Statistics for EES
Standard error

Dirk Metzler

April 13, 2026

Contents

Contents

1 The standard error SE

The Standard Error

$$SE = \frac{\text{sd}}{\sqrt{n}}$$

describes the variability of the sample mean.

n: sample size sd: sample standard deviation

1.1 example: drought stress in sorghum

drought stress in sorghum

References

[BB05] V. Beyel and W. Brüggemann. Differential inhibition of photosynthesis during pre-flowering drought stress in Sorghum bicolor genotypes with different senescence traits. *Physiologia Plantarum*, 124:249–259, 2005.

- 14 sorghum plants were not watered for 7 days.
- in the last 3 days: transpiration was measured for each plant (mean over 3 days)
- the area of the leaves of each plant was determined

transpiration rate

=

(amount of water per day)/area of leaves

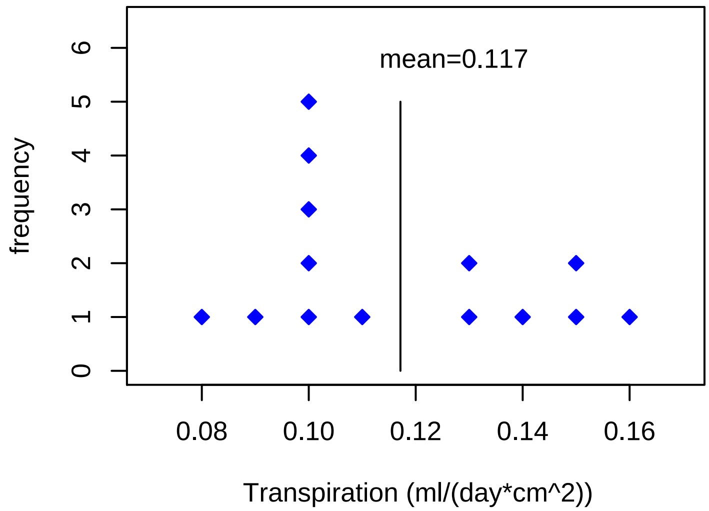
$$\left[\frac{\text{ml}}{\text{cm}^2 \cdot \text{day}}\right]$$

Aim: Determine mean transpiration rate μ under these conditions.

If we had many plants, we could determine μ quite precisely.

Problem: How accurate is the estimation of μ with such a small sample? ($n = 14$)

drought stressed sorghum (variety B, $n = 14$)



transpiration data: $x_1, x_2, \ldots, x_{14}$

$$\overline{x} = \big(x_1 + x_2 + \cdots + x_{14}\big)/14 \; = \frac{1}{14}\sum_{i=1}^{14} x_i$$

$$\overline{x} = 0.117$$

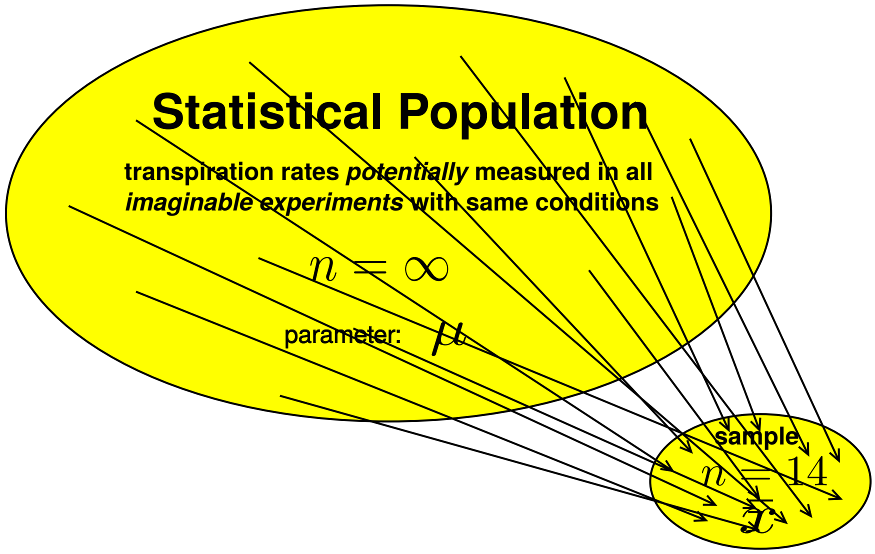
our estimation:

$$\mu \approx 0.117$$

how accurate is this estimation?

