Graduate Lecture
Dark Energy - Observational Evidence and Theoretical Modeling
Lectures I+II

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Contents

Chapter 1

Introduction

$\hbar = c = k = 1$.

1.1 Bibliography

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Chapter 2

Observational Evidence for Accelerated Expansion from Supernovae

2.1 The Luminosity Distance

2.1.1 The Robertson-Walker Metric and Friedman Equation

We start with the Robertson-Walker Metric for a homogenous and isotropic Universe

$$\boxed{ds^2 = dt^2 - [a(t)]^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right] ,} \qquad (2.1)$$

which is given in spherical coordinates and with the curvature scale K in units of inverse length square. The scale factor $a(t)$ is dimensionless.

The Friedman equations are given by

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} , \qquad (2.2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} , \qquad (2.3)$$

where G is Newton's gravitational constant, Λ the cosmological constant and ρ the sum of the energy densities of all the constituents of the Universe. We can combine the two Friedmann equations to obtain

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} , \qquad (2.4)$$

which if we multiply this by a^3 and note that the volume $V \propto a^3$ is the equation for the conservation of energy with

$$dE + pdV = 0\,,$$

where we recognise that the pressure does work in the expansion.

2.1.2 The Critical Density

We will first introduce some simplifying notations, where their meaning will become clear during the course of this section. First we introduce the Hubble parameter

$$\boxed{H(t) \equiv \frac{\dot{a}}{a}\,, \qquad (2.5)}$$

which is the (normalized) expansion rate of the universe. Furthermore we can formally associate an energy density with the cosmological constant

$$\boxed{\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}\,. \qquad (2.6)}$$

In this notation the 1st Friedmann equation reads like

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\sum_i \rho_i + \rho_\Lambda\right)\,, \qquad (2.7)$$

where the index i is a label for the kind of particle fluid we study, like matter or radiation. Note that in general we have to sum over all the 'particle' species or energy components in the universe in order to obtain the total energy-momentum tensor. In order to obtain a flat universe we require $K = 0$ and hence

$$\rho_{\text{tot}} \equiv \sum_i \rho_i + \rho_\Lambda = \frac{3H^2}{8\pi G} \equiv \rho_{\text{crit}}\,.$$

We can define then

$$\boxed{\Omega_i \equiv \frac{\rho_i}{\rho_{\text{crit}}}\,, \qquad (2.8)}$$

which is the energy density in units of the critical density $\rho_{\rm crit}$. In this way we can define quantities like Ω_Λ, $\Omega_{\rm m}$ (for matter) and $\Omega_{\rm r}$ for radiation. Note that we define these quantities time dependent and not only at $t_{\rm today}$, if we want to specify the values today we will add an index 0, ie. $\Omega_{i,0}$[1]. With this notation the 1st Friedmann equation becomes

$$\frac{K}{a^2H^2} = \sum_i \Omega_i + \Omega_\Lambda - 1$$

and if we define $\Omega_k \equiv -K/(aH)^2$

$$\boxed{1 = \sum_i \Omega_i + \Omega_\Lambda + \Omega_k\,. \quad (2.9)}$$

Note that the sign of the definition of Ω_k varies in the literature.

In the following we will only discuss models with pressureless matter with $p = 0$. In general the flat cosmologies we discuss here, are called *Friedmann-Robertson-Walker or FRW models.* For pressureless matter we obtain with the energy conservation equation Eqn. 2.4

$$\rho_{\rm m} = \rho_{\rm m,0}\left(\frac{a}{a_0}\right)^{-3} ,$$

where $\rho_{\rm m,0}$ is the energy density in matter today and a_0 is the scale factor today. Note that we choose

$$\boxed{a_0 \equiv 1 \qquad (2.10)}$$

in the rest of the lecture, unless otherwise noted. The 1st Friedmann equation for a flat (K=0) universe can then be written as

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm m,0}a^{-3} + \frac{\Lambda}{3}\,. \qquad (2.11)$$

[1] Note that in most articles and books $\Omega_{\rm m}$ and Ω_Λ etc. refer actually to the densities today.

2.1.3 Redshift

In order to study the influence of the expansion of the universe on light emitted by a distant galaxy and received by an observer at the origin we exploit the fact that propagation of light in general relativity is along a null geodesic. If we put the observer at the origin with $r = 0$ and choose a radial null geodesic we get

$$ds^2 = d\theta = d\phi = 0$$

and hence from Eqn. 2.1

$$\frac{dt}{a(t)} = \pm \frac{dr}{(1 - Kr^2)^{1/2}}\,, \tag{2.12}$$

where the + sign corresponds to an emitted light ray and the − sign to a received one. For light ray emitted at time t_1 and a distance r_1 which is received at the origin at time t_0 we obtain

$$\begin{aligned} \int\limits_{t_1}^{t_0} \frac{dt}{a(t)} &= -\int\limits_{r_1}^{0} \frac{dr}{(1-Kr^2)^{1/2}} = \frac{1}{|K|^{1/2}} \int\limits_{0}^{|K|^{1/2} r_1} \frac{dr^*}{(1-kr^{*2})^{1/2}} \\ &= \frac{1}{|K|^{1/2}} S_k^{-1}\left(|K|^{1/2} r_1\right)\,, \end{aligned} \tag{2.13}$$

with

$$S_k(x) = \begin{cases} \sin(x) & \text{if } K > 0 \quad \text{or} \quad \Omega_\mathrm{k} < 0\,, \\ x & \text{if } K = 0 \quad \text{or} \quad \Omega_\mathrm{k} = 0\,, \\ \sinh(x) & \text{if } K < 0 \quad \text{or} \quad \Omega_\mathrm{k} > 0\,, \end{cases}$$

where we have used for the second equation the substitution $r^* = |K|^{1/2} r$ with $K = k|K|$. Now in order to understand how the frequency ν_0 (wavelength) of the received light behaves in relation to the emitted frequency ν_1, we consider two succesive wavefronts. The time when a second wavefront arrives $t_0 + dt_0$ which has been emitted after a short time dt_1 is again given by

$$\int_{t_1+dt_1}^{t_0+dt_0} \frac{dt}{a(t)} = \frac{1}{|K|^{1/2}} S_k^{-1}(|K|^{1/2} r_1)\,,$$

where the right hand side does not change because of Weyl's postulate that the 'substratum' (galaxies) have constant coordinates. So we finally find the relation between the time difference of the two signals

$$\frac{dt_0}{a(t_0)} = \frac{dt_1}{a(t_1)}$$

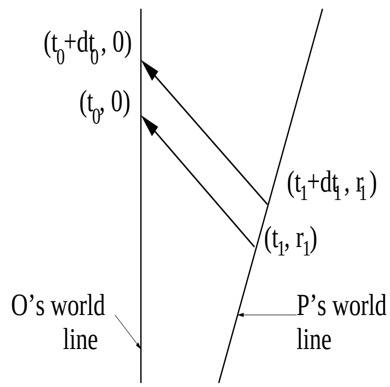

Figure 2.1: Propagation of light rays.

