

Handout on Mathematics for EES students – Continuous processes

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1 Bacterial population in continuous time

Size of the time steps

A population grows with a rate of 0.8 per hour? What does this mean?

Size of the time steps

Solution: we take very short time steps of Δt hours.

Then $X_n = x(n \cdot \Delta t) = x(t)$ is the population size at time t .

If we consider $x(t)$ as a continuous function, we have:

$$x(t + \Delta t) = x(t) + 0.8 \cdot \Delta t \cdot x(t)$$

or

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = 0.8 \cdot x(t)$$

With $\Delta t \rightarrow 0$ we obtain $0.8 \cdot x(t) = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$

This is the definition of the derivative of the function x with respect to time: $\frac{dx(t)}{dt}$ or $x'(t)$ or $\dot{x}(t)$

Linear differential equation

The equation

$$x'(t) = 0.8 \cdot x(t)$$

is called *linear differential equation*.

Remember: $\exp'(x) = \exp(x)$ and $\exp'(\lambda x) = \lambda \exp(\lambda x)$

So $x'(t) = 0.8 \cdot x(t)$ has the solution $x(t) = \exp(0.8t)$ We have an exponential growth.

Generally:

Linear differential equation $x'(t) = \lambda x(t) \Rightarrow$ exponential growth with $x(t) = x_0 e^{\lambda t}$

λ is called the “Malthusian parameter”.

Example: model of diffusion through a membrane

Slightly more general: $x'(t) = \lambda x(t) + c$

To solve it we try: $x(t) = x_0 e^{\mu t} + b$ and search for μ and b .

$$\Rightarrow x'(t) = x_0 e^{\mu t} \cdot \mu$$

Thus in the LDE: $x_0 e^{\mu t} \cdot \mu = \lambda(x_0 e^{\mu t} + b) + c = \lambda x_0 e^{\mu t} + \lambda b + c$

This is true for $\lambda = \mu$ and $\lambda b + c = 0 \Leftrightarrow b = -\frac{c}{\lambda}$

$$\text{Solution: } x(t) = x_0 e^{\lambda t} - \frac{c}{\lambda}$$

Example: Diffusion through a membrane with $\dot{x}(t) = \alpha(\beta - x(t)) = -\alpha x(t) + \alpha\beta$

α : speed of diffusion, β : concentration in medium

2 Phase-line diagrams

Ordinary Differential Equation and Phase-line diagrams

Ordinary differential equation (ODE) for $x : t \mapsto x(t)$:

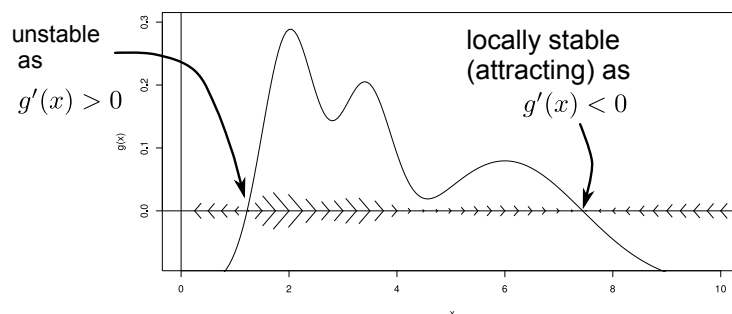
$$\dot{x} = g(x)$$

where

$$g(x) = g(x(t))$$

and

$$\dot{x} = x'(t) = \frac{dx}{dt} = \frac{dx(t)}{dt}$$



Ordinary Differential Equation and Phase-line diagrams

Ordinary differential equation (ODE):

$$x'(t) = \dot{x} = g(x(t)) \text{ or } \dot{x} = g(x)$$

To find the long-term state of the process we plot $g(x)$.

When $g(x) = 0$ you have a fixed point, as the speed of $x(t)$ is null.

Then you mark with arrows the movement of $x(t)$ on the x -axis. It is to the right (increasing) when $g(x) > 0$ and to the left (decreasing) when $g(x) < 0$. This way you can see the stability of the fixed points.

You can also note that: if $g'(x) > 0$ the point is unstable and if $g'(x) < 0$ the point is stable.

Phase-line diagrams - Example

Example: Diffusion through a membrane

$$\begin{aligned}
 \dot{x}(t) &= \alpha \cdot (\beta - x(t)) && \text{with } \alpha > 0 \\
 g(x) &= \alpha \cdot (\beta - x) \\
 g(x) = 0 &\Leftrightarrow x = \beta
 \end{aligned}$$

Only fixed point

Derivative: $g'(x) = -\alpha < 0$ for all x , thus the point is stable.

Interpretation: diffusion until the concentration is equal on both sides of the membrane.