How much does $\overline{x}$ (our estimation) deviate from μ (the true mean value)?

We consider our 14 plants as
random sample
from a very large "population" of possible values.



We estimate the population mean
μ
by the sample mean
$\overline{x}$.

Each single observation X is a *random variable* and the population mean μ is the *expected value* of this random variable. (Will be formally defined later in the semester.)

μ is a parameter.

$\overline{x}$ is a statistic.

parameter: real or theoretical value within mathematical model

- example: μ, the expected value of each single observation X
- non-random (in classical frequentistic stats)
- assumed usually in stats: there is a true value that is unknown

statistic: a function of the sampled data (that is, calculated from the data)

- example: $\overline{x}$
- are random variables because data is also random due to
  - randomly sampling from natural variation
  - random process
  - measurement error

estimator: statistic to estimate the value of a parameter

- example: $\overline{x}$ is an estimator for μ

Another example

The **statistics**

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2} \qquad \text{and} \qquad \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2}$$

are **estimators** for the population standard deviation σ, which is a **parameter**.

Each new sample gives a new value of $\overline{x} = (x_1 + x_2 + \cdots + x_n)/n$.

$\overline{x}$ depends on randomness: it is a *random variable*

Problem: How variable is $\overline{x}$?

More precisely: What is the typical deviation of $\overline{x}$ from μ?

What does the variability of $\overline{x}$ depend on?

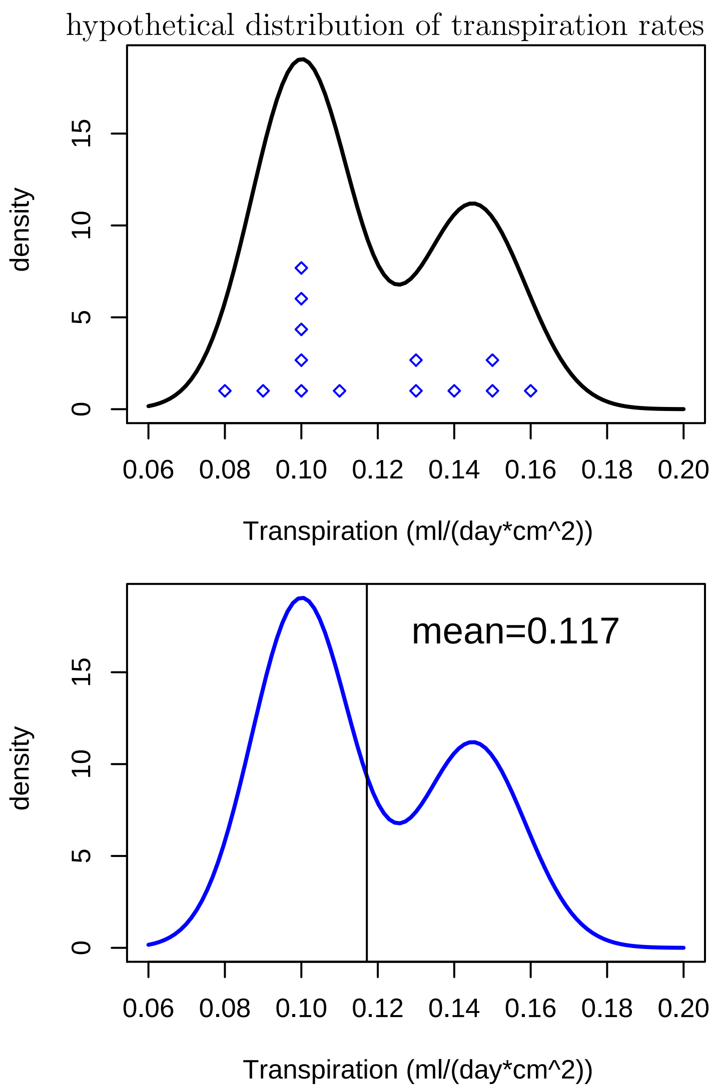
What does the variability of $\overline{x}$ depend on?