and hence the relation of the emitted (ν_1) and received (ν_0) frequencies is given by

$$\frac{\nu_0}{\nu_1} = \frac{dt_1}{dt_0} = \frac{a\,(t_1)}{a\,(t_0)}\,,$$

which is usually expressed by the redshift parameter

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a\,(t_0)}{a\,(t_1)} - 1\,, \tag{2.14}$$

where λ_1 and λ_0 are the wavelength corresponding to ν_1 and ν_0. Light from a distant object is usually redshifted[2]. Note that if we put the observer at t_0 today and use $a_0 = 1$ we obtain

$$a = \frac{1}{1+z} \tag{2.15}$$

2.1.4 Proper and Angular Diameter Distance

In general there is a world time and one can define the absolute distance between 'substratum' particles by looking at their position at the *same* world time. If we set $dt = d\theta = d\phi = 0$ in Eqn. 2.1 and assume one particle is at the origin and the other at r_1 we obtain as the *proper* distance

$$d_p = a(t) \int_0^{r_1} \frac{dr}{(1 - Kr^2)^{1/2}}\,,$$

[2]Note that in a collapsing universe it is actually blueshifted.

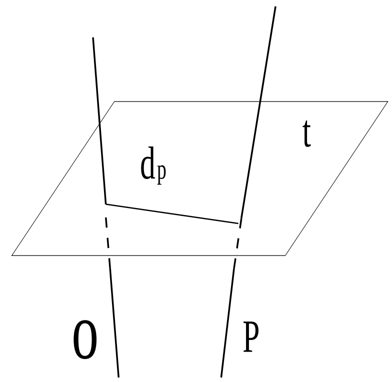

Figure 2.2: Distance between two fluid particles.

however this requires a *synchronous* measurement of the distance which is of no practical use. One more practical method would be to compare the known absolute luminosity of an object with its observed apparent luminosity or the true diameter with the observed angular diameter.

In this section we consider the second method, while in the next section we will concentrate on the luminosity measurements. We calculate in the following the angular diameter observed at the origin at $t = t_0$ of a light source of of true proper diameter D at $r = r_1$ and $t = t_1$. We choose the

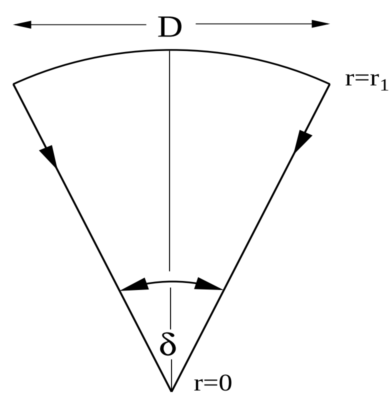

Figure 2.3: Angular diameter distance.

coordinate system like in Fig. 2.3. The light travels then on a cone with a half angle $\theta = \delta/2$. The proper diameter of the source is then given by Eqn. 2.1

$$D = a(t_1) r_1 \delta \qquad \text{for}\, \delta \ll 1\,,$$

so we obtain for the angular diameter of the source

$$\delta = \frac{D}{a(t_1)r_1}\,.$$

In Euclidean geometry the angular diameter of a source of diameter D at a distance d is $\delta = D/d$, so we define in *general* the angular diameter distance

$$d_{\mathrm{A}} \equiv \frac{D}{\delta}, \qquad (2.16)$$

and hence we can write

$$d_{\mathrm{A}} = a(t_1)r_1 = \frac{r_1}{1+z}\,.$$

Since we are studying the propagation of light r_1 is given by Eqns. 2.12-2.13 and we obtain

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^z \frac{dz}{H(z)} = \frac{1}{|K|^{1/2}} S_k^{-1}(|K|^{1/2} r_1)\,,$$

where the first equation was obtained by substituting the time integration with a redshift integration and using

$$\frac{dz}{dt} = -\frac{\dot{a}}{a^2} = -\frac{H}{a}\,.$$

and we finally obtain with $|K|^{1/2} = H_0\sqrt{\Omega_{\mathrm{k},0}}$

$$d_A(z) = \frac{1}{\sqrt{|\Omega_k|}H_0(1+z)} S_k\left(H_0\sqrt{|\Omega_k|}\int_0^z \frac{dz}{H(z)}\right)\,. \qquad (2.17)$$

From Eqn. 2.7 we see that the angular diameter distance depends via the Hubble parameter on the cosmological parameters like H_0, $\Omega_{\Lambda,0}$ and $\Omega_{\mathrm{m},0}$. If on could observe the angular diameter distance really accurately one could measure these parameters and also the curvature or general geometry of the universe. An excellent probe in this way in the anisotropies in cosmic microwave background radiation. One can calculate a typical size of an overdense region at the time the microwave photons start to stream free and we also know the the distance to this last scattering surface. We can compare this with the observed angular size (in form of the anisotropy power spectra) and hence obtain a very accurate measurement of the curvature of the universe.

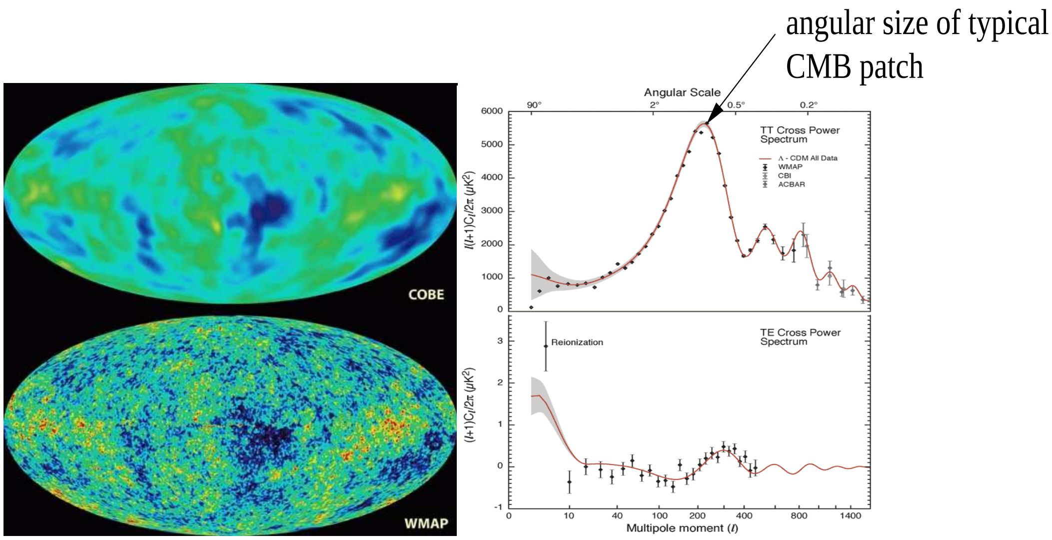

Figure 2.4: Angular anisotropy power spectrum of the cosmic microwave background as observed by the WMAP team (2003).

