MAX PLANCK INSTITUTE
FOR SOLAR SYSTEM RESEARCH

Introduction to Rayleigh–Bénard Convection (RBC) and Rotating RBC

MagIC workshop, Oct. 27-29, Göttingen

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International Max Planck Research School

Max Planck Institute for Solar System Research

27th Oct 2025

Thermal convection in nature

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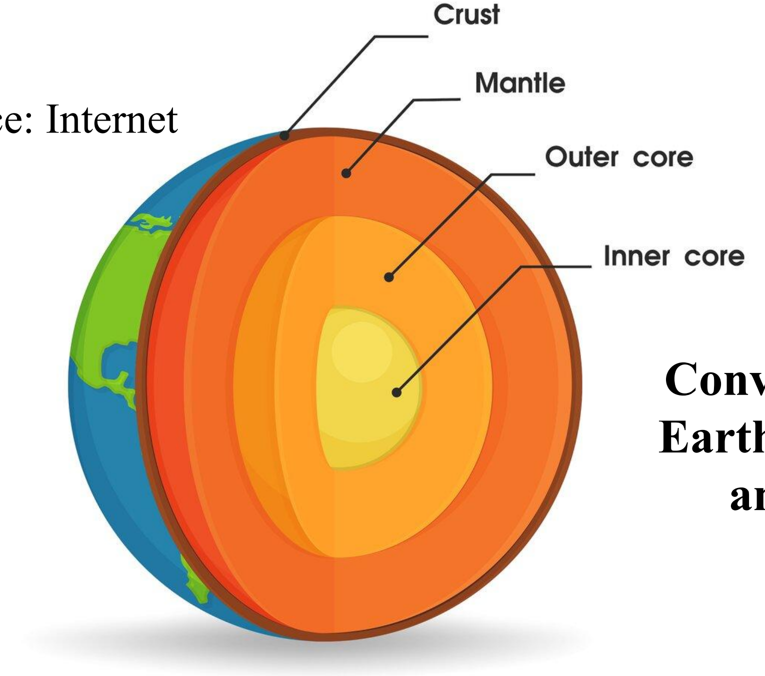
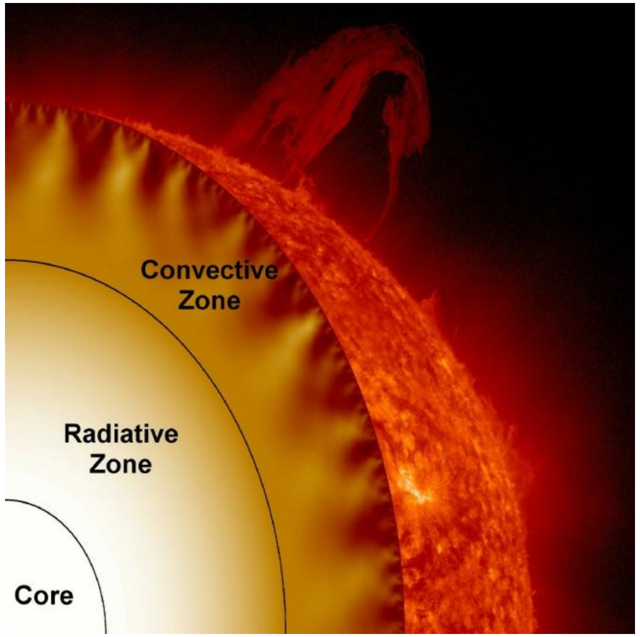
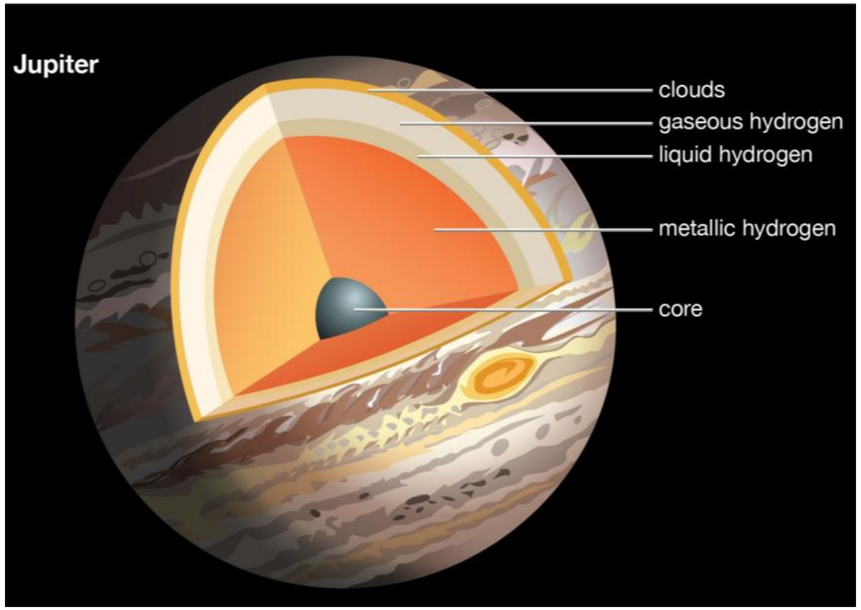


