However, b does not need to be of affine growth in $x \in \mathbb{R}$.

Let us take an $\mathcal{F}_0$-measurable random variable ξ with $E[|\xi|] < \infty$.

Existence of unique strong solutions (Meyer-Brandis, Proske and K., '21)

Let (C.1)-(C.4) hold for $\alpha = 1$ and $\sigma(\cdot, 0) = 0$. Then we have pathwise uniqueness for (2) and there is a unique strong solution X^ξ such that

$$X_0^\xi = \xi \quad \text{a.s.}$$

Moreover, X^ξ is integrable and its first absolute moment function $\mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto E[|X_t^\xi|]$ is locally bounded.

(i) As we show in our paper [4], all these methods are extendible to the McKean-Vlasov SDE

$$dX_t = b(t, X_t, \mathcal{L}(X_t))\, dt + \sigma(t, X_t)\, dW_t \quad \text{for } t \geq 0,$$

where the product measurable drift b is defined on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R})$ instead of $\mathbb{R}_+ \times \mathbb{R}$.

In such a setting, $\mathcal{P}_p(\mathbb{R})$ is the Polish space of all Borel probability measures μ on $\mathbb{R}$ with finite p-th moment

$$\int_{\mathbb{R}} |x|^p \, \mu(dx),$$

equipped with the p-th Wasserstein metric for $p \geq 1$.

(ii) If the diffusion σ should depend on the law of the solution, we provide methods in another work [5] to handle the McKean-Vlasov SDE

$$dX_t = b(t, X_t, \mathcal{L}(X_t))\,dt + \sigma(t, X_t, \mathcal{L}(X_t))\,dW_t \quad \text{for } t \geq 0,$$

where the product measurable drift b and diffusion σ are defined on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{P}_p(\mathbb{R})$ for $p \geq 2$ instead of $\mathbb{R}_+ \times \mathbb{R}$.

(iii) As an application in mathematical finance, in a joint work with Brigo and Graceffa [1] the SDE

$$dV_t = \big(k(t) - l_0(t)V_t + l(t)V_t^{\alpha}\big)\,dt + \lambda(t)V_t^{\beta}\,dW_t \qquad (3)$$

for $t \in [0, T]$ with $T > 0$, $\alpha \geq 1$ and $\beta \geq \frac{1}{2}$ yields the dynamics of a squared volatility process.

Thereby, k, l_0, l and λ are real-valued continuous functions on $[0, T]$ such that

$$l \leq 0 \quad \text{and} \quad \frac{\lambda^2}{2} \leq k.$$

Note that pathwise uniqueness holds for (3) and there is a unique strong solution starting at a positive deterministic value.

Support representations

In the sequel, let us consider the SDE (2) on a finite time horizon:

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \quad \text{for } t \in [0, T] \tag{4}$$

with initial condition $X_0 = x_0$ a.s., where W is replaced by its restriction to $[0, T] \times \Omega$ and $x_0 \in \mathbb{R}$.

First, we recall that the linear space $C([0, T], \mathbb{R})$ of all real-valued continuous paths on $[0, T]$, equipped with the supremum norm

$$\|x\|_\infty := \sup_{t \in [0,T]} |x(t)|,$$

is a separable Banach space.

The image measure $P \circ W^{-1}$ of W in $C([0,T],\mathbb{R})$, which is its law, admits full support in the sense that

$$\mathrm{supp}(P \circ W^{-1}) = \{x \in C([0,T],\mathbb{R}) \,|\, x(0) = 0\}.$$

That is, for any path $x \in C([0,T],\mathbb{R})$ starting at 0 the probability

$$P(\|W - x\|_\infty < \varepsilon)$$

that W remains in the open ball with center x and radius ε is positive for any $\varepsilon > 0$.

Under the conditions below, this property of the driver W carries over to the solutions to (4).

Full support theorem for SDEs (Stroock and Varadhan, '72)

Let b and σ be bounded and Lipschitz continuous in $x \in \mathbb{R}$, uniformly in $t \in [0, T]$, σ be continuous and

$$\sigma \neq 0.$$

Then the unique solution X to the SDE (4) satisfies

$$\operatorname{supp}(P \circ X^{-1}) = \{x \in C([0, T], \mathbb{R}) \,|\, x(0) = x_0\}.$$

Flow of mild solutions

To study the support of solutions to (4) when σ may have zeros, let

$$H([0, T], \mathbb{R})$$

denote the separable Banach space of all absolutely continuous paths $h : [0, T] \to \mathbb{R}$ such that

$$\int_0^T \dot{h}(t)^2 \, dt < \infty,$$

endowed with the Cameron-Martin norm

$$\|h\|_H := |h(0)| + \left(\int_0^T \dot{h}(s)^2 \, ds \right)^{\frac{1}{2}}.$$

Under the conditions of the last result and the hypothesis that σ is of class $C_b^{0,2}$, each $h \in H([0,T],\mathbb{R})$ induces an ODE

$$\dot{x}(t) = \big(b - (1/2)\rho\big)(t, x(t)) + \sigma(t, x(t))\dot{h}(t) \tag{5}$$

for $t \in [0,T]$ with initial condition $x(0) = x_0$ and the correction term

$$\rho := \frac{\partial \sigma}{\partial x} \cdot \sigma.$$

We readily see that (5) admits a unique mild solution x_h. That is,

$$x_h(t) = x_0 + \int_0^t \big(b - (1/2)\rho\big)(s, x_h(s))\, ds + \int_0^t \sigma(s, x_h(s))\, dh(s)$$

for all $t \in [0,T]$, and $x_h \in H([0,T],\mathbb{R})$.

