Imprecise probability foundations – Extra material

SIPTA School 2026, Munich, Germany

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Overview

Kick-off

Classical probability theory

Multivariate probability — exercise

Interpretation of probability

Frequentist & Bayesian approaches

Limitations of probability theory

Fréchet's bounds & independence
Aversions of real agents

Probability intervals

Credal sets

The robust Bayesian approach
The completely independent product is not convex

Acceptability & Desirability

Changing what shape $\mathcal{D}$ can take

Interval expectation & probability

Learning: categorical prediction

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Multivariate probability — exercise

Consider the (joint) random variable $\boldsymbol{X} = (X_1, X_2, X_3)$ with $\mathcal{X} = \{0, 1\}^3$.
The joint probabilities are determined by the following information:

- X_1 and X_2 are independent
- It holds that $P(X_1 = 0) = 1/2$ and $P(X_2 = 0) = 1/5$
- The conditional probabilities $P(X_3 \mid X_1, X_2)$ are determined by the following table:

x_1	0	0	1	1
x_2	0	1	0	1
$P(X_3 = 0 \mid X_1 = x_1, X_2 = x_2)$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

Derive that $P(X_1 = 0 \mid X_2 = 0, X_3 = 0) = \frac{5}{8}$.

As part of your calculation, you should derive and write a general expression for $P(X_1 \mid X_2, X_3)$ in terms of the (symbolic) probabilities that are given—before filling in the specific numbers.

Overview

The frequentist approach

- Associated to the frequentist interpretation
- Used in classical statistics

- The **likelihood function** describes the data-generating process:

$$L_{\boldsymbol{x}}(\theta) = P(\boldsymbol{X} = \boldsymbol{x} | \Theta = \theta),$$

where θ represents the process-specific parameters.

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  where θ represents the process-specific parameters.

- Typical reasoning techniques & tools:
  - Maximum likelihood estimation
  - Confidence intervals
  - Hypothesis tests

The Bayesian approach

- Associated to the subjective interpretation
- Used in Bayesian statistics and uncertainty quantification

- Centered around the use of Bayes's rule for updating belief:

$$\underbrace{P(\Theta = \theta | \boldsymbol{X} = \boldsymbol{x})}_{\textbf{posterior}} \propto L_{\boldsymbol{x}}(\theta) \underbrace{P(\Theta = \theta)}_{\textbf{prior}},$$

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- Typical reasoning techniques & tools:
  - Posterior mean and maximum a posteriori estimation
  - Credible intervals
  - Probabilities for hypotheses

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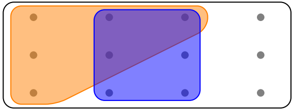
Unknown dependence and Fréchet's bounds

- Nature of dependence between two events A and B often not known



independence

- Fréchet's bounds:

$$P(A)P(B) = P(A \cap B) \in \Big[\max\{0, P(A) + P(B) - 1\}, \min\{P(A), P(B)\}\Big]$$

$$P(A \cup B) \in \Big[\max\{P(A), P(B)\}, \min\{1, P(A) + P(B)\}\Big]$$

- How does independence fit in here?

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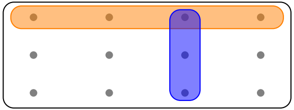

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Are real agents irrational?

Risk aversion

- Urn with 20 red, 20 gray balls
- Two gambles:

	R	G
f_{RG}	\$50	\$50
f_{R}	\$100	\$0

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- Two urns; the agent must choose one:
  - E 20 black balls, 20 white balls
  - U unknown proportion of black and white balls
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- Rational agents must choose a pmf for U; in case they choose $p_{\mathrm{B}} = p_{\mathrm{W}} = 1/2$ they are *indifferent* between urns E and U
- Real agents *strictly prefer* urn E over urn U

Overview

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- Interpretation of probability
- Limitations of probability theory
- Probability intervals
- Credal sets
  - The robust Bayesian approach
  - The completely independent product is not convex
- Acceptability & Desirability
- Interval expectation & probability