2.1.5 Luminosity Distance and Deceleration Parameter

As mentioned before another way to measure distance is via comparing the known absolute luminosity of an object with the the observed apparent luminosity. For a telescope mirror with radius b as shown in Fig. 2.5 the

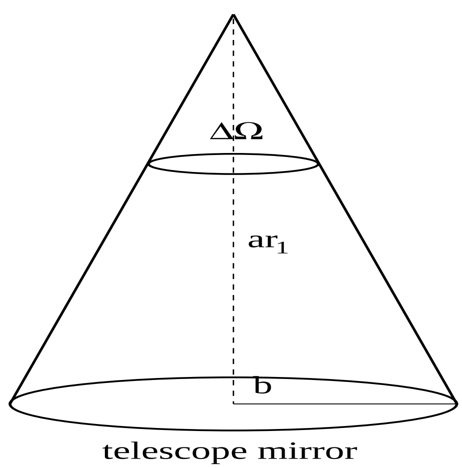

Figure 2.5: The luminosity distance.

solid angle is given by

$$\Delta\Omega = \frac{\pi b^2}{a^2(t_0) r_1^2}$$

and the fraction of isotropically emitted photons that reach telescope is given by ratio of solid angle $\Delta\Omega$ to total solid angle 4π

$$\frac{\Delta\Omega}{4\pi} = \frac{\pi b^2}{4\pi a^2(t_0) r_1^2}$$

If the source has an absolute (or bolometric[3]) luminosity $\mathcal{L}$, which is the total power emitted by the source (in a specified band), the question is what is the received power ? Let us look at a single photon. Photons which are emitted with energy $h\nu_1$ are redshifted to $h\nu_1 a(t_1)/a(t_0) = h\nu_0$. Furthermore photons emitted at intervals δt_1 are received at intervals $\delta t_0 = \delta t_1 a(t_0)/a(t_1)$. So for a single photon we get

$$\begin{aligned}
\text{emitted power}: \quad P_{\rm em} &= \tfrac{h\nu_1}{\delta t_1} \\
\text{received power}: \quad P_{\rm rec} &= \tfrac{h\nu_0}{\delta t_0} \\
&= \tfrac{h\nu_1}{\delta t_1} \tfrac{a^2(t_1)}{a^2(t_0)} \ ,
\end{aligned}$$

hence for the total received power P, we get

$$P = \mathcal{L} \left(\frac{a^2(t_1)}{a^2(t_0)} \right) \frac{A}{4\pi a^2(t_0) r_1^2} \ ,$$

where we have used $A = \pi b^2$ for the total mirror area. Now the total apparent luminosity or bolometric flux density is given by

$$\mathcal{F} \equiv \frac{P}{A} = \frac{\mathcal{L} a^2(t_1)}{4\pi r_1^2} \ , \qquad (2.18)$$

where we applied $a(t_0) = 1$. In Euclidean space the flux density is given by $\mathcal{F} = \mathcal{L}/(4\pi d^2)$ and this is now generalized to define the *luminosity distance*

[3]The term bolometric is usually applied when the luminosity is calculated over an entire bandwidth $\Delta\nu$.

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_{\rm L}^2} \, . \tag{2.19}$$

Therefore we obtain

$$d_{\rm L} = \frac{r_1}{a} = (1+z) r_1 = (1+z)^2 d_{\rm A} \, ,$$

so we finally obtain

$$d_{\rm L}(z) = \frac{1+z}{\sqrt{|\Omega_k|} H_0} S_k \left(H_0 \sqrt{|\Omega_k|} \int_0^z \frac{dz}{H(z)} \right) \, . \tag{2.20}$$

It is interesting to note that for low redshifts $z \ll 1$ and small r_1 we have

$$d_A \simeq d_{\rm L} \simeq d_P \simeq r_1$$

and the distinction becomes important only for objects billions of light years away. Therefore we draw our attention to the redshift dependence of the scale factor at late times (or small redshifts). We can Taylor expand the scale factor around $t = t_0$ and obtain

$$a(t) = a(t_0) \left[1 + H_0 \, (t - t_0) - \frac{1}{2} q_0 H_0^2 (t_0 - t)^2 + \cdots \right] \, , \tag{2.21}$$

where we used the definition of the Hubble constant $H_0 = \dot{a}(t_0)/a(t_0)$ and we defined the *deceleration parameter*

$$q_0 = - \frac{\ddot{a}(t_0)}{a(t_0) H_0^2} \, . \tag{2.22}$$

As the name already suggests the deceleration parameter quantifies if the expansion of the universe is accelerating ($q_0 < 0$) or decelerating ($q_0 > 0$). It is quite convenient to express the cosmological models in terms of q_0 and H_0 but we leave this as an Exercise !.

If we use this expansion in Eqn. 2.12 for the propagation of light we obtain on for left hand side

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \frac{1}{a(t_0)} \int_{t_1}^{t_0} \left[1 + H_0 (t_0 - t) + \left(1 + \frac{q_0}{2} \right) H_0^2 (t_0 - t)^2 + \cdots \right]$$

and for the right hand side

$$\int_0^{r_1} \frac{dr}{(1-Kr^2)^{1/2}} \approx \frac{1}{|K|^{1/2}} \int_0^{|K|^{1/2} r_1} \left(1 + \frac{1}{2} k r^{*2}\right) dr^* = r_1 + \mathcal{O}(r_1^3)$$

and we obtain

$$r_1 = \frac{1}{a(t_0)} \left[t_0 - t_1 + \frac{1}{2} H_0 \left(t_0 - t_1\right)^2 + \cdots \right] .$$

Furthermore we obtain for the redshift

$$z = \frac{1}{a} - 1 = H_0 (t_0 - t_1) + \left(1 + \frac{q_0}{2}\right) H_0^2 \left(t_0 - t_1\right)^2 + \cdots$$

and hence

$$r_1 = \frac{1}{a(t_0) H_0} \left[z - \frac{1}{2} \left(1 + q_0\right) z^2 + \cdots \right] .$$

Finally we can write the expansion of the luminosity distance for low redshifts

$$d_{\mathrm{L}} = H_0^{-1} \left[z + \frac{1}{2} \left(1 - q_0\right) z^2 + \cdots \right] . \tag{2.23}$$

This expansion will play a vital rôle for the calibration of the magnitude - redshift relation for Supernovae as we will discuss it in Section 2.2.

2.2 Distance vs. Redshift with Type Ia Supernovae

We will now study an application of what we have learned so far. The analysis of the distance - redshift relation with Type Ia Supernovae and the what we can learn about the cosmological parameters H_0, $\Omega_{\Lambda,0}$, $\Omega_{\mathrm{m},0}$ and $\Omega_{\mathrm{k},0}$.

