Graduate Lecture
Dark Energy - Observational Evidence and Theoretical Modeling
Lectures IV

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Chapter 3

Dark Energy Models

In order to explain the Type Ia Supernovae data discussed in the previous section it is necessary that the expansion of the universe is accelerating. Hence the deceleration parameter q_0 has to be negative.

3.1 Generalized Equation of State

As it is obvious from the discussion in beginning of Section 2.1.2 the cosmological constant can be viewed as another fluid component in the universe, like matter or radiation. If we write the 2nd Friedman equation 2.3 in terms of ρ_Λ it takes the generic form

$$\frac{\ddot{a}}{a} = -\sum_i \frac{4\pi G}{3} \left(\rho_i + 3p_i\right) \, ,$$

where the summation runs over *all* fluid components we obtain for consistency reasons

$$p_\Lambda = -\rho_\Lambda \, ,$$

which means the pressure in a cosmological constant fluid is negative. If we use the conservation of energy for this fluid we obtain from Eqn. 2.4

$$\dot{\rho}_\Lambda = -3\left(\rho_\Lambda + p_\Lambda\right)\frac{\dot{a}}{a} = 0 \, ,$$

and hence as we see already from the definition of ρ_Λ in Eqn. 2.6 that $\rho_\Lambda = \text{const.}$. As a matter of fact we could have started with this and than showed with Eqn. 2.4 that the pressure has to be the negative of the energy density.

Now in general the behaviour of simple fluids (or gases) is governed by their *equation of state*

$$p = w\rho\,, \qquad (3.1)$$

with w the constant equation of state factor. Note that "ordinary" cold dark matter has an equation of state factor $w = 0$, since it is pressureless. Relativistic matter like radiation has a pressure $p = \rho/3$ and hence $w = 1/3$. While as argued before a cosmological constant has an equation of state of $w = -1$.

Let us now explore the question what type of fluid to get accelerated expansion of the universe if we drop the cosmological constant. From Eqn. 2.4 we obtain for a fluid with an equation of state factor w

$$\rho_{\rm de}(a) = \rho_{\rm de,0} a^{-3(1+w)}\,, \qquad (3.2)$$

with $a_0 = 1$ where we introduced the label "de" for *dark energy*. The phrase "dark energy" was coined to describe a component which does not gravitationally clump and has no large interactions with ordinary and cold dark matter. As before we can define the densities in units of the critical density $\rho_{\rm crit}$ and obtain the quantities $\Omega_{\rm de}$ and $\Omega_{\rm de,0}$.

If we assume we have a flat ($K = 0$) universe which has only the dark energy component it is straight forward to show

$$a(t) = \left[\frac{3(1+w)}{2} H_0 t\right]^{\frac{2}{3(1+w)}}$$

where this solution is only valid for $w \neq -1$[1]. From the 2nd Friedmann equation 2.3 we obtain in this case

$$\frac{\ddot{a}_0}{a_0} = -\frac{\Omega_{\rm de,0} H_0^2}{2}\left(1 + 3w\right)\,,$$

where $\Omega_{\rm de,0} = 1$ because we assumed $K = 0$. Therefore we obtain for the deceleration parameter from Eqn. 2.22

$$q_0 = -\frac{\ddot{a}_0}{a_0 H_0^2} = \frac{1+3w}{2}$$

[1]Note however that fluids with $w < -1$ are very unphysical since they lead to negative energy densities which are unstable.

and from the condition $q_0 < 0$ for acceleration we obtain $w < -1/3$. If we include a matter component this condition generalises in a flat universe to $w < -1/(3\Omega_{\rm de,0})$ Exercise !.