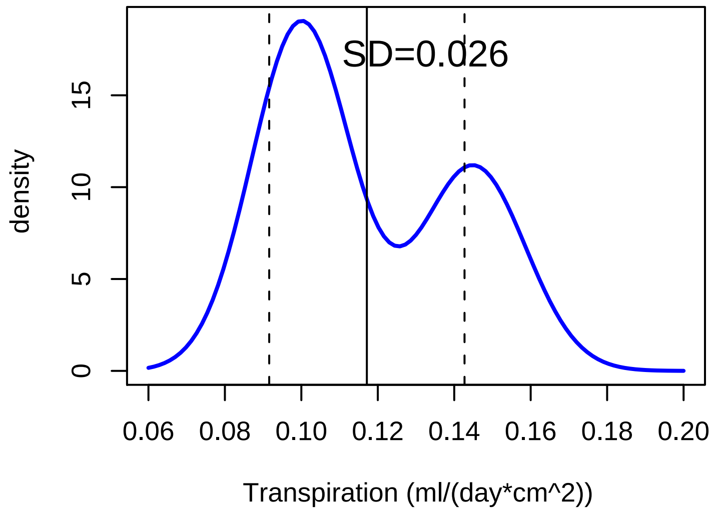
1. On the the variability of the single observations $x_1, x_2, \ldots, x_n$
2. On the sample size n

The larger n, the smaller is the variability of $\overline{x}$.

To explore this dependency we perform a (Computer-)Experiment.

Experiment: Take a population, draw samples and examine how $\overline{x}$ varies.