However in order to do this we need to introduce the notion of magnitudes.

2.2.1 Cosmological Magnitudes

When we discussed the luminosity distance in Section 2.1.5 we introduced the notion of of bolometric flux, which is related to the bolometric brightness. The brightness in general is the intensity of a radiating source, ie. the energy flux per solid angle and per unit frequency. The bolometric brightness again is integrated over a frequency wave band. Now the definition of magnitudes is an ancient concept. Hipparchus (150 BC) divided stars into six classes of brightness he called magnitudes. The brightest stars were called first magnitude and the faintest sixth. With quantitative measurements it was found that each jump in magnitude corresponded to a fixed *ratio* in flux, hence the magnitude scale is logarithmic. This is not too surprising since the eye has an approximately logarithmic response to light, which enables a large dynamic range. It was found that a difference of five magnitudes corresponds to a factor 100 in brightness and we have

$$\frac{b}{B} = 100^{(M-m)/5} = 10^{(m_2-m_1)/2.5} \,.$$

Instead of using the brightness ratio we could have also used the ratio of the received flux. We can now build up the magnitude ladder with a standard candle. A standard candle is an object which has always the same emitted luminosity $\mathcal{L}$. We obtain then with Eqn. 2.18

$$M - m = 2.5 \log \frac{d_{\mathrm{L},0}^2}{d_{\mathrm{L}}^2} = 5 \log \frac{d_{\mathrm{L},0}}{d_{\mathrm{L}}} \,,$$

where M is the intrinsic magnitude of the standard candle at some close by distance $d_{\mathrm{L},0}$. In astronomical situations this distance is usually chosen to $10\,\mathrm{pc}$[4]. So usually one obtains

$$m = M + 5 \log d_{\mathrm{L}} \,.$$

where d_{L} is given in units of $10\,\mathrm{pc}$. However in cosmological situation this is a rather small distance and a more natural unit is $1\,\mathrm{Mpc}$. If we measure the distance in this unit the apparent magnitude is given by

$$\boxed{m = M + 5 \log d_{\mathrm{L}} + 25 \,. \qquad (2.24)}$$

If we use the approximation for $z \ll 1$ for the luminosity distance in Eqn. 2.23 we obtain

$$m = M - 5 \log H_0 + 5 \log cz + \cdots + 25 \,. \qquad (2.25)$$

Note that we explicitly write the speed of light c in this equation. This approximation only depends on the Hubble constant H_0 but not on other cosmological parameters. So nearby objects can be used to calibrate for the intrinsic magnitude M.

[4]The unit 1 pc is defined to be the distance of an object which produces one arcsec of a parallax angle for one astronomical unit (AU), which is the distance from the sun to the earth. $1\ \mathrm{pc} = 3.09 \times 10^{16}\ \mathrm{m}$.

2.2.2 Type Ia Supernovae as Standardizable Candles – Phillips Relation

In order to study the magnitude-redshift relation to very large distances, one needs a very bright standard candle. Type Ia Supernovae explosions are a good candidate for such a standard candle. Since Supernovae are almost as bright as their host galaxies they can be observed to large distances. An example how bright these objects are can be seen in Fig. 2.6. Observationally

Figure 2.6: Type Ia Supernovae 1998aq in NGC3982 (picture taken by H. Dahle). This is a spiral galaxy in Ursa Major of visual brightness 11.8 mag. The Supernovae itself was estimated to reach 11.4 mag. The galaxy is at a distance of ≈ 20.5 Mpc (Stetson & Gibson 2001).

Type I Supernovae are distinct from Type II that they have *no* hydrogen lines in their maximum light spectrum. Additionally Type Ia show a strong Si absorption feature at 6150 Å.

Type Ia Supernovae are probably the product of mass being accreted to a white dwarf in a close binary system. A white dwarf is a an approximately earth size star which is only supported by its electron degeneracy pressure (Pauli principle). Chandrasekhar showed that there is an upper mass limit which can be supported by electron degenarcy pressure which is called the Chandrasekhar mass which is

$$M_{\mathrm{Ch}} = 1.44 M_{\odot}\,.$$

Sometimes there is too much mass accreted onto the white dwarf and its starts to exceed the Chandrasekhar mass limit. In this case the degenerate electron pressure can no longer support the star and it collapses. The collapse energy drives nuclear reaction which build up ^{56}Ni which β-decays into ^{56}Co which in turn β-decays into ^{56}Fe.

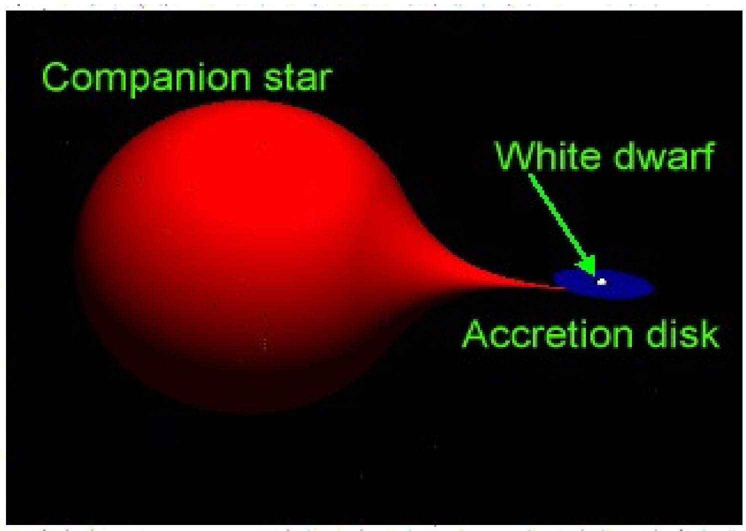

Figure 2.7: Model of close binary system which might be the progenitor to a Type Ia Supernovae explosion [Picture take from Paul Rickers web page].

These thermonuclear explosions lead to typical typical brightening and fading of the Supernovae, which in case of the Type Ia is governed by a two exponential whose timescale is governed by the two β-decays. Note the the β-decay of ^{56}Ni has a halftime of $\tau_{\mathrm{Ni}} = 17.6$ days. In Fig. 2.8 we see a typical SNe observation, where the discovery was made from the ground and the follow up with the Hubble Space Telescope. The brightening and fading gives rise to a typical lightcurve for Type Ia Supernovae as shown in Fig. 2.9. One problem with Type Ia SNe is however that, although they have a narrow range of absolute peak magnitudes M, there is a slight variation.

However Phillips (1993) discovered that there is a tight relation between the peak magnitude and the decay time. This relation is not well understood yet from a theoretical point of view but basically the time scale and the overall energy of the Supernovae explosion depend both on the amount of Ni which is present in the progenitor. With the Phillips relation it is possible to normalize the peak flux and also "stretch" the time axis so that all Type Ia

Figure 2.8: The brightening and fading of SNe 1998ay.