We can now as described in the previous chapter estimate the best fit values on w, $\Omega_{\rm m,0}$ and $\Omega_{\Lambda,0}$. However for this analysis it is usually assumed the universe is flat and $\Omega_{\Lambda,0} = 1 - \Omega_{\rm m,0}$ is not a free parameter. In Fig. 3.1

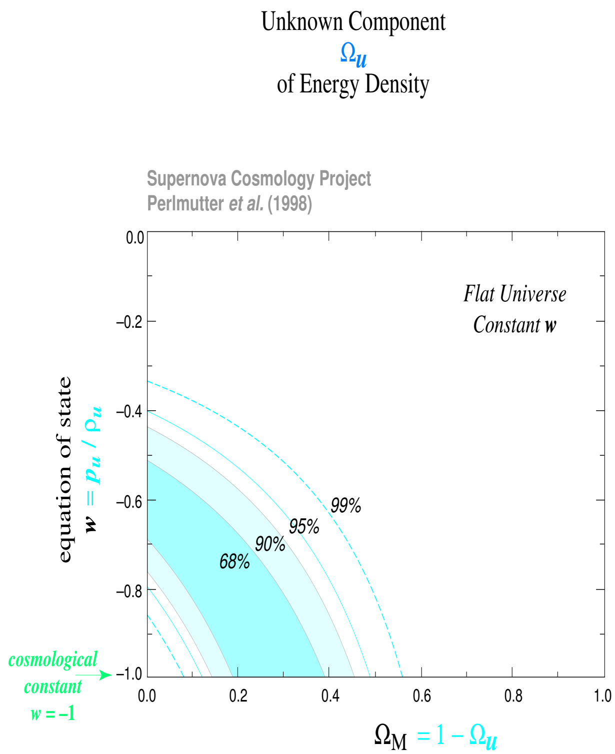

Figure 3.1: Joint likelihood contours in the $\Omega_{\rm m,0} - w$ plane. The plot is from the Perlmutter et al. (1998) analysis for the Supernovae Cosmology Project.

we show the result of the parameter estimation procedure as performed by the SCP collaboration (1998). Again we recognize a non-vanishing $\Omega_{\rm de,0} = 1 - \Omega_{\rm m,0}$ component and a $w < -1/3$ on the 99% level, which is a clear indication that the expansion of the universe is accelerating.

3.2 Scalar Fields and Fine Tuning

In the last Section we have shown that an equation of state for the dark energy component with $w < -1/3$ is sufficient to explain accelerated expansion. For a cosmological constant with $w = -1$ the energy density remains constant over the entire evolution of the universe. One way to interpret the cosmological constant is that it corresponds to an energy of the vacuum. This is can be seen directly from the Einstein equation, since the presence of Λ leads to a curvature of the universe without the presence of any other energy component. The energy density in the cosmological constant with current measurements is

$$\Omega_\Lambda = 0.7 \rightarrow \rho_\Lambda \approx 10^{-48}\ \text{GeV} \approx 10^{-121}\ M_{\text{pl}}^4\,,$$

where the Planck units are the characteristic scale for the initial conditions of the universe, when the system becomes governed by a still absent theory of quantum gravity[2]. The Planck mass is defined, where the de Broglie wavelength of a particle becomes equal to its Schwarzschild radius

$$\frac{2\pi\hbar}{m_{\text{Pl}}c} = \frac{2Gm_{\text{Pl}}}{c^2}\,.$$

Note that in the notes here it is more convenient to talk in terms of the *reduced* Planck mass

$$\boxed{M_{\text{Pl}} = \sqrt{\frac{\hbar c}{8\pi G}} \approx 2 \times 10^{18}\ \text{GeV}\,, \qquad (3.3)}$$

and hence the initial conditions for the cosmological constant need to be fined tuned to a quite unnatural number, which is about 120 (!) orders of magnitude lower than the natural expected value. This is one of the biggest embarrassments of modern cosmology.

Now as mentioned above the cosmological constant can be viewed as the vacuum energy present in the universe. If we believe in supersymmetric fundamental theories there is *no* vacuum energy, which is one of it strength. This is because each fundamental particle has a fermionic or bosonic partner which cancels the vacuum energy exactly to zero. However we know that supersymmetry must be broken at some stage in the universe, because we do not observe it at low energies today. Models for supersymmetry breaking

[2] Although string theory looks as a very promising candidate.

roughly predict a scale of 1 TeV which is still too large to explain the observed values.