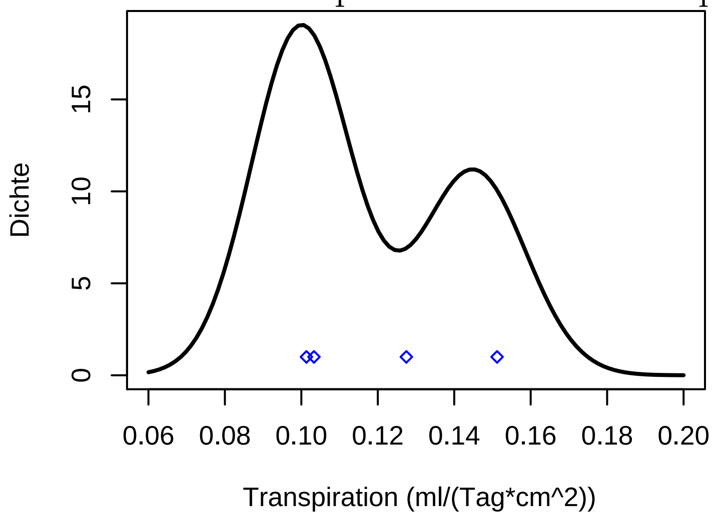


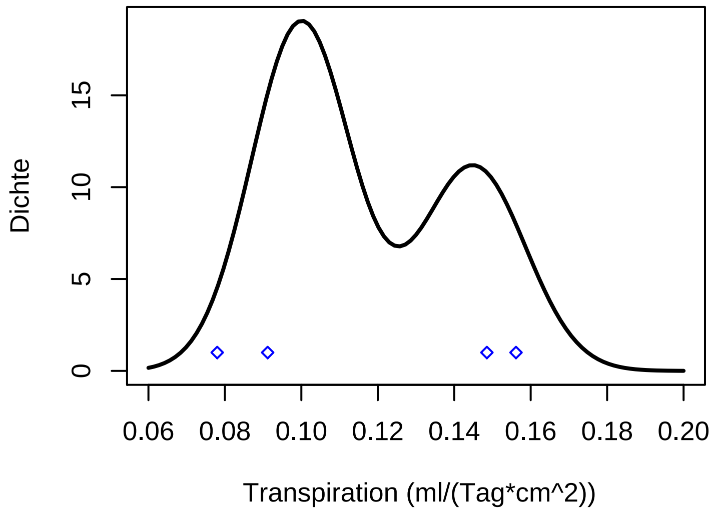


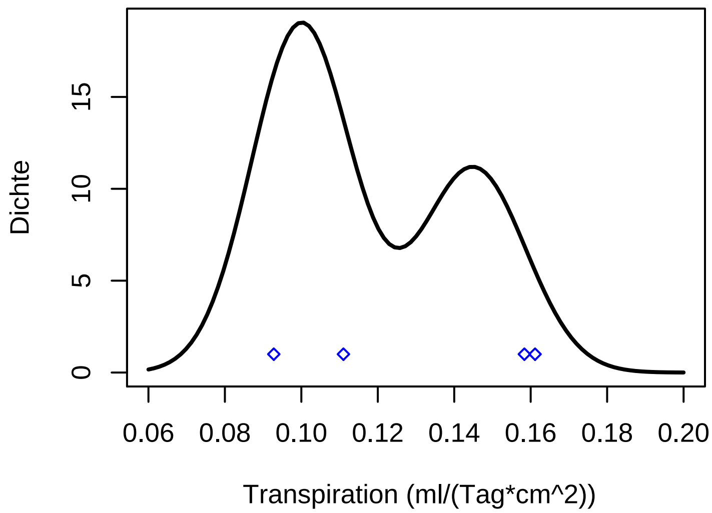
At first with small sample sizes:

$$n = 4$$

sample of size 4 second sample of size 4 third sample of size 4

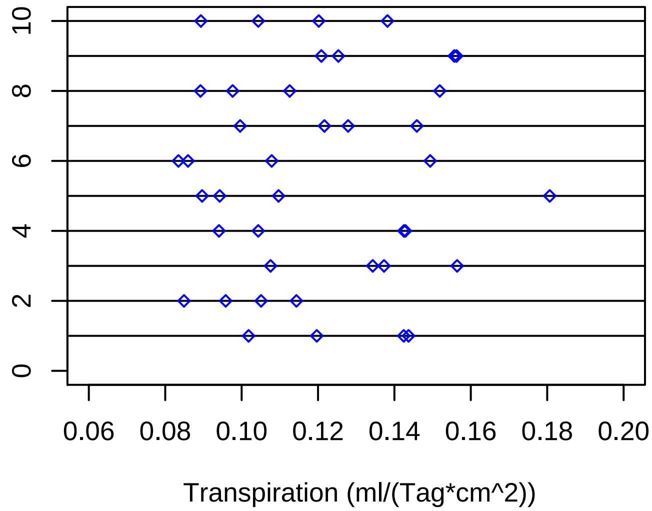


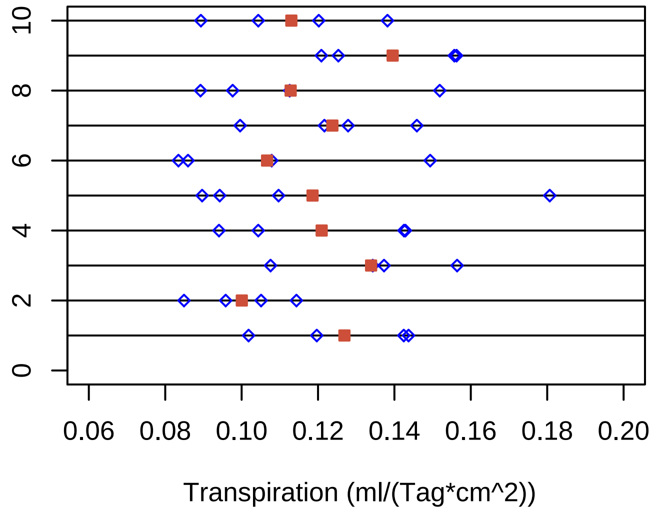




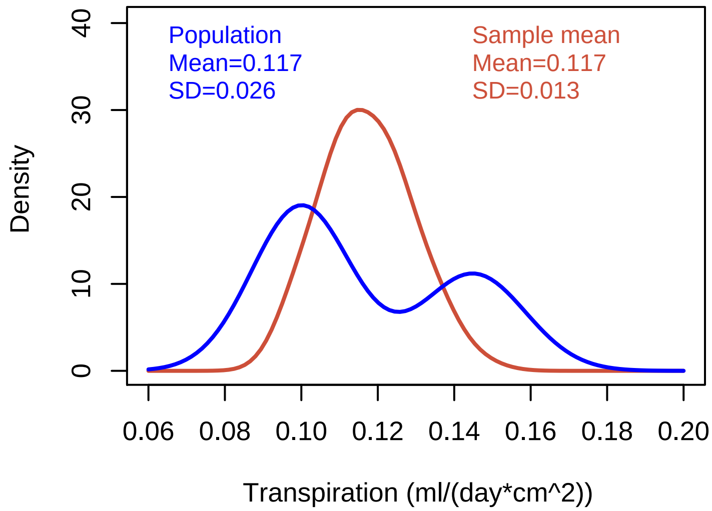
How variable are the sample means?

10 samples of size 4 and the corresponding sample means





distribution of sample means (sample size $n = 4$)

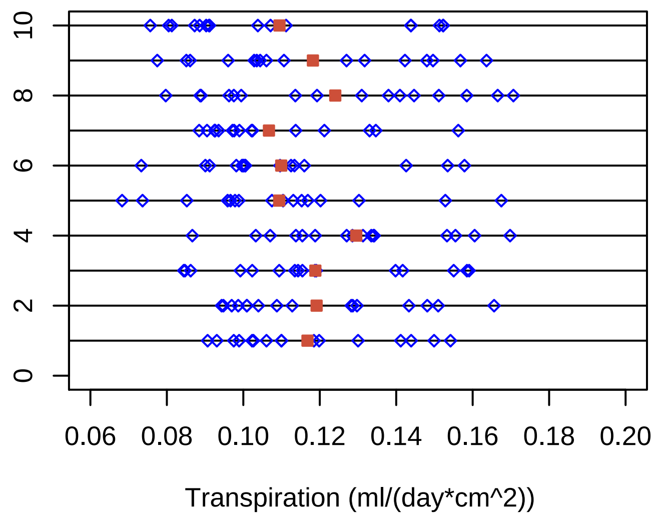


$$\text{population: standard deviation} = 0.026$$

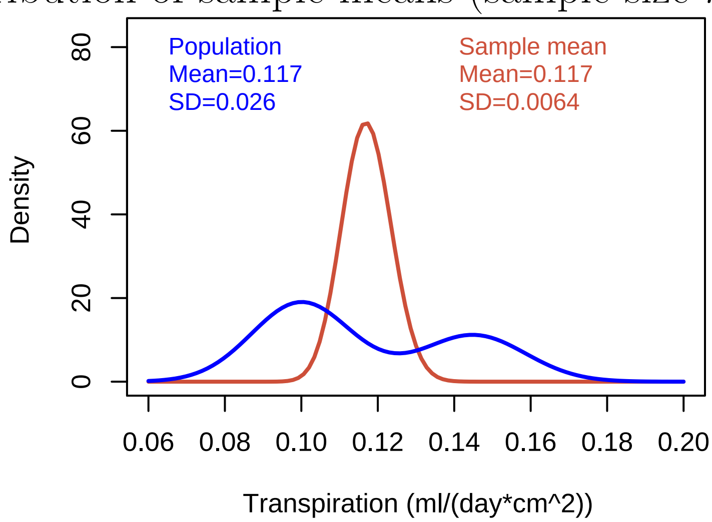
$$\text{sample means } (n = 4): \quad \begin{aligned} \textit{standard deviation} &= 0.013 \\ &= 0.026/\sqrt{4} \end{aligned}$$