Summary of the above approach

- To study the size of a population in a continuous manner, we express this size as a function of time.
- We define the Ordinary Differential Equation that models the speed of growth of this population.
- By solving this equation, we find the function describing the population size as a function of time.
- We use the method of Phase-line Diagrams to find whether the system has fixed points and whether they are stable.

3 Logistic growth

Definition and fixed points

The continuous Verhulst model:

$$\dot{x} = g(x) = \frac{\lambda}{K} \cdot x \cdot (K - x) = \lambda \cdot x \cdot \left(1 - \frac{x}{K}\right)$$

Stability:

$$\begin{aligned} g'(x) &= \lambda \left(1 - \frac{x}{K}\right) + \lambda x \left(-\frac{1}{K}\right) \\ &= \lambda \cdot \left(1 - \frac{x}{K} - \frac{x}{K}\right) \\ &= \lambda \cdot \left(1 - 2\frac{x}{K}\right) \end{aligned}$$

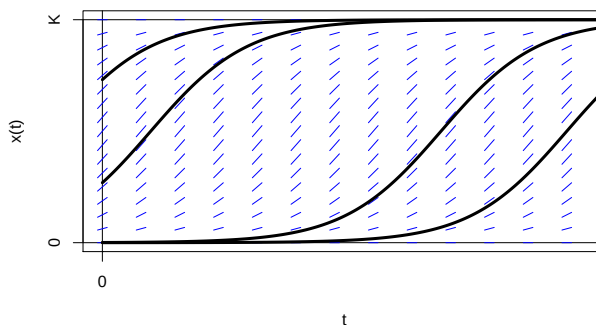
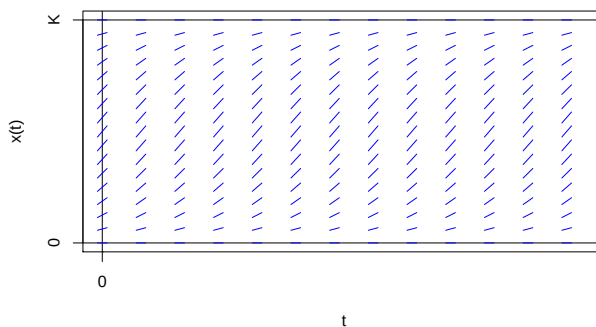
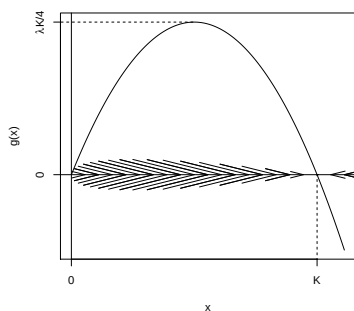
$$\begin{aligned} g'(0) &= \lambda > 0 \Rightarrow \text{fixed point } x = 0 \text{ unstable} \\ g'(K) &= -\lambda < 0 \Rightarrow \text{fixed point } x = K \text{ stable} \end{aligned}$$

$\lambda > 0$: initial growth rate and

$K > 0$: carrying capacity

Fixed points: $g(x) = 0 \Leftrightarrow$

$$\begin{cases} x = 0 \\ \text{or} \\ 1 - \frac{x}{K} = 0 \Leftrightarrow x = K \end{cases}$$



Logistic function

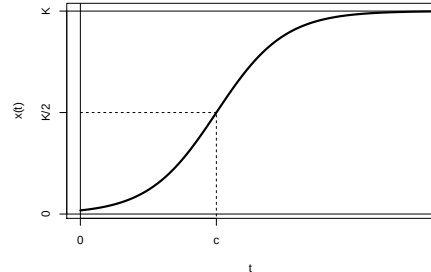
The solution of the ODE $\dot{x} = \lambda x \cdot (1 - x/K)$ is the **logistic function**

$$x(t) = \frac{K}{1 + e^{-\lambda \cdot (t-c)}},$$

where c depends on the initial condition $x(0)$.

$$x(0) = \frac{K}{1 + e^{\lambda \cdot c}}$$

$$\Rightarrow c = \ln \left(\frac{K}{x(0)} - 1 \right) / \lambda$$



Logistic function is really the solution

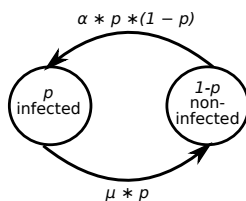
Show that $x(t) = \frac{K}{1 + e^{-\lambda \cdot (t-c)}}$ fulfills $\dot{x} = \lambda x \cdot (1 - x/K)$:

$$\begin{aligned} \dot{x} = x'(t) &= K \cdot \frac{-1}{(1 + e^{-\lambda \cdot (t-c)})^2} \cdot e^{-\lambda \cdot (t-c)} \cdot (-\lambda) \\ &= \lambda \cdot \frac{K}{1 + e^{-\lambda \cdot (t-c)}} \cdot \frac{e^{-\lambda \cdot (t-c)}}{1 + e^{-\lambda \cdot (t-c)}} \\ &= \lambda \cdot x(t) \cdot \frac{e^{-\lambda \cdot (t-c)}}{1 + e^{-\lambda \cdot (t-c)}} \\ &= \lambda \cdot x(t) \cdot \left(1 - \frac{1}{1 + e^{-\lambda \cdot (t-c)}} \right) \\ &= \lambda \cdot x(t) \cdot (1 - x(t)/K) \\ &= \lambda \cdot x \cdot (1 - x/K) \end{aligned}$$

q.e.d.

Application in epidemiology – model

Model the evolution of a human population with infection by Malaria and transmission by vector. After Sir Ronald Ross (1857-1932, Nobel Prize in 1902) who found that malaria was propagated by Anopheles and proposed to fight against them.



With

α : rate of infection and

μ : rate of loss of infection.

Simplification: No death and No immunity

Ordinary Differential equation:

$$\dot{p} = \alpha p(1 - p) - \mu p$$

Application in epidemiology – demonstration

Demonstrate that this is a logistic growth process.

We search for

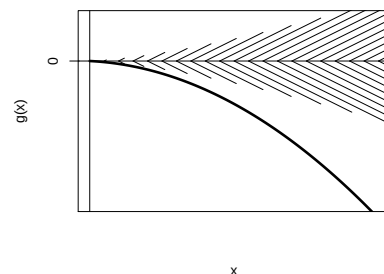
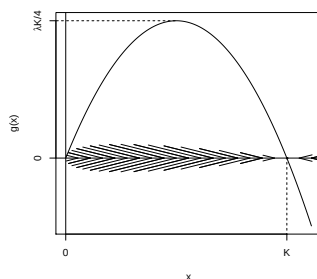
$$\begin{aligned}
 \dot{p} &= \lambda p \left(1 - \frac{p}{K}\right) \\
 \dot{p} &= p(\alpha(1-p) - \mu) \\
 &= p(\alpha - \mu - \alpha p) \\
 &= p \left(\alpha - \mu - \frac{\alpha p}{\alpha - \mu} (\alpha - \mu) \right) \\
 &= p(\alpha - \mu) \left(1 - \frac{\alpha p}{\alpha - \mu} \right) \\
 \Rightarrow \lambda &= \alpha - \mu \quad \text{and} \quad K = \frac{\alpha - \mu}{\alpha}
 \end{aligned}$$

Application in epidemiology – long-term behaviour

If $\alpha - \mu < 0 \Rightarrow K < 0$ Negative carrying capacity – no infection.

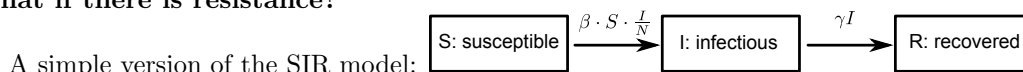
For $\alpha > \mu$ logistic growth
with $p \rightarrow K$ for $t \rightarrow \infty$: the infection remains.

For $\alpha < \mu$, $p \rightarrow 0$
for $t \rightarrow \infty$: the infection dies out.



Conclusion: no need to completely eradicate the vector. It is enough to maintain the population small enough so that $\alpha < \mu$ and then the infection dies out.

What if there is resistance?



$$N = S + R + I \quad , \quad \frac{dS}{dt} = -\frac{\beta \cdot S \cdot I}{N} \quad , \quad \frac{dI}{dt} = \frac{\beta \cdot S \cdot I}{N} - \gamma \cdot I \quad , \quad \frac{dR}{dt} = \gamma \cdot I$$

$\mathcal{R}_0 = \beta/\gamma$ is the basic reproduction rate (don't confuse with $R(0)$!), that is:

If almost nobody is resistant, an infected infects on average $\mathcal{R}_0$ others before (s)he recovers (or dies).

I reaches its maximum when $\frac{dI}{dt} = 0$, that is

$$\begin{aligned}
 0 &= I \cdot (\beta S/N - \gamma) \\
 0 &= \beta S/N - \gamma \\
 S/N &= \gamma/\beta = 1/\mathcal{R}_0
 \end{aligned}$$

More about the SIR model in the section on multidimensional processes.

Some of what you should be able to explain

- What is a rate? (e.g. per hour or per day)
- What is a differential equation?
 - What is a solution of a differential equation?
 - Why using differential equations and not just their solution to define a model?
- How to find fixed points and analyse their stability.
- What do the stable and unstable fixed points tell you about the system?

- logistic growth and carrying capacity
- (simplified) SIR model and what it predicts