Still looking for a field which has a vacuum energy of the cosmological constant might still bring valuable insights. The simplest field we can thin about is a scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,,$$

which has the usual form of kinetic minus potential energy, with the action

$$A = \int d^4x\,\sqrt{-g}\mathcal{L}\,.$$

Note the $\sqrt{-g}$ factor is the Jacobian due to the integration over the 4-dimensional space-time volume in the action. Now in order to look at the cosmological consequences we need the energy-momentum tensor for a the scalar field. It can be obtained by applying Noether's theorem[3]. The conserved quantity corresponding to infinitesimal changes in time and space parameters is

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^\nu\phi)}\frac{\partial\phi}{\partial\phi^\mu} - \mathcal{L}g_{\mu\nu}\,.$$

If we assume we have a homogeneous scalar field, which we have to have from a cosmological point of view in order to fulfill the cosmological principle, we can show that in Minkowski space ($g_{\mu\nu} = \eta_{\mu\nu}$) we have for the energy density

$$T_{00} = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{3.4}$$

and for the momentum density (pressure)

$$T_{ij} = p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{3.5}$$

From this we see that the equation of state is given by

$$w = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}\,.$$

If the kinetic part is much smaller than the potential energy ($\dot{\phi}^2/2 \ll V(\phi)$) the equation of state factor $w \to -1$ if $V \neq 0$. This again stating that a

[3]Noether's theorem is powerful tool which states that each symmetry of the Lagrangian has a corresponding conserved quantity. Symmetry in time results in energy conservation and the homogeneity in space in momentum conservation.

cosmological constant corresponds to a constant vacuum energy. Hence in order to obtain accelerated expansion we need a scalar field whose kinetic energy is negligible compared to the potential[4]. In Fig. 3.2 we see two

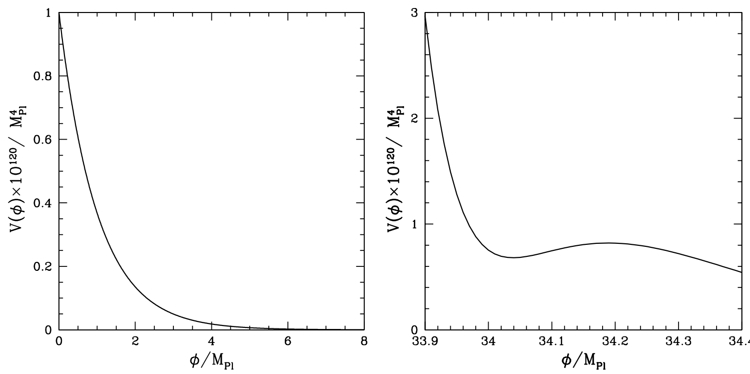

Figure 3.2: Typical potential for scalar field dark energy models (Quintessence). On the left a "slow roll" configuration and on the right a "false vacuum" configuration.

typical potentials for dark energy configurations. On the left is a "slow roll" configuration where the scalar field is still dynamically evolving, but its kinetic energy is negligible compared to the potential. On the right is a configuration where the scalar field is actually frozen in, in a "false vacuum" state (false vacuum, because the "true " vacuum represents the lowest energy state). In general dark energy models with (canonical) scalar field are called *Quintessence* to describe the fifth element character (besides, gravitational, electro-magnetic, weak and strong interactions). Besides of describing a dynamical approach there is hope that these fields can be linked with fundamental theories, like string theory.

3.2.1 The Exponential Potential

One of the earliest studies of scalar fields and their influence on the evolution on the *late* universe, was done in 1988 for an exponential potential (Ratra & Peebles; Wetterich). In general we obtain from the conservation of energy

[4]This is the same requirement as for so called inflationary models, which describe a phase of exponential expansion in the *early* universe. As a matter of fact the dark energy scalar field dark energy models we are going to discuss represent some sort of *late* time inflation.