Increase the sample size from 4to 16

10 samples of size 16 and the corresponding sample means



distribution of sample means (sample size $n = 16$)



population: standard deviation $= 0.026$

sample mean ($n = 16$): $$\begin{aligned} \textit{standard deviation} &= 0.0065 \\ &= 0.026/\sqrt{16} \end{aligned}$$

Theorem 1. *If $X_1, X_2, \ldots X_n$ are independent $\mathbb{R}$-valued random variables with expected value μ and variance σ^2, we get for $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$:*

$$\mathbb{E}\overline{X} = \mu$$

and

$$Var\ \overline{X} = \frac{1}{n}\sigma^2,$$

that is,

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

In particular: The standard error $\frac{s}{\sqrt{n}}$ is a estimator for the standard deviation of the $\sigma_{\overline{X}}$ sample mean $\overline{X}$ of $(X_1, X_2, \ldots, X_n)$.
The sample standard deviation s is an estimator of the standard deviation σ in the entire population.

Proof: Linearity of the expected value implies

$$\begin{aligned}\mathbb{E}\overline{X} &= \mathbb{E}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(X_i\big) \\ &= \frac{1}{n}\sum_{i=1}^{n}\mu = \mu.\end{aligned}$$

The independce of X_i helps to simplify the variance:

$$\begin{aligned}\text{Var}\ \overline{X} &= \text{Var}\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big) = \frac{1}{n^2}\text{Var}\Big(\sum_{i=1}^{n} X_i\Big) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}\big(X_i\big) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n}\sigma^2\end{aligned}$$

General Rule 1. *Let $\overline{x}$ be the mean of a sample of size n from a distribution (e.g. all values in a population) with standard deviation σ. Since $\overline{x}$ depends on the random sample, it is a random variable. Its standard deviation $\sigma_{\overline{x}}$ fulfills*

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}.$$

Problem: σ is unknown

Idea: Estimate σ by sample standard deviation s:

$$\sigma \approx s$$

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} \approx \frac{s}{\sqrt{n}} =: \text{SEM}$$

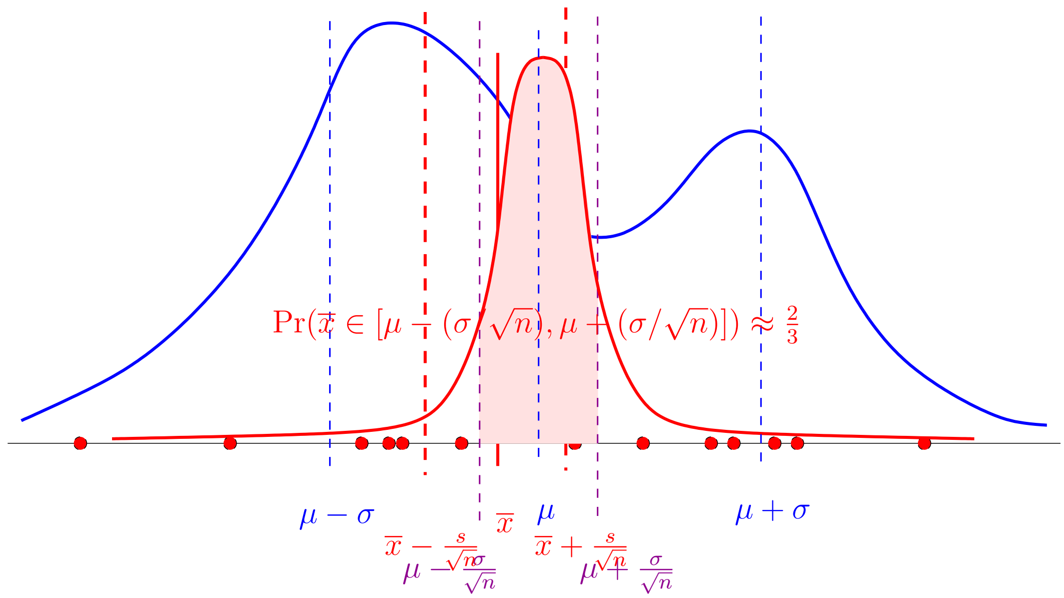
SEM stands for *Standard Error of the Mean*, or *Standard Error* for short.