The robust Bayesian approach

- Generalization of the classical Bayesian approach
- Tries to overcome criticisms of precision associated to the subjective interpretation
- Still centered around the use of Bayes's rule for updating belief:

$$\mathcal{C}^{\Theta|\boldsymbol{x}} = \left\{ P(\Theta|\boldsymbol{X} = \boldsymbol{x}) \propto L_{\boldsymbol{x}}(\Theta)P(\Theta) : P(\Theta) \in \mathcal{C}^{\Theta} \right\},$$

  where
  - $\mathcal{C}^{\Theta}$ is called the *set of priors* (many classes exist)
  - $\mathcal{C}^{\Theta|\boldsymbol{x}}$ is called the *set of posteriors*
- Typical inferences:
  - Lower and upper posterior means
  - Credible intervals (with lower bound guarantees)
  - Probability bounds for hypotheses

The completely independent product is not convex

Problem setup

- Random variables X_1 and X_2 with outcome spaces $\mathcal{X}_1 = \{0, 1\}$ and $\mathcal{X}_2 = \{-, +\}$
- Linear-vacuous marginal credal sets

$$\mathcal{C}^{X_1} := \mathcal{C}^{p^{X_1}, \frac{1}{5}} \text{ with } p^{X_1} = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \quad \text{and} \quad \mathcal{C}^{X_2} := \mathcal{C}^{p^{X_2}, \frac{1}{4}} \text{ with } p^{X_2} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

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Show that their completely independent joint credal set is not convex

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Demonstration

- We construct a counterexample to convexity, i.e., we construct a convex mixture of elements of the joint that lies outside the joint because it does not factorize

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- We construct a counterexample to convexity, i.e., we construct a convex mixture of elements of the joint that lies outside the joint because it does not factorize
- An element q of $\mathcal{C}^{\boldsymbol{X}} = \mathcal{C}^{X_1} \otimes \mathcal{C}^{X_2}$ is defined for every $r^{X_1} \in \mathcal{P}_{X_1}$ and $r^{X_2} \in \mathcal{P}_{X_2}$ by

$$q_{(x_1,x_2)} = (q^{X_1} \otimes q^{X_2})_{(x_1,x_2)} = q^{X_1}_{x_1} q^{X_2}_{x_2} = (\tfrac{4}{5} p^{X_1}_{x_1} + \tfrac{1}{5} r^{X_1}_{x_1})(\tfrac{3}{4} p^{X_2}_{x_2} + \tfrac{1}{4} r^{X_2}_{x_2})$$

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Demonstration

$r_0^{X_1} = r_-^{X_2} = 0$

	0	1	q^{X_2}
$-$			$1/4$
$+$			$3/4$
q^{X_1}	$2/5$	$3/5$	

$r_0^{X_1} = r_-^{X_2} = 1$

	0	1	q^{X_2}
$-$			$1/2$
$+$			$1/2$
q^{X_1}	$3/5$	$2/5$	

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$r_0^{X_1} = r_-^{X_2} = 0$

	0	1	q^{X_2}
$-$	$^2/_{20}$	$^3/_{20}$	$^1/_4$
$+$	$^6/_{20}$	$^9/_{20}$	$^3/_4$
q^{X_1}	$^2/_5$	$^3/_5$	

$r_0^{X_1} = r_-^{X_2} = 1$

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$$r_0^{X_1} = r_-^{X_2} = 0 \qquad\qquad r_0^{X_1} = r_-^{X_2} = 1$$

$$\frac{1}{2}\begin{array}{c|cc|c} & 0 & 1 & q^{X_2} \\ \hline - & 2/20 & 3/20 & 1/4 \\ + & 6/20 & 9/20 & 3/4 \\ \hline q^{X_1} & 2/5 & 3/5 & \end{array} + \frac{1}{2}\begin{array}{c|cc|c} & 0 & 1 & q^{X_2} \\ \hline - & 3/10 & 2/10 & 1/2 \\ + & 3/10 & 2/10 & 1/2 \\ \hline q^{X_1} & 3/5 & 2/5 & \end{array} = \begin{array}{c|cc|c} & 0 & 1 & q^{X_2} \\ \hline - & 8/40 & 7/40 & \\ + & 12/40 & 13/40 & \\ \hline q^{X_1} & & & \end{array}$$

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$\frac{1}{2}$

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$=$

not factorizing

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Changing what shape $\mathcal{D}$ can take