in Eqn. 2.4
$$\ddot{\phi} + 3H\dot{\phi} + V' = 0\,, \tag{3.6}$$
with the prime denoting the derivative with respect to the field and
$$H^2 = \frac{1}{3M_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_n\right)\,, \tag{3.7}$$
Let us start with the simple example where there is *no* other component. We then would like to answer the question if there are any potentials which would lead to an equation of state $p_\phi = w\rho_\phi$ with w constant. By subtracting and adding Eqns. 3.4 and 3.5 we obtain
$$V = \frac{1-w}{2}\rho_\phi$$
and
$$\dot{\phi}^2 = (1+w)\rho_\phi\,.$$
If we then use again Eqn. 3.4 we get
$$\dot{\phi}^2 = 2\frac{(1+w)}{1-w}V$$
and from the time derivative of this equation we obtain
$$\ddot{\phi} = \frac{1+w}{1-w}V'\,,$$
where we have used $\dot{V} = V'\dot{\phi}$. From Eqn. 3.7 with $\rho_n = 0$ we obtain then
$$H^2 = \frac{2V}{3M_{\rm Pl}^2}\frac{1}{1-w}\,.$$
Combining this into the equation of motion, Eqn. 3.6 for the scalar field ϕ we obtain finally
$$2V' + \frac{V}{M_{\rm Pl}}\sqrt{12(1+w)} = 0\,,$$
which is a simple 1st order differential equation we can solve with the ansatz
$$V(\phi) = V_0 e^{-\lambda\phi/M_{\rm Pl}}\,. \tag{3.8}$$
We then obtain
$$\lambda = \sqrt{3(1+w)}\,,$$

and if we require $-1 < w < 1$ we get $\lambda < \sqrt{6}$. With this we can easily obtain the solution for a generic exponential potential $V = V_0 \exp[-\lambda\phi/M_{\rm Pl}]$

$$\begin{aligned} \phi(t) &= \phi_0 + \frac{2M_{\rm Pl}}{\lambda}\ln(tM_{\rm Pl})\,, \\ w &= \frac{\lambda^2}{3} - 1\,, \\ \rho_\phi &\propto a^{-\lambda^2}\,, \\ a &\propto t^{2/\lambda^2}\,. \end{aligned} \tag{3.9}$$

Note that the second last relation is a trivial consequence of $\rho_\phi \propto a^{-3(1+w)}$ for constant w. These are attractor solutions, where small perturbations around it decay like t^{-1} and t^{1-6/λ^2}. To show this is an Exercise ! in the stability of nonlinear differential equations which is beyond the scope of this lecture. For $\lambda > \sqrt{6}$ there is not a single attractor and $\rho_\phi \propto a^{-6}$, with $w \to 1$, which corresponds to kinetic domination of the energy density.

We will now consider the behaviour when a second component with

$$\dot{\rho}_n + nH\rho_n = 0\,,$$

is present, with $n = 3$ ($w = 0$) for matter and $n = 4$ ($w = 1/3$) for radiation with $\rho_n \propto a^{-n}$. There are now two different cases: Those potentials in which the scalar energy density scales slower than a^{-n} ($\lambda < \sqrt{n}$) and those where the scalar energy density scales faster ($\lambda > \sqrt{n}$). Adding an extra component increases the damping term in Eqn. 3.6 and it follows that the scaling in $\rho_\phi \propto 1/a^{\lambda^2-\delta}$ is always slower (than without an extra component ρ_n) with $\lambda^2 \geq \delta \geq 0$. For $\lambda < \sqrt{n}$ the dark energy component scales slower than the other component and will eventually become dominant and reaches the attractor solution in Eqns. 3.9. For $\lambda > \sqrt{n}$ there is a different behaviour. If the field would scale like in the $\rho_n = 0$ case it would be arbitrarily damped (by the present ρ_n component and hence its kinetic energy will be so far reduced that it reaches the $w \to -1$ branch and begins to catch up again and the final behaviour is that the field mimics the dominant component with the attractor

$$\begin{aligned} \Omega_{\rm de} &\equiv \frac{\rho_\phi}{\rho_\phi + \rho_n} = \frac{n}{\lambda^2}\,, \\ \rho_\phi &\propto \frac{1}{a^n}\,, \\ w &= \frac{n}{3} - 1\,. \end{aligned} \tag{3.10}$$

