PF-SDM for Shape and Time Series Analysis

Topic Background

Understanding and quantifying shape dynamics is a central challenge in biomedical imaging, as morphological features often encode critical information about development, function, and disease. However, many existing shape descriptors either fail to jointly capture geometric and topological structure or lack the smoothness and interpretability required for reliable medical applications. This motivates the study of the Push-Forward Signed Distance Morphometric (PF-SDM), a framework designed to provide a compact, robust, and mathematically well-structured representation of closed shapes dynamics. By enabling access to differential-geometric quantities and supporting the fusion of spatial intensity information with temporal shape dynamics, PF-SDM opens new possibilities for principled, interpretable analysis of complex biological shapes.

1 Description of the Project I

Let $S \subseteq \Omega_s \subseteq (-1,1)^d$ be a given shape modeled as a manifold of co-dimension one, and a reference shape $S_r \subseteq \Omega_r \subseteq (-1,1)^d$. Moreover, let $\Psi_S : \Omega_S \to \Omega_r$ be a C^1-diffeomorphism between the shape domain Ω_S and the reference one Ω_r.

The PF-SDM framework, consists on computing the *Signed Distance Function (SDF)* $\phi_S : \Omega_S \to \mathbb{R}$ of the shape S, and deform it to the reference domain S_r via its push-forward $(\phi_S)_{\#\Psi_S} : \Omega_r \to \mathbb{R}_+$. The main challenge for this approach is computing a deformation map Ψ_S for non-convex shapes, that preserves the geometrical structure of the shape domain Ω_S.

1.1 Potential Tasks

1. Given a shape deformation map $\psi_S : S \to S_r$, can we find a conformal extension $\Psi_S : \Omega_S \to \Omega_r$ for non-convex shapes?

2. Compute the (linear or non-linear) elastic extension Ψ_S minimizing a functional of the form

$$\mathcal{L}[u] := \int_\Omega W(\nabla u)\, dx - \int_\Omega \log(\det \nabla u)\, dx + \int_{\partial\Omega} \|\psi_S - \psi\|^2\, ds. \tag{1}$$

3. Analyze the geometrical properties of the elastic extension Ψ_S.

4. Analyze the convergence rates for a given finite-dimensional approximation $\hat{u}_\theta \in \mathcal{S}_m(\Omega_S)$.

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Type:
M.Sc. Mathematics

Research area:
Shape Analysis/ Numerical Analysis

Programming language:
Python

Required skills:
Numerical Analysis

5. The SDF is obtained by solving the Eikonal equation

$$\begin{cases} |\nabla \phi|^2 = 1, & \text{in } \Omega_S, \\ \phi|_S = 0, & \text{on } S. \end{cases} \tag{2}$$

Can we formalize a modified PDE that gives us directly the PF-SDF $(\phi_S)_{\#\psi_S}$?

Description of the Project II

Assume that we have already a deformation map $\Psi_S : \Omega_S \to \Omega_r$. One way in which we can obtain a shape descriptor, is by computing the PF-SDF normalized Fourier coefficients $\mathbf{c}_S(r, t) \in \mathbb{R}^{N_f}$, at (r, t). This yields a shape invariant distance of the form

$$d_\phi(S_1, S_2) := ||\mathbf{c}_{S_1} - \mathbf{c}_{S_1}||_2. \tag{3}$$

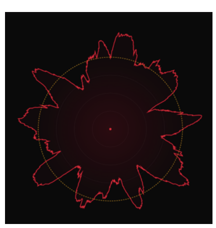
1. Given shapes with different medial axes, define a clustering algorithm that distinguishes between different classes.

2. Estimate the probability that such algorithm detects the different medial axes topologies, using the normalized coefficients $\mathbf{c}_S$ as shape descriptors.

3. We can generalize the notion of the PF-SDM for analyzing time series. Concretely, given a time series X_t described by

$$X_{t+1} = F_i(X_t) + \eta_{t+1}, \tag{4}$$

where η_{t+1} is some sampling noise, and $i \in \{1, 2\}$. Here F_1 and F_2 are two different models that define different classes. We can understand it as a closed shape as show in Fig. 3. The clustering task consists in given a trajectory $(X_t)_{t\in[T]}$,



build a classifier that decides its class (1 or 2). Analyze the PF-SDM framework applied to clusteing time series.

4. What geometrical information can we extract from time-series?

5. Compare with other time series clustering algorithms.