- Replacing the generating rules in Axioms D3 and D4
- Adding additional generating rules

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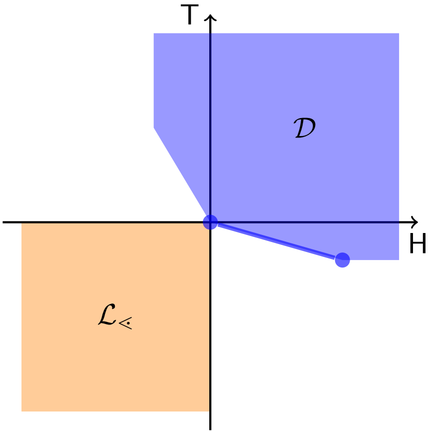
- *Example*: replace positive scaling and combination by *convexity*,

  if $f, g \in \mathcal{D}$ and $\mu \in [0, 1]$,
  then $\mu \cdot f + (1 - \mu) \cdot g \in \mathcal{D}$

  This means replacing the *positive linear hull* posi by the *convex hull* co



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The categorical prediction task

What is the color of the next ball to be drawn from an urn?

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What is the color of the next ball to be drawn from an urn?

Setup

- Outcome space $\mathcal{X}$ (of ball colors: Red, Green, Black, White,...)
- We have $n \in \mathbb{N}$ observations $\boldsymbol{x} = (x_1, \ldots, x_n)$
  (sequence of colors of previous balls drawn)
- Draw inferences for or make decisions related to the next observation X_{n+1}
  (color of next ball drawn)

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Exchangeability assumption

- *The order of the observations is irrelevant* (inferences should remain the same when $(\mathrm{R}, \mathrm{G}, \mathrm{R})$ or $(\mathrm{R}, \mathrm{R}, \mathrm{G})$ has been observed)
- Observed occurrence vector $\boldsymbol{n} \in \mathbb{N}^{\mathcal{X}}$ with $n_z := \sum_{k=1}^{n} \delta_{zx_k}$ and $n_S := \sum_{x \in S} n_x$ (if $\boldsymbol{x} = (\mathrm{R}, \mathrm{G}, \mathrm{R}, \mathrm{W})$ then $n_\mathrm{R} = 2$, $n_\mathrm{G} = 1$, $n_\mathrm{W} = 1$, and $n_{\neg\mathrm{W}} = 3$)

Learning lower expectations for categorical prediction

Idea

Add epistemic uncertainty using $s > 0$ pseudo-observations (of unknown color) and predict according to the 'observed' frequency

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Predictive inference

- Outcome prediction: $\underline{P}_s(\{x\}|\boldsymbol{n}) = \frac{n_x}{s+n}$ and $\overline{P}_s(\{x\}|\boldsymbol{n}) = \frac{s+n_x}{s+n}$

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Predictive inference

- Event prediction: $\underline{P}_s(S|\boldsymbol{n}) = \frac{n_S}{s+n}$ and $\overline{P}_s(S|\boldsymbol{n}) = \frac{s+n_S}{s+n}$

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- Lower expectation for $f \in \mathcal{L}$:

$$\underline{E}_s(f|\boldsymbol{n}) = \frac{n}{s+n} E_p(f) + \frac{s}{s+n} \min_{x\in\mathcal{X}} f(x) \quad \text{with } p_x := \frac{n_x}{n}$$

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Properties of $\underline{E}_s(\cdot|\boldsymbol{n})$

- linear-vacuous model; vacuous for $n = 0$; more precise with more observations
- does not depend on a specific categorization $\mathcal{X}$
- immediate prediction model of the *imprecise Dirichlet-multinomial model* ID(M)M

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Example setup

- $\boldsymbol{x} = (\mathrm{R}, \mathrm{G}, \mathrm{R}, \mathrm{W}, \mathrm{W})$
- $s = 2$

Inferences

- $\underline{P}_2(\{B, W\}|\boldsymbol{n}) = ?$
- $\overline{P}_2(\{B, W\}|\boldsymbol{n}) = ?$

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- $s = 2$

Inferences

- $\underline{P}_2(\{B, W\}|\boldsymbol{n}) = 2/7$
- $\overline{P}_2(\{B, W\}|\boldsymbol{n}) = 4/7$