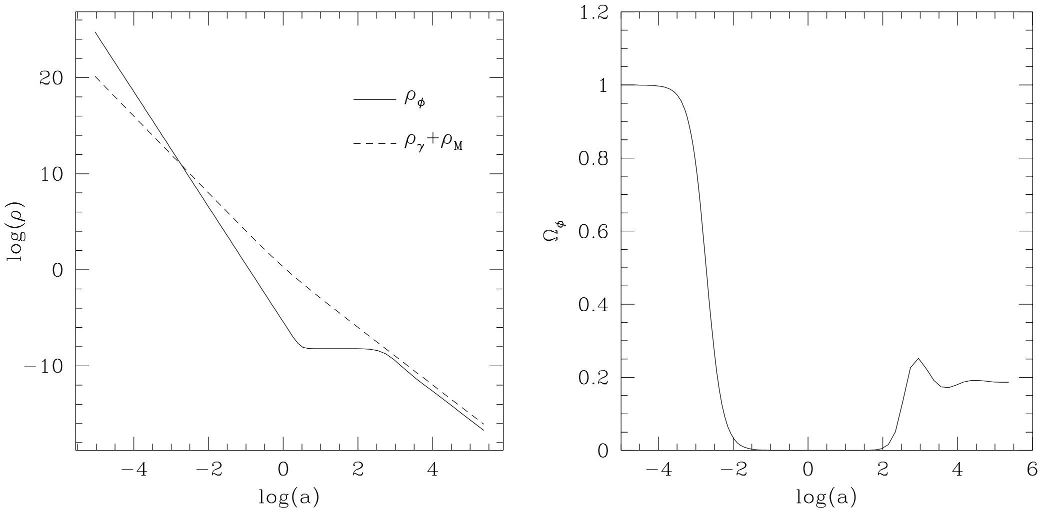

Figure 3.3: Attractor behaviour for exponential dark energy (Ferreira & Joyce 1998). In the left panel we plot the evolution of the energy density in the scalar field (ρ_ϕ) and in a component of radiation-matter as a function of scale factor for a situation in which the scalar field (with $\lambda = 4$) initially dominates, then undergoes a transient and finally locks on to the scaling solution. In the right panel we plot the evolution of the fractional density in the scalar field.

In Fig. 3.3 we show the behaviour for $\lambda = 4$ how the attractor works. Initially the scalar field is domination over radiation and matter and is kinetically dominant and scales like $1/a^6$, until the energy density in radiation is undershot. Then it turns around scaling much slower than radiation or matter until it has caught up and settles down to the fraction given in Eqns. 3.10.

The big advantage of this attractor solutions is that they can start of on an energy scale at early times which is of the order of the Planck scale $\rho_{\phi,i} = \mathcal{O}(M_{\rm Pl}^4)$ and still reaches the attractor. However the attractor given here with $\Omega_{\rm de} = n/\lambda^2$ can *not* explain a universe where the dark energy component dominates. But this is exactly what is required in a flat universe with $\Omega_{\rm de,0} = 0.7$ and $\Omega_{\rm m,0} = 0.3$.

3.3 Tracker Solution

We have seen in the previous Section that although while a exponential potential provides an elegant way to avoid the fine tuning of initial conditions, it unfortunately does not explain why the matter and dark energy density today roughly coincide. One would want that the energy density in the dark energy component somehow tracks below the the other components for most of the evolution of the universe and then suddenly dominates and leads to an accelerated expansion.

The difference of the tracker solutions to the previously discussed exponential potential is that its energy density is changing steadily with ϕ and evtl. manages to overtake the background fields. So we can write down the following two conditions:

(a) As for the self-adjusting exponential potential a wide range of initial conditions should be drawn towards a common cosmic history; but
(b) these tracking solutions should not "self-adjust to th background equation of state, but, instead, maintain some finite difference in the equation-of-state such that the dark energy ultimately dominates and the universe enters a period of acceleration.

Two potentials which fulfill this are

$$V(\phi) = M^{4+\alpha}\phi^{-\alpha}$$

and

$$V(\phi) = M^4 e^{M_{\text{Pl}}/\phi}\,,$$

where M is a free parameter which needs to be adjusted in order to obtain $\Omega_{\Lambda,0} = 0.7$ today. The tracker solutions fulfils

$$V'' = (9/2)\left(1 - w^2\right)\left[(\alpha+1)/\alpha\right] H^2 \tag{3.11}$$

at all times. Sine ρ_ϕ should begin to dominate today we need ϕ to be $\mathcal{O}(M_{\text{Pl}})$ since $V'' \approx \rho_\phi/\phi^2$ and $H^2 \approx \rho_\phi/M_{\text{Pl}}^2$. In order to obtain $\Omega_{\Lambda,0} = 0.7$ or $\rho_{\phi,0} \approx 10^{-47}$ GeV we obtain with $V(\phi) \approx \rho_\phi$ imposes the constraint $M \approx \left(\rho_{\phi,0} M_{\text{Pl}}^{\alpha}\right)^{1/(\alpha+4)}$. For low values of α the mass M has to be as small as 1 meV. However $M > 1$ GeV - comparable to particle physics scales - is possible for $\alpha \geq 2$. In Fig. 3.4 we show the evolution of the dark energy density and the equation of state for the exponential tracker. If initially ρ_ϕ is smaller than the tracker solution the field remains frozen until H^2 decreases