SNe fit a universal lightcurve as shown in Figure 2.10. Hence if we know the "intrinsic" , normalized magnitude of a Type Ia Supernovae and its decay time (sometimes measured as the magnitude after 15 days) we can work out the intrinsic magnitude of this particular SNe. With spectral information of the host galaxy we can work out the redshift of the SNe and hence draw an apparent magnitude - redshift diagram.

If we have a sample of low redshift Type Ia SNe we can use Eqn. 2.25, measure the apparent magnitude and redshift and hence work out

$$\mathcal{M} \equiv m - \log cz = M - 5 \log H_0 + 25\,, \tag{2.26}$$

which is a measure of the absolute magnitude. If we know this for all SNe we can write

$$m = \mathcal{M} + 5 \log \mathcal{D}_{\rm L}\,, \tag{2.27}$$

with $\mathcal{D}_{\rm L} = H_0 d_{\rm L}$ the Hubble constant free luminosity distance. In Fig. 2.11 we show the measured magnitude redshift relation and some theoretical predications. We see that a flat matter dominated universe (short dashed line) is systematically under-predicting the magnitudes and hence is not a good fit. However the presence of a cosmological constant improves the fit considerably.

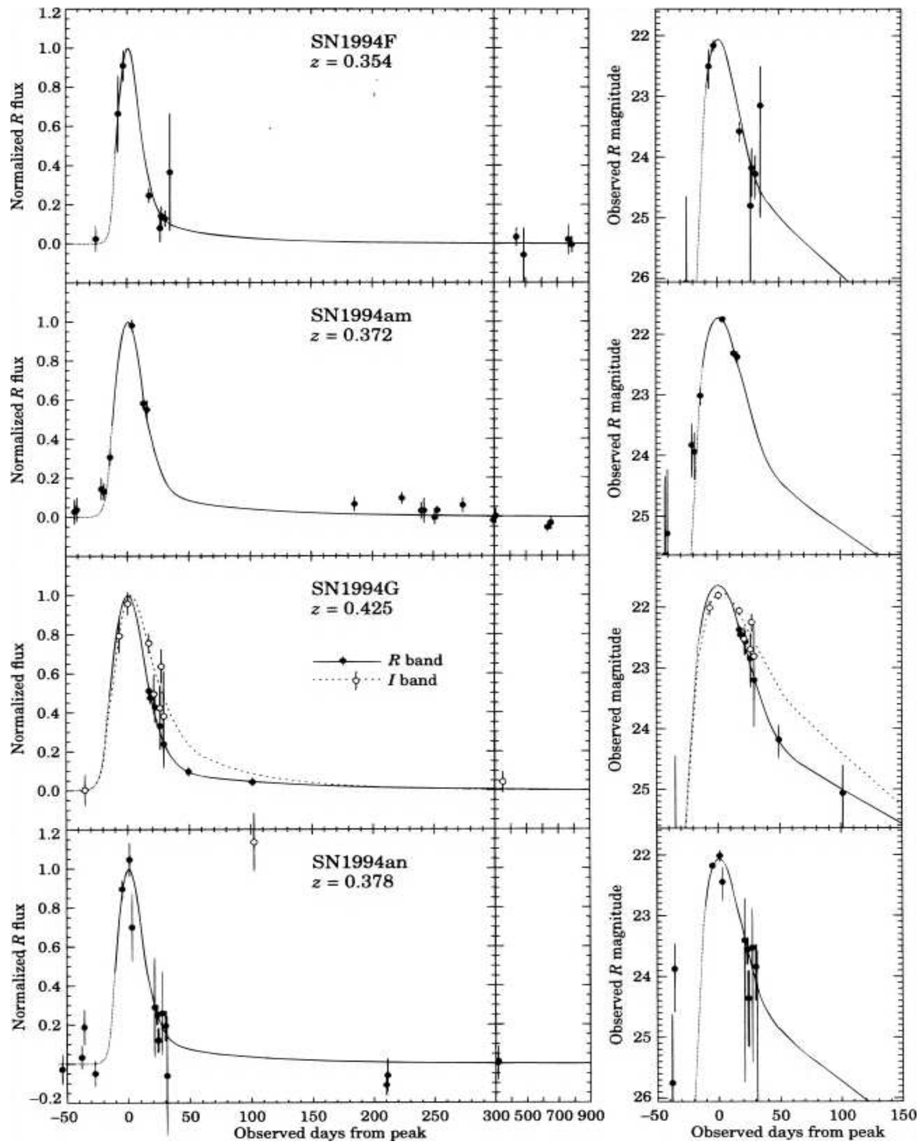

Figure 2.9: Various lightcurves of Type Ia SNe as discussed by Perlmutter et al. (1997).

2.3 Parameter Estimation

In order to quantify which cosmological model fits the data the best we have to address a parameter estimation problem. The topics discussed in

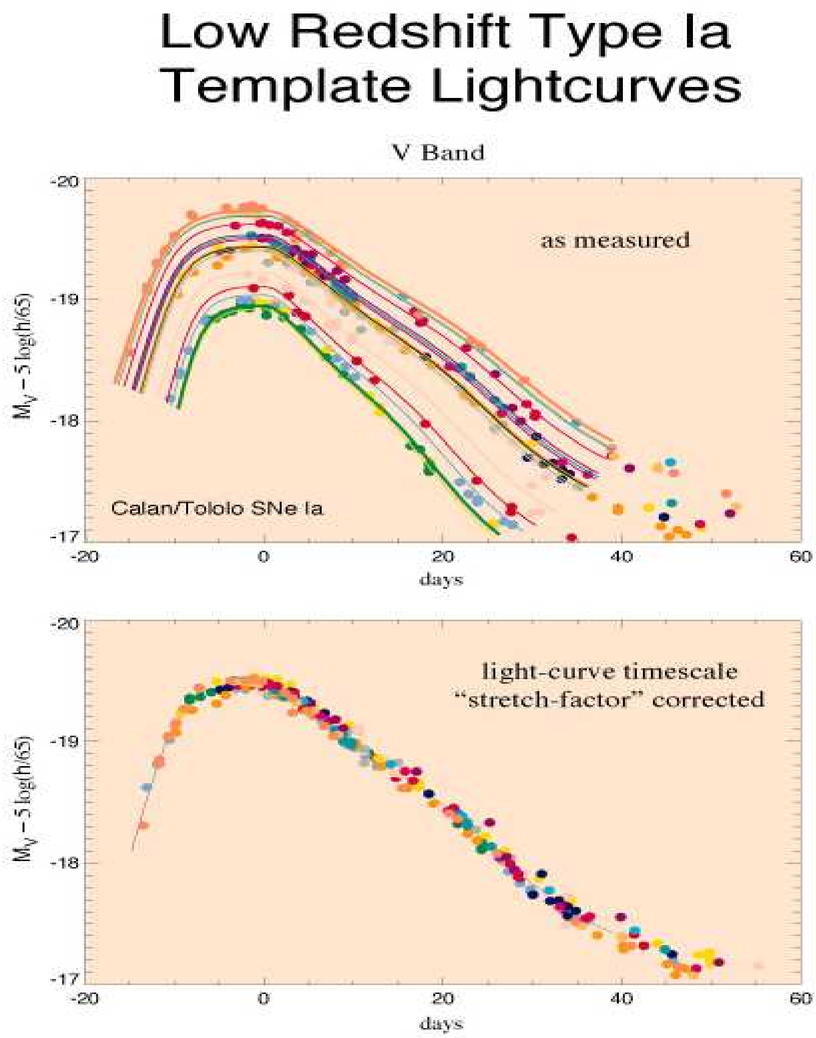

Figure 2.10: Stretch factor corrected lightcurves from SCP.

this Section apply in general for the estimation of parameters and are hence a valuable tool for every physicist who has to deal with data.