Note: The statistic SEM$= \frac{s}{\sqrt{n}}$ is an estimator
for the parameter $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$.

The distribution of $\overline{x}$

Observation

Even if the distribution of x is asymmetric and has multiple peaks, the distribution of $\overline{x}$ will be bell-shaped (at least for larger sample sizes n.)

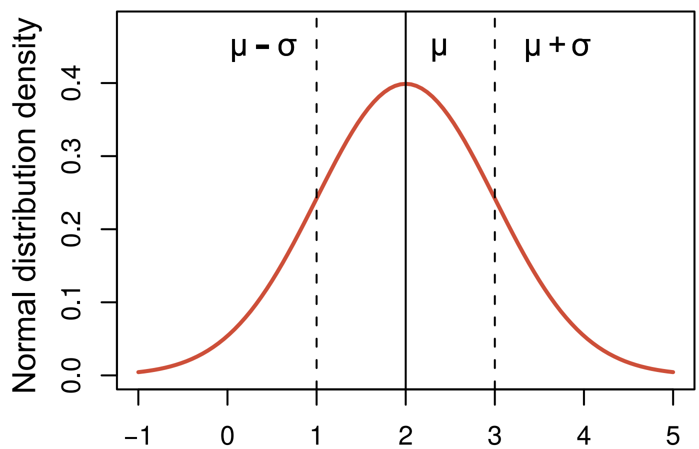


$\Pr(\overline{x} \in [\mu - (\sigma/\sqrt{n}), \mu + (\sigma/\sqrt{n})]) \approx \frac{2}{3}$
$\mu - \sigma$
$\overline{x}$
μ
$\mu + \sigma$
$\overline{x} - \frac{s}{\sqrt{n}}$
$\overline{x} + \frac{s}{\sqrt{n}}$
$\mu - \frac{\sigma}{\sqrt{n}}$
$\mu + \frac{\sigma}{\sqrt{n}}$

The distribution of $\overline{x}$ is approximately of a certain shape:

the normal distribution.

Density of the normal distribution

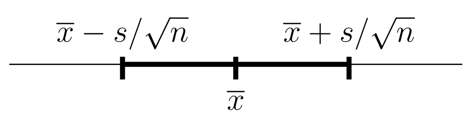


The normal distribution is also called *Gauß distribution* (after Carl Friedrich Gauß, 1777-1855)

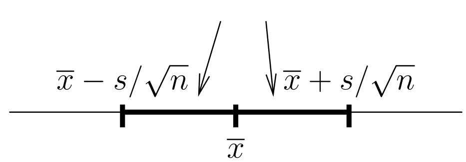
2 Taking standard errors into account

Important consequence

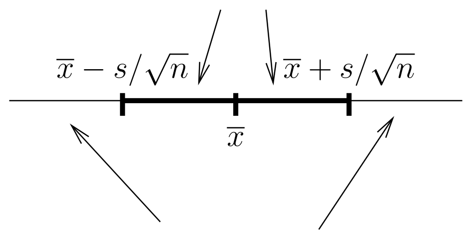
Consider the interval



This interval contains μ with probability of ca. $2/3$



This interval contains μ with probability of ca. $2/3$



probability that μ is outside of interval is ca. $1/3$

Thus:

It may happen that $\overline{x}$ deviates from μ by more than $s/\sqrt{n}$.