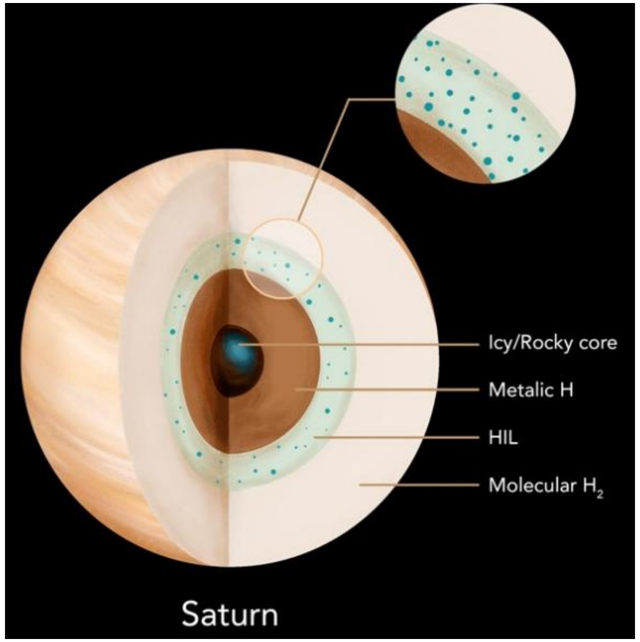
Convection in the Earth's outer core and mantle



Solar convection

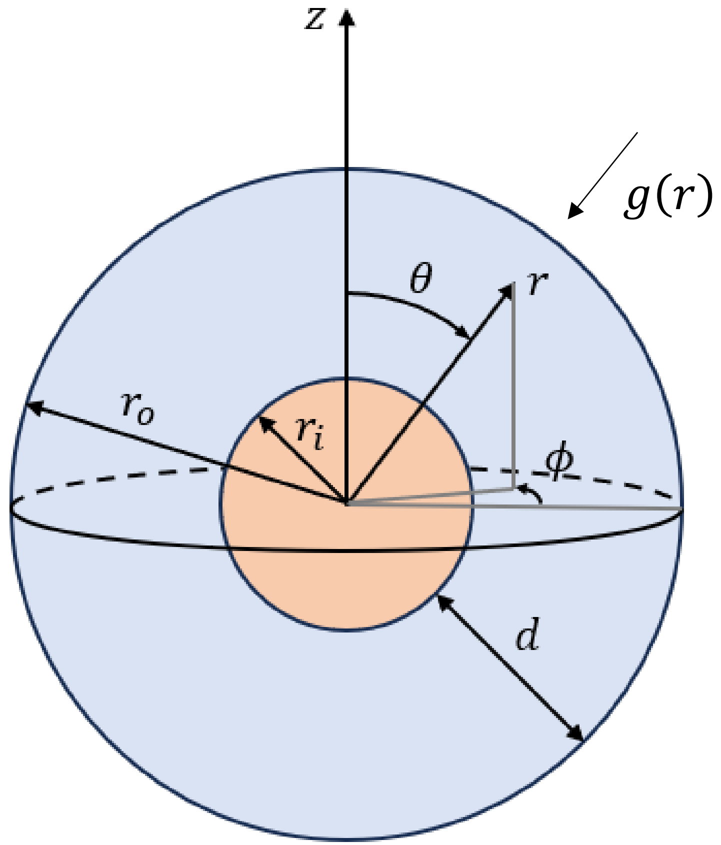


Convection in Jupiter’s molecular layer



Convection in Saturn’s molecular layer

Model for thermal convection



- Governing equations (under Oberbeck-Boussinesq approximation):

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_0} \nabla p + \alpha g (T - T_{ref}) \boldsymbol{e_r} + \nu \nabla^2 \boldsymbol{u},$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa \nabla^2 T.$$

- $u(r, \theta, \phi, t)$: Velocity field, $T(r, \theta, \phi, t)$: Temperature field.
- $p(r, \theta, \phi, t)$: Reduced pressure field (including hydrostatic pressure).