Let us assume that we have a sample of Type Ia SNe with a given magnitude m_i and uncertainty in the magnitude $\sigma_{m,i}$, which is typically of the order $\sigma_m = 0.15$ mag. Furthermore we know the redshift z_i of the Su-

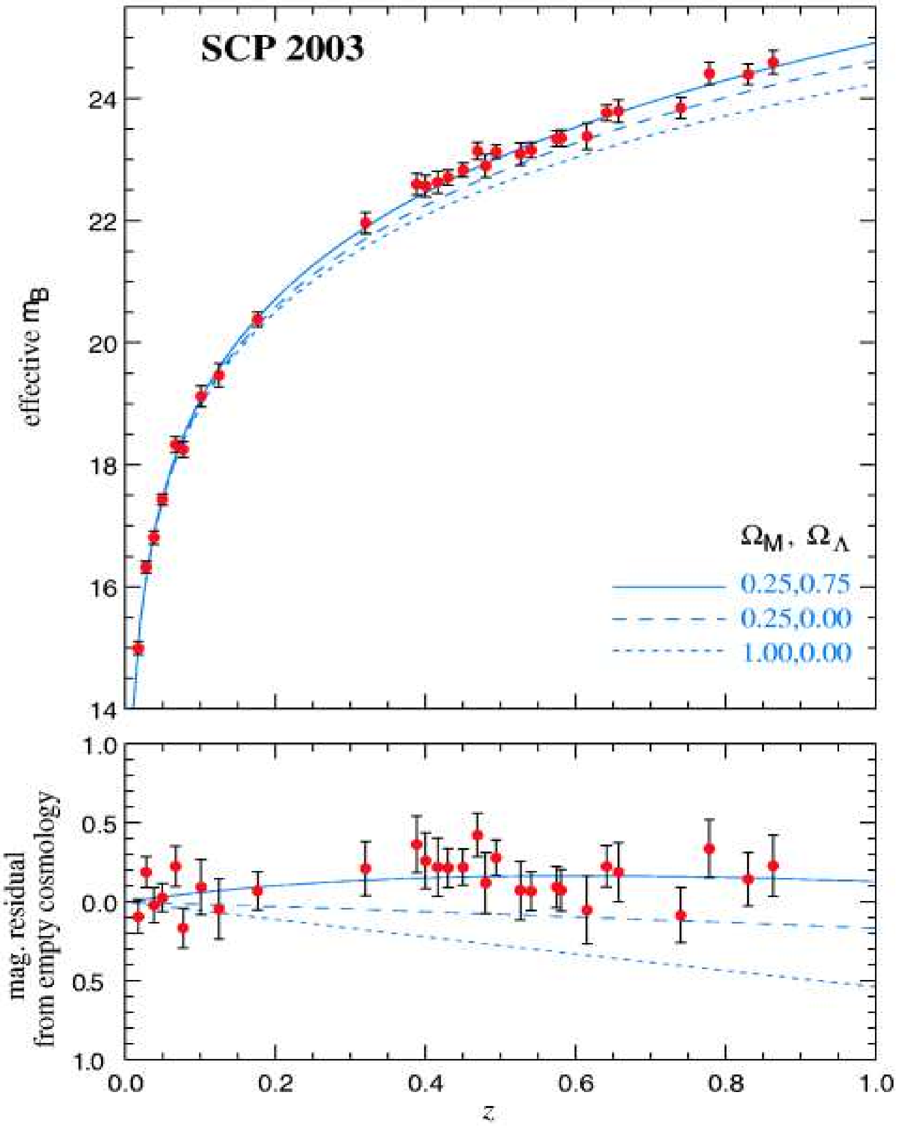

Figure 2.11: Magnitude - Redshift diagram from Knop et al. (2003). The data points are from the Supernovae Cosmology project at high redshifts and from the Calan/Tolo survey at low redshifts. The lower panel shows the relative magnitudes to an empty (Milne) universe with $\Omega_{\mathrm{k},0} = 1$ and $\Omega_{\Lambda,0} = \Omega_{\mathrm{m},0} = 0$.

pernovae. In general this redshift has an errorbar as well, but it can be neglected in comparison to the magnitude uncertainty. We can than compare the measurement with the theoretical prediction of Eqn. 2.27 for each set of parameters $(\Omega_{\rm m,0}, \Omega_{\Lambda,0}, \mathcal{M})$. There are two ways to tackle the absolute magnitude $\mathcal{M}$. We could first just look at the low redshift SNe sample from Calan/Tololo and use Eqn. 2.26 to measure the absolute magnitude. Note that this equation does *not* depend on the cosmological parameters. Secondly we could view $\mathcal{M}$ as a free parameter like the cosmological parameters $(\Omega_{\rm m,0}, \Omega_{\Lambda,0})$ and try to find the best fit value for it.

We will follow the second approach here. In order to get a compact notation we define the parameter vector

$$\theta \equiv (\Omega_{\rm m,0}, \Omega_{\Lambda,0}, \mathcal{M})\,.$$

If we assume that the errors in the magnitude follow a Gaussian distribution we can obtain the best fit parameters by maximising the posterior probability (likelihood)

$$L(\theta) \propto \exp\left[-\frac{1}{2}\chi^2\right]$$

with

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{m(z_i;\theta) - m_i}{\sigma_{m,i}}\right)^2 ,$$

where N is the number of data points. One can then numerically minimize Eqn. 2.3 and obtain the best fit values $\hat{\theta}$. As a matter of fact by calculating $L(\theta)$ over the entire sensible parameter range we obtain the posterior distribution.