Example: Comparison of mean values

Example: Galathea

Galathea: Carapace lengths in a sample

Males: $\overline{x}_1 = 3.04$ mm $s_1 = 0.9$ mm $n_1 = 25$

Females: $\overline{x}_2 = 3.23$ mm $s_2 = 0.9$ mm $n_2 = 29$

The females are apparently larger.

Is this significant?

Or could it be just *random*?

How precisely do we know the true mean value?

Males: $\overline{x}_1 = 3.04$ mm $s_1 = 0.9$ mm $n_1 = 25$

$$s_1/\sqrt{n_1} = 0.18 \text{ [mm]}$$

We have to assume uncertainty in the magnitude of ± 0.18 (mm) in $\overline{x}_1$

How precisely do we know the true mean value?

Females: $\overline{x}_2 = 3.23$ mm $s_2 = 0.9$ mm $n_2 = 29$

$$s_2/\sqrt{n_2} = 0.17 \text{ [mm]}$$

It is not unlikely that $\overline{x}_2$ deviates from the true mean by more than ± 0.17 (mm).

The difference of the means

$$3.23 - 3.04 = 0.19 \text{ [mm]}$$

is not much larger than the expected inaccuracies.

It could also be due to pure random that $\overline{x}_2 > \overline{x}_1$

MORE PRECISELY:

If the true means are actually equal $\mu_{Females} = \mu_{Males}$ it is still quite likely that the sample means $\overline{x}_2$ are $\overline{x}_1$ that different.

In the language of statistics:

The difference of the mean values is (statistically) *not significant.*

not significant = can be just random

Application example:

Planning an experiment:

How many observations do I need?

(How large should n be?)

If you know which precision you need for (the standard error $s/\sqrt{n}$ of) $\overline{x}$

and if you already have an idea of s

then you can estimate the value of n that is necessary: $s/\sqrt{n} = g$

(g = desired standard error)

Example: Stressed transpiration values in another sorghum subspecies: $\overline{x} = 0.18$ $s = 0.06$ $n = 13$

How often do we have to repeat the experiment to get a standard error of ≈ 0.01?

Which n do we need?

$$\text{Solution: desired: } s/\sqrt{n} \approx 0.01$$
$$\text{From the previous experiment we know: } s \approx 0.06, \quad \text{so: } \sqrt{n} \approx 6$$
$$n \approx 36$$

Summary

- Assume a population has mean value μ and standard deviation σ.
- We draw a sample of size n from this population with sample mean $\overline{x}$.
- $\overline{x}$ is a random variable with mean value μ and standard deviation $\sigma/\sqrt{n}$.
- Estimate the standard deviation of $\overline{x}$ by $s/\sqrt{n}$, where s is the standard deviation computed from the sample with the formula with $n-1$.
- $s/\sqrt{n}$ is the *Standard Error (of the Mean).*
- Deviations of $\overline{x}$ of the magnitude of $s/\sqrt{n}$ are usual. They are *not significant:* they can be random.

Parameters and estimators

Parameter	**Estimator** from sample $x_1, \ldots, x_n$	**unbiased?**
Population mean μ	$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$	✓
Population variance σ^2	$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$	—
	$\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$	✓
Population SD σ	$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$	—
	$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$	—
Standard Deviation of $\bar{x}$	Standard Error $s/\sqrt{n}$	—

Some of what you should be able to explain

- Concepts: parameters, statistics, estimators
- Why is the sample mean $\overline{x}$ a random variable?
- distribution properties of $\overline{x}$

- What is the standard error and how is it different from …
  - … sd?
  - … the standard deviation $\sigma_{\overline{x}}$ of the mean?
- When calculating the standard error from data why must I once divide by n (or $\sqrt{n}$) and another time by $n-1$ (or $(\sqrt{n-1})$?
- Applications of the standard error in descriptive data analysis and experimental design.