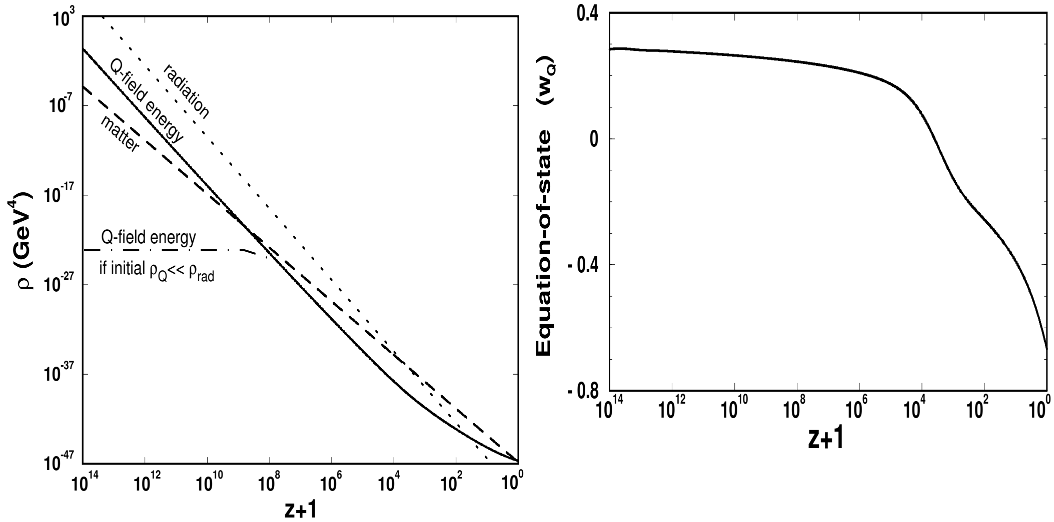

Figure 3.4: Left: Evolution of energy densities for exponential tracker. Right: The evolution of the equation of state [Zlatev et al. 1998].

that the tracker equation 3.11 is fulfilled. Then the field rolls down the potential maintaining Eqn. 3.11. If the energy density ρ_ϕ is larger than the tracker solution, the field starts rolling down the potential immediately and very fast, so that the kinetic energy dominates and shifts as a^{-6} ($w = 1$) until ϕ falls below the tracker and is frozen until it follows it. The equation of state initially is minutely smaller than radiation ($w = 1/3$) and then drops at matter radiation equality below zero and approaches $w \to -1$ when the dark energy becomes to dominate.

To conclude our discussion about dark energy we mention that there is now a plethora of valid dark energy models and we show in Fig. 3.5 the low redshift evolution of the dark energy equation for a sample of models. One of the biggest challenges in modern cosmology is to test which of these models fits the data best and to find out more about the nature of dark energy.

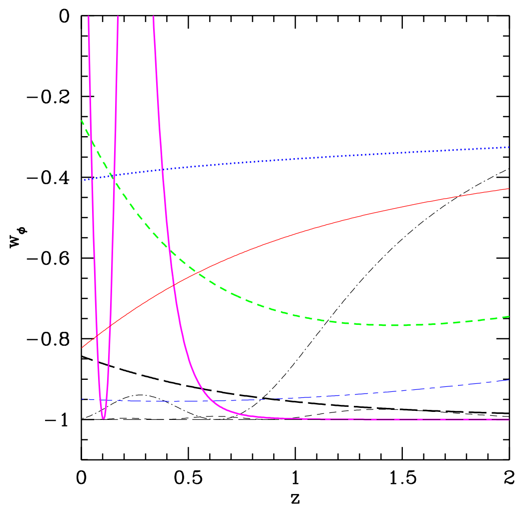

Figure 3.5: Low redshift evolution of the equation of state for a sample of dark energy models [Weller and Albrecht 2001].