Since from a cosmological point of view we are not interested in the absolute magnitude $\mathcal{M}$ we can marginalize over it and obtain the 2-dimensional probability distribution

$$\tilde{L}(\Omega_{\rm m,0}, \Omega_{\Lambda,0}) = \int d\mathcal{M}\; L(\Omega_{\rm m,0}, \Omega_{\Lambda,0}, \mathcal{M})\,.$$

In fact this can be even done analytically because $\mathcal{M}$ is just a linearly added parameter. If we define

$$\begin{aligned} c_1 &\equiv \sum_{i=1}^{N} \frac{1}{\sigma_{m,i}^2} \\ f_0 &\equiv \sum_{i=1}^{N} \frac{5\log \mathcal{D}_L(z_i) - m_i}{\sigma_{m,i}^2} \end{aligned}$$

$$f_1 \equiv \sum_{i=1}^{N} \frac{(5 \log \mathcal{D}_L(z_i) - m_i)^2}{\sigma_{m,i}^2} ,$$

we obtain

$$\tilde{\chi}^2 = f_1 - \frac{f_0^2}{c_1} \qquad (2.28)$$

2.3.1 Sampling the Likelihood by a Grid Based Method

We start with a Fortran 90 example of how to calculate the χ^2 values.

```
MODULE STATISTICS

CONTAINS

!  calculate standard chi2 function
FUNCTION CHI2(omegami,omegali,Minti)
  USE COSMOLOGY
  USE SNDATA
  REAL, INTENT(IN) ::  omegami,omegali,Minti
  INTEGER ::  I
  REAL ::  sum
  REAL ::  CHI2

  omegam=omegami
  omegal=omegali
  Mint=Minti
  omegak=1.0-omegam-omegal
  IF (NOB(omegam,omegal)) THEN
    sum = 0.0
    DO I=1,N
      sum=sum+(m(i)-mag(z(i)))**2/dm(i)**2
    END DO
    CHI2 = sum
  ELSE
    CHI2 = 1.0E30 !  assign zero likelihood if nobtest fails
  END IF
  RETURN
END FUNCTION CHI2

!  calculate chi2 function; with analytic marginalization over Mint
```

```
FUNCTION CHI2ANA(omegami,omegali)
  USE COSMOLOGY
  USE SNDATA
  REAL, INTENT(IN) ::  omegami,omegali
  INTEGER ::  I
  REAL ::  c1,f0,f1
  REAL ::  CHI2ANA

  omegam=omegami
  omegal=omegali
  Mint=0.0 !  note we set this zero in order to calc 5.0*log(DL)
  omegak=1.0-omegam-omegal
  IF (NOB(omegam,omegal)) THEN
    c1=0.0
    f0=0.0
    f1=0.0
    DO I=1,N
      c1=c1+1.0/dm(i)**2
      f0=f0+(mag(z(i))-m(i))/dm(i)**2
      f1=f1+(mag(z(i))-m(i))**2/dm(i)**2
      END DO
    CHI2ANA = f1-f0*f0/c1
  ELSE
    CHI2ANA = 1.0E30 !  assign zero likelihood if nobtest fails
  END IF
  RETURN
END FUNCTION CHI2ANA
END MODULE STATISTICS
```

The likelihood is simply calculated by looping over the parameters and calculating the χ^2 values for each grid point. The function `CHI2` calculates the χ^2 in the classical way, while `CHI2ANA` is using the analytical marginalization over the intrinsic magnitude $\mathcal{M}$. The function `mag(z)` calculates the theoretical magnitudes for a given model, the arrays `z(i)`, `m(i)` and `dm(i)` hold the data points. Also note the logical function `NOB`, which sorts out the models for which no big bang occurs which are given by the condition

$$\Omega_{\Lambda,0} \geq 4\Omega_{\mathrm{m},0} \left\{ \mathrm{coss} \left[\frac{1}{3} \mathrm{coss}^{-1} \left(\frac{1-\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}} \right) \right] \right\}^3 , \tag{2.29}$$

with "coss" being defined as cosh for $\Omega_{\mathrm{m},0} < 1/2$ and cos for $\Omega_{\mathrm{m},0} > 1/2$. If this condition is fullfilled, the argument in the square root of the definition of the Hubble parameter can become negativ for certain redshifts. We hence just assign a zero likelihood for these models. We can now discuss how we loop over the different parameters:

```
DO OMEGAM = 0.0,1.5,0.1
  DO OMEGAL = -1.0,2.0,0.1
    omegak = 1.0-omegam-omegal
    DO Mint = 15.5,16.5,0.01
      test=chi2(omegam,omegal,Mint)
      write(11,FMT='(3F10.2,F18.10)') Mint,OMEGAM,OMEGAL,exp(-0.5*(test-testmin))
!  FIND MINIMUM
      if (test<chi2min) then
        chi2min=test
        omegammin = omegam
        omegalmin = omegal
        Mintmin = Mint
      END IF
    END DO
  END DO
END DO
```

and for the analytically marginalized fit:

```
DO OMEGAM = 0.0,1.5,0.1
  DO OMEGAL = -1.0,2.0,0.1
    omegak = 1.0-omegam-omegal
    test=chi2ana(omegam,omegal)
    write(11,FMT='(2F10.2,F18.10)') OMEGAM,OMEGAL,exp(-0.5*(test-testmin))
!  FIND MINIMUM
    if (test<chi2min) then
      chi2min=test
      omegammin = omegam
      omegalmin = omegal
    END IF
  END DO
  write(11,*)
END DO
```

We choose for our analysis the 2004 Riess et al. compilation of various SNe observations. Note that we use the entire sample. The likelihoods at

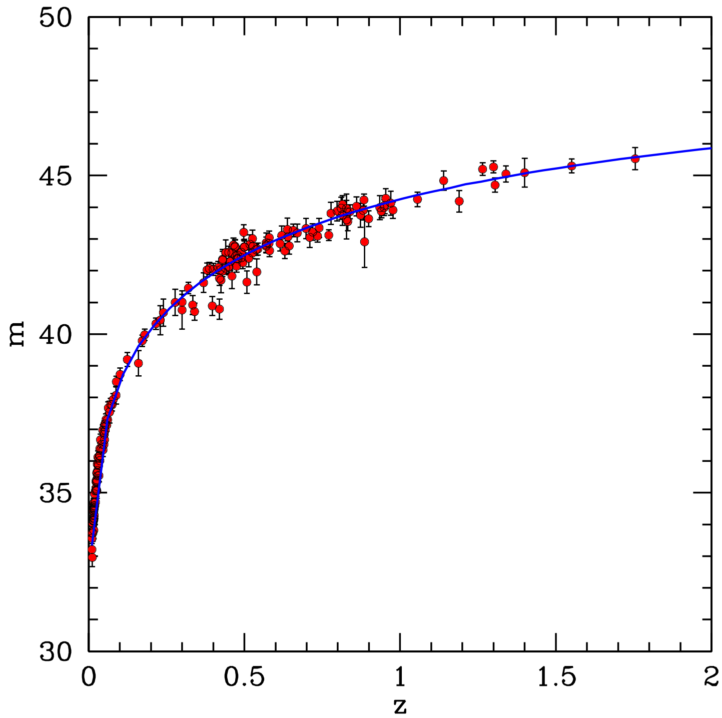

Figure 2.12: Riess et al. (2004) data compilation with the best fit model estimated here: $\Omega_{\mathrm{m},0} = 0.5$, $\Omega_{\Lambda,0} = 1.1$. Note that the normilaziton of the magnitude is arbitrary.

the gridpoints are shown in Figure 2.13 The same in Figure 2.14 but for the analytically marginalization over $\mathcal{M}$.