- Physical control parameters:

$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$: thermal expansion, $g(r)$: gravity,

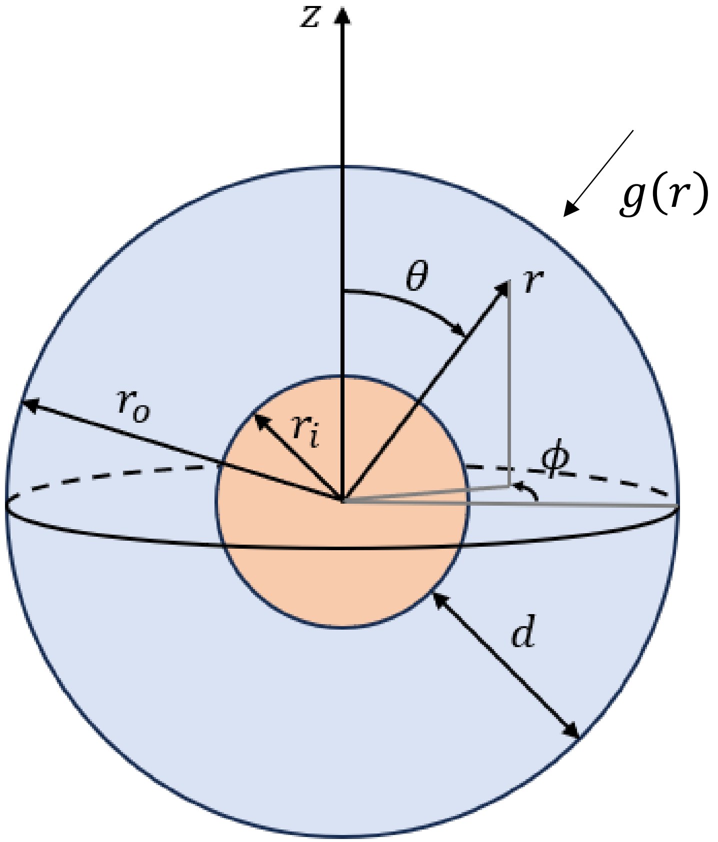
ν: kinematic viscosity, κ: thermal diffusivity,

- Key assumptions in Oberbeck-Boussinesq approximation:

Incompressibility: $u \ll v_c$, Small vertical scale: $L \ll v_c^2/g$, Small temperature difference: $\rho = \rho(T_0)[1 - \alpha(T - T_0)]$,

Also, constant physical parameters: ν, κ, α.

Model for thermal convection



- Non-dimensionalize the governing equation (MagIC):

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{Ra}{Pr} g T \boldsymbol{e_r} + \nabla^2 \boldsymbol{u},$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T.$$

- Length scale: $d = r_o - r_i$,
- Velocity scale: ν/d,
- Time scale: d^2/ν,
- Pressure scale: $\rho_0(\nu/d)^2$,
- Gravity scale: $g(r_o)$
- Temperature scale: $\Delta T = T_{hot} - T_{cold}$.

- Input parameters:

$$Ra = \frac{buoyancy\ term}{viscous\ term} = \frac{\alpha g_o \Delta T d^3}{\nu\kappa}, \qquad Pr = \frac{\nu}{\kappa}, \qquad \eta = \frac{r_i}{r_o}.$$

- Boundary conditions:

Thermal B.C.: $T_i = 1, \qquad T_o = 0.$

$$\frac{\partial T}{\partial r}(r_i) = \frac{\partial T}{\partial r}(r_o) = Const$$

Mechanical B.C.:

No-slip: $u_i = u_o = 0$

Stress-free: $\sigma_{r\theta} = \sigma_{r\phi} = 0$

$$\sigma_{\varphi r} = \mu\left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r}\right)$$

$$\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),$$

Response parameters

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T.$$

Take the temporal and horizontal average:

$$\frac{d}{dr}\boxed{\left(\overline{\langle u_r T \rangle_s} - \frac{1}{Pr}\frac{d\overline{\langle T \rangle_s}}{dr}\right)} = 0$$

Dimensionless total heat flux

- Response parameters:

Where, $\nabla^2 T_c = 0.$

$$Nu = \frac{total\ heat\ flux}{conductive\ heat\ flux} = \frac{\overline{\langle u_r T \rangle_s} - \frac{1}{Pr}\frac{d\overline{\langle T \rangle_s}}{dr}}{-\frac{1}{Pr}\frac{dT_c}{dr}},$$

$$Re = \frac{u_{rms} L}{\nu} = \overline{\sqrt{\langle |\boldsymbol{u}|^2 \rangle}}.$$

$$\bar{f} = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} f\,\mathrm{d}t \quad \langle f \rangle = \frac{1}{V}\int_V f(r,\theta,\phi)\,\mathrm{d}V \quad \langle f \rangle_s = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(r,\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Exact relations in Boussinesq equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \frac{Ra}{Pr} g\,T\,\boldsymbol{e}_r + \Delta\boldsymbol{u}$$

$$\boldsymbol{u}\cdot\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\boldsymbol{u}\cdot\nabla p + \boxed{\frac{Ra}{Pr} gTu_r + \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}}$$

$$-\overline{\langle \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle} = \overline{\left\langle \frac{Ra}{Pr} gTu_r \right\rangle}$$

$g(r)\sim 1/r^2$

$$\boxed{-\overline{\langle \boldsymbol{u}\cdot\nabla^2\boldsymbol{u}\rangle} = \varepsilon_U = \frac{3}{1+\eta+\eta^2}\frac{Ra}{Pr^2}(Nu-1)}$$

Kinetic energy dissipation rate

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \frac{1}{Pr}\Delta T$$

$$T\left(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T\right) = \boxed{T\frac{1}{Pr}\nabla^2 T}$$

$$\overline{\langle (\nabla T)^2\rangle} = \overline{\left\langle \nabla^2\left(\frac{T^2}{2}\right)\right\rangle}$$

$$\boxed{\overline{\langle (\nabla T)^2\rangle} = \varepsilon_T = \frac{3\eta}{1+\eta+\eta^2} Nu}$$

Thermal dissipation rate

The onset of convection

$$Ra = \frac{\alpha g_o \Delta T d^3}{\nu\kappa} > Ra_c, \quad Nu > 1, \quad Re > 1.$$

Determine Ra_c:

- Linear Stability Analysis (LSA): Solve the eigenvalue problem of the linearized equations.
- Direct time integration of the linearized equations: Integrate the linearized equation in time and measure the growth rate of each mode (MagIC can do).
- Ra_c for spherical shells:

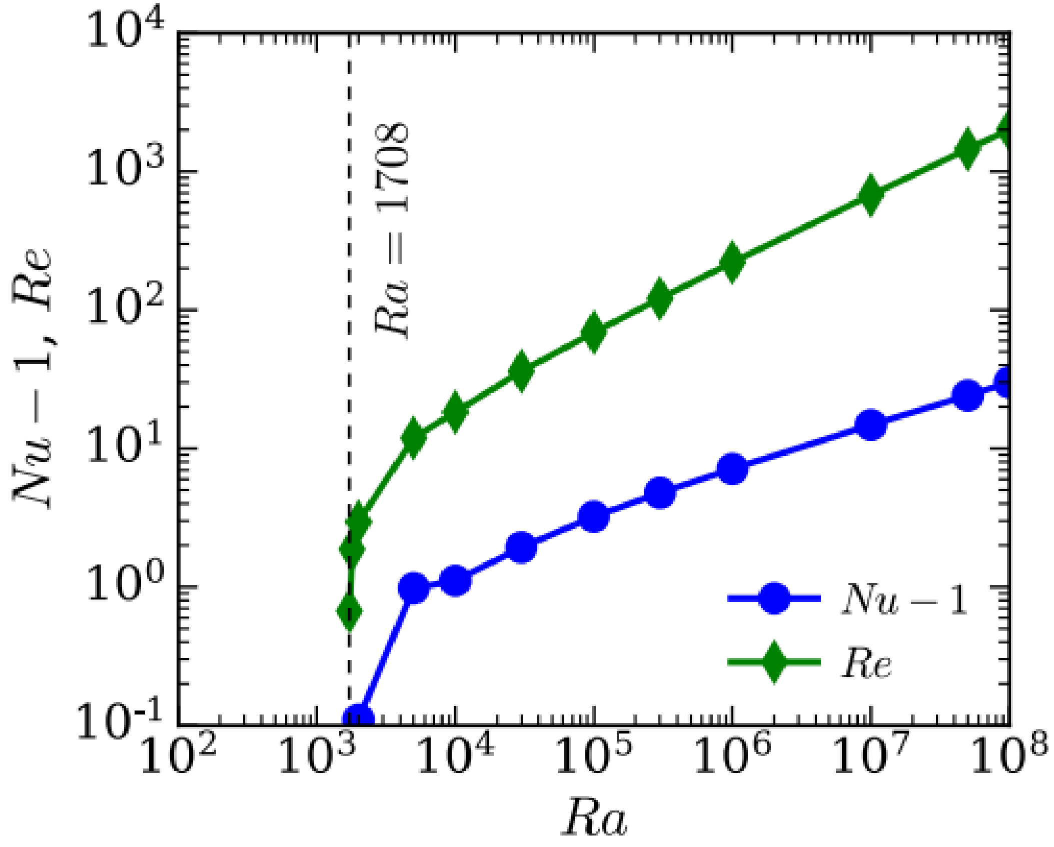
  $Ra_c \approx 575$ for no-slip boundaries with $\eta = 0.2$.

  $Ra_c \approx 1375$ for no-slip boundaries with $\eta = 0.8$.

- Ra_c for horizontal plates[1]:

  $Ra_c \approx 1708$ for no-slip boundaries.

  $Ra_c \approx 647$ for stress free boundaries.



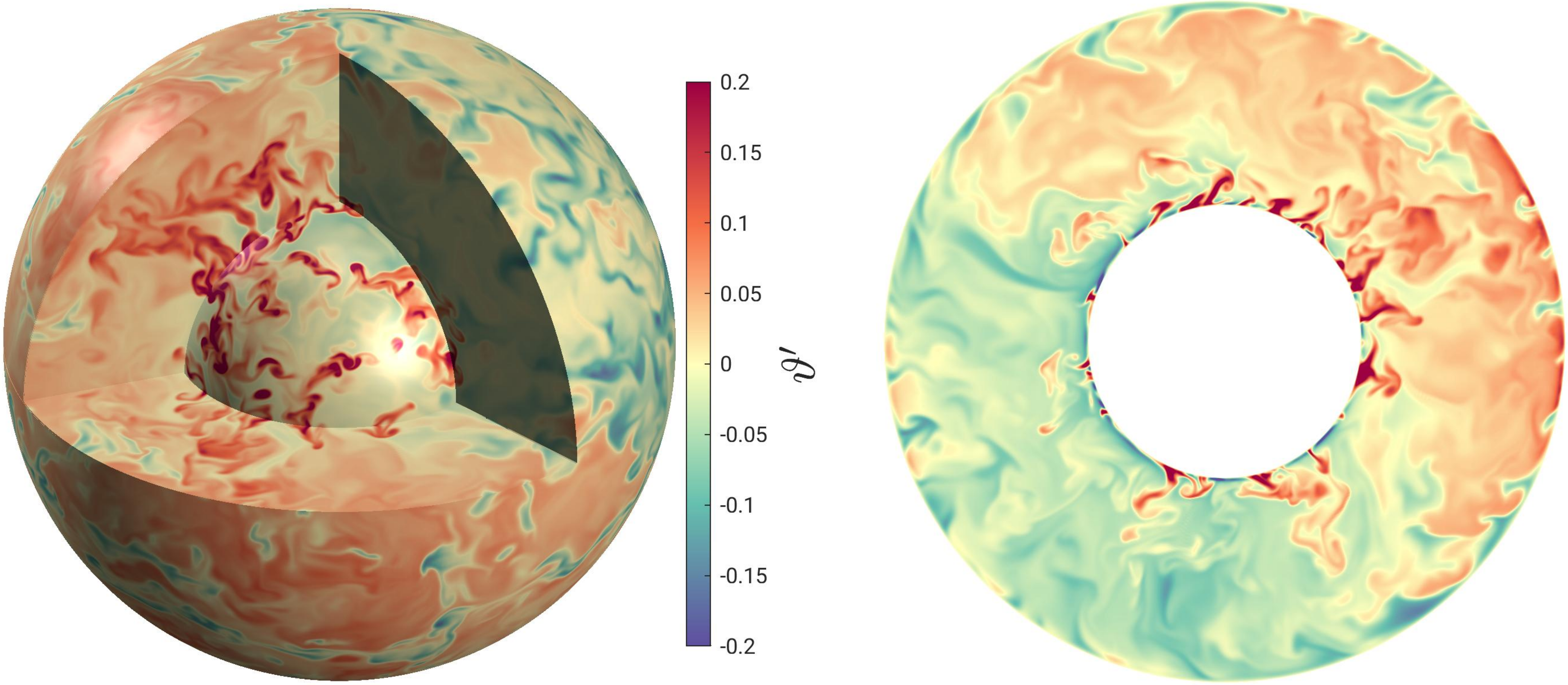
Simulations in periodic box with $Pr = 1$[2]

[1] Chandrasekhar S. Hydrodynamic and hydromagnetic stability[M]. Courier Corporation, 2013.

[2] Mohammad Anas. Effect of Rotation and Confinement on Heat Transfer in Rayleigh–Bénard Convection (PhD thesis), IIT, 2024.

Flow structures in turbulent convection

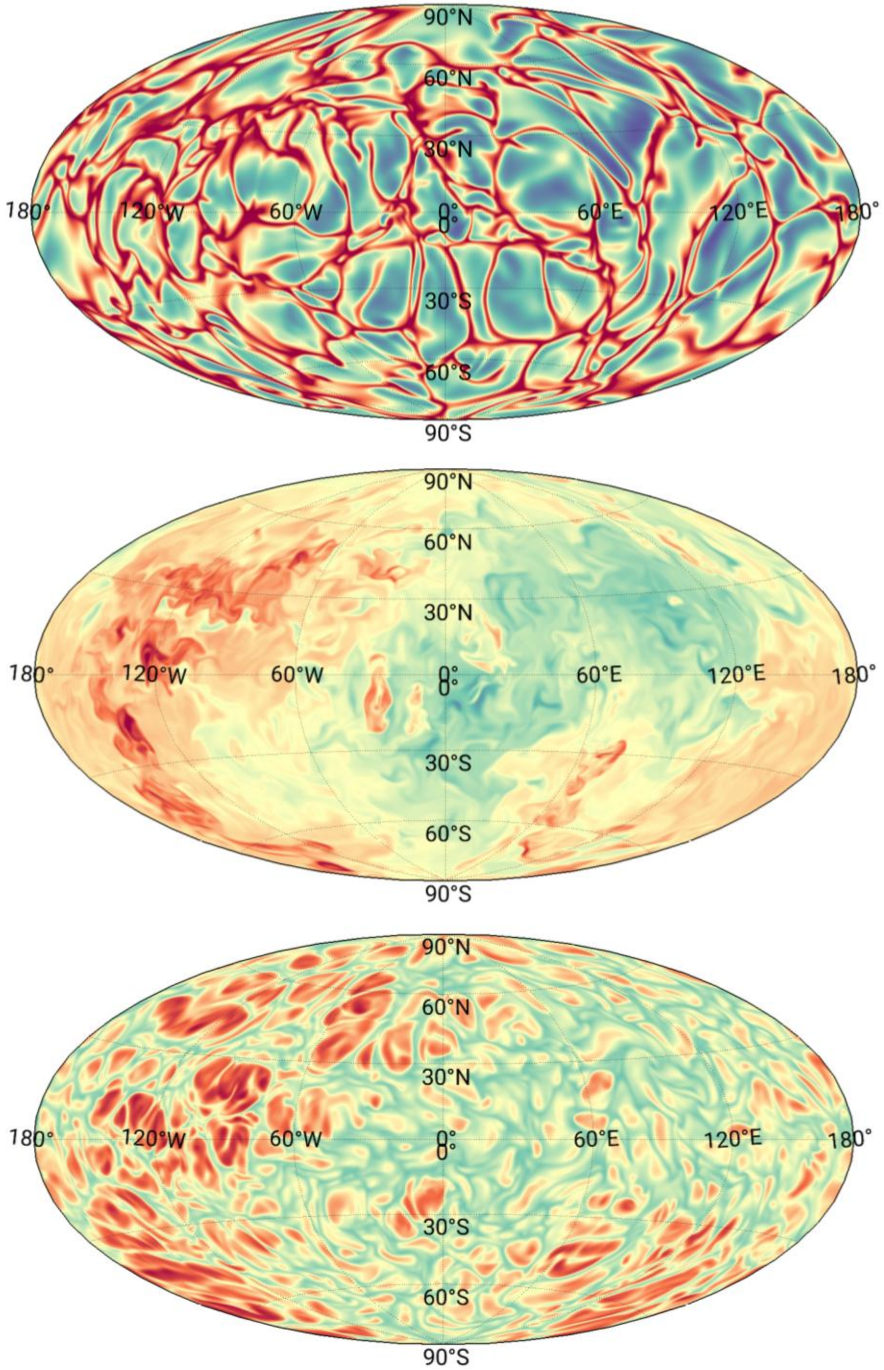
No-slip and isothermal B.C.



$Pr = 1, \eta = 0.4, Ra = 3 \times 10^7$

$Ra_c \approx 825$

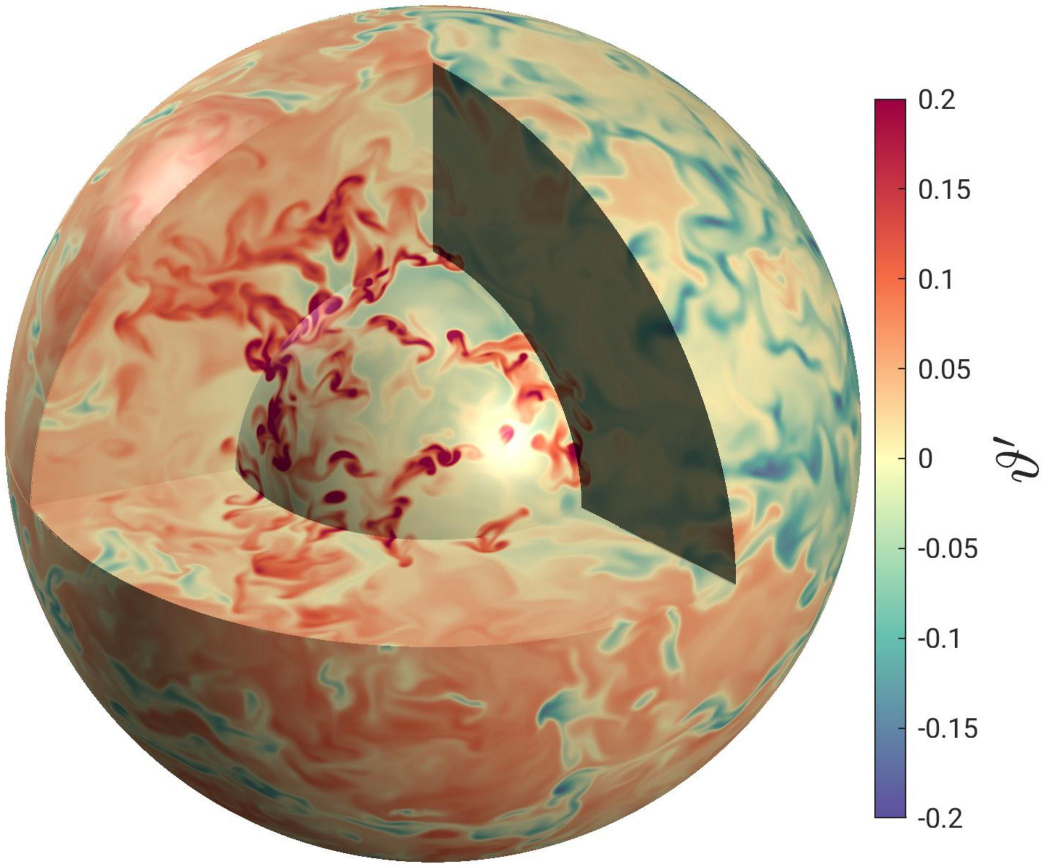
Sheet-like plumes near the boundary



$r = r_i + \lambda_\vartheta^i$