Support theorem for SDEs (Stroock and Varadhan, '72)

Let b be bounded and Lipschitz continuous in $x \in \mathbb{R}$, uniformly in $t \in [0, T]$, and σ be of class $C_b^{1,2}$. Then

$$\mathrm{supp}(P \circ X^{-1}) = \overline{\{x_h \mid h \in H([0, T], \mathbb{R})\}} \quad \text{in } C([0, T], \mathbb{R}).$$

That is, for every $x \in C([0, T], \mathbb{R})$ the probability

$$P(\|X - x\|_\infty < \varepsilon)$$

is positive for any $\varepsilon > 0$ if and only if there is a sequence $(h_n)_{n \in \mathbb{N}}$ in $H([0, T], \mathbb{R})$ such that $\lim_{n \uparrow \infty} \|x - x_{h_n}\|_\infty = 0$.

For $\alpha \in]0, 1]$ we consider the non-separable Banach space

$$C^{\alpha}([0, T], \mathbb{R})$$

of all real-valued α-Hölder continuous paths on $[0, T]$, equipped with the α-Hölder norm

$$\|x\|_{\alpha} := |x(0)| + \sup_{s,t\in[0,T]:\, s\neq t} \frac{|x(s) - x(t)|}{|s - t|^{\alpha}}.$$

Then $H([0, T], \mathbb{R}) \subsetneq C^{\frac{1}{2}}([0, T], \mathbb{R})$ and we set

$$C^{0}([0, T], \mathbb{R}) := C([0, T], \mathbb{R}) \quad \text{and} \quad \| \cdot \|_{0} := \| \cdot \|_{\infty},$$

by convention.

Support theorem for SDEs (Ben Arous, Gradinaru and Ledoux, '94)

Under the same conditions as in the previous result,

$$\operatorname{supp}(P \circ X^{-1}) = \overline{\{x_h \mid h \in H([0,T],\mathbb{R})\}} \quad \text{in } C^\alpha([0,T],\mathbb{R})$$

for any $\alpha \in]0, \frac{1}{2}[$.

The case when both b and σ are time-independent was established independently by Millet and Sanz-Solé ('94) with different methods.

Based on the functional Itô formula, this support characterisation extends to the path-dependent SDE

$$dX_t = b(t, X)\,dt + \sigma(t, X)\,dW_t \quad \text{for } t \in [0, T] \tag{6}$$

with initial condition $X_0 = x_0$ a.s., where the product measurable drift b and diffusion σ are defined on

$$[0, T] \times C([0, T], \mathbb{R}) \quad \text{instead of} \quad [0, T] \times \mathbb{R}.$$

In addition, b and σ are required to be non-anticipative, which means that

$$b(t, x) = b(t, x^t) \quad \text{and} \quad \sigma(t, x) = \sigma(t, x^t)$$

for all $t \in [0, T]$ and $x \in C([0, T], \mathbb{R})$.

Under the conditions of the next result, every $h \in H([0,T],\mathbb{R})$ induces a path-dependent ODE

$$\dot{x}(t) = \big(b - (1/2)\rho\big)(t,x) + \sigma(t,x)\dot{h}(t) \tag{7}$$

for $t \in [0,T]$ with initial condition $x(0) = x_0$ and the correction term

$$\rho := \partial_x \sigma \cdot \sigma$$

that involves the vertical derivative $\partial_x \sigma$ of σ. Moreover, there is a unique mild solution x_h to (7) and the resulting flow map

$$H([0,T],\mathbb{R}) \to H([0,T],\mathbb{R}), \quad h \mapsto x_h$$

is Lipschitz continuous on bounded sets.

Support theorem for path-dependent SDEs (Cont and K., '20)

Let b be bounded and Lipschitz continuous in $x \in C([0,T],\mathbb{R})$, uniformly in $t \in [0,T]$, and σ be of class $\mathbb{C}_b^{1,2}$ and

$$\text{together with} \quad \partial_x \sigma \quad \text{be Lipschitz continuous}$$

in the sense of functional Itô calculus. Then the unique solution X to (6) satisfies

$$\operatorname{supp}(P \circ X^{-1}) = \overline{\{x_h \,|\, h \in H([0,T],\mathbb{R})\}} \quad \text{in } C^\alpha([0,T],\mathbb{R})$$

for any $\alpha \in [0, \frac{1}{2}[$.

Finally, a support characterisation for the path-dependent stochastic Volterra integral equation

$$X_t = x_0 + \int_0^t b(t,s,X)\,ds + \int_0^t \sigma(t,s,X)\,dW_s$$

for $t \in [0,T]$, where the non-anticipative product measurable drift b and diffusion σ are defined on

$$[0,T]^2 \times C([0,T],\mathbb{R}) \quad \text{instead of} \quad [0,T] \times C([0,T],\mathbb{R}),$$

is derived in a consecutive work [3], under an absolute continuity condition on b and σ.

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Thank you for your attention!