2.3.2 Joint Likelihoods

A simple program like this can lead to the plot shown in Figure 2.15. Fig. 2.15 shows the joint joint likelihood contour where different contours correspond to different likelihood levels. The best fit value is roughly at $\Omega_{\mathrm{m},0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, but from a statistical point of view models with in the 68% $(1 - \sigma)$ or even the 95% $(2 - \sigma)$ contour are still viable. However even on the 99% level the cosmological constant is positive and

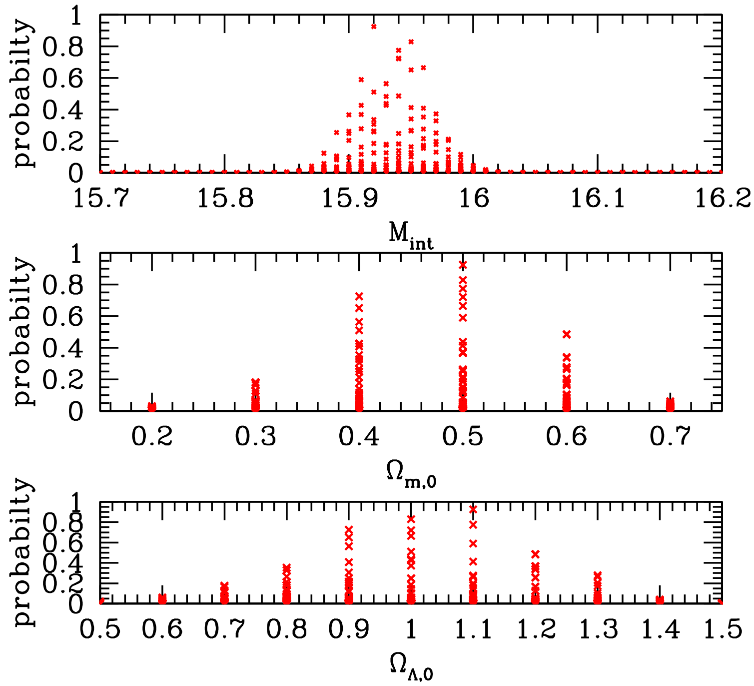

Figure 2.13: Likelihoods at the grid point for the 3 parameters. Note that $\mathcal{M} = M_{int}$ and that the peak probability is normalized to 1.

non-vanishing. In 1997 Supernovae Cosmology Project and the High-z Supernovae Search team (Perlmutter et al. and Riess et al.) reported similar results, which led to a renewed interest into the cosmological constant. Historically Einstein introduced the cosmological constant in order to balance the gravitational effects of matter and obtain a static universe. After Hubble's discovery that the universe is expanding Einstein abandoned the idea of a static universe and the cosmological constant.

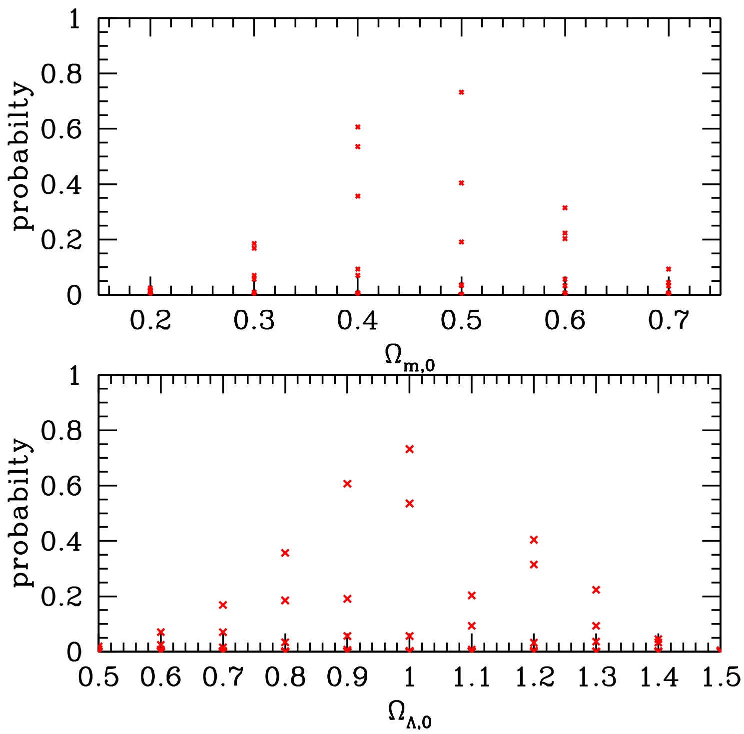

Figure 2.14: Likelihoods at the grid point for the 2 parameters. Again the peak probability is normalized to 1.

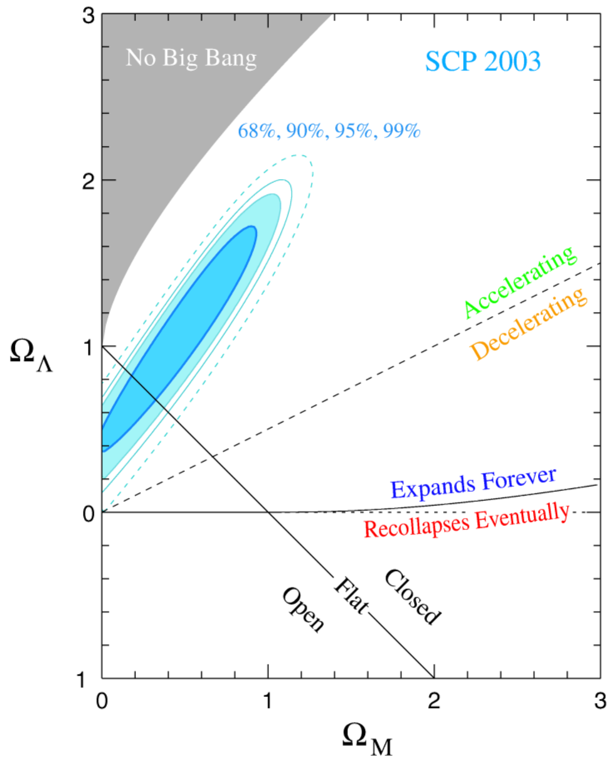

Figure 2.15: Joint likelihood contours in the $\Omega_{m,0} - \Omega_{\Lambda,0}$ plane. The plot is from the Knop et al. (2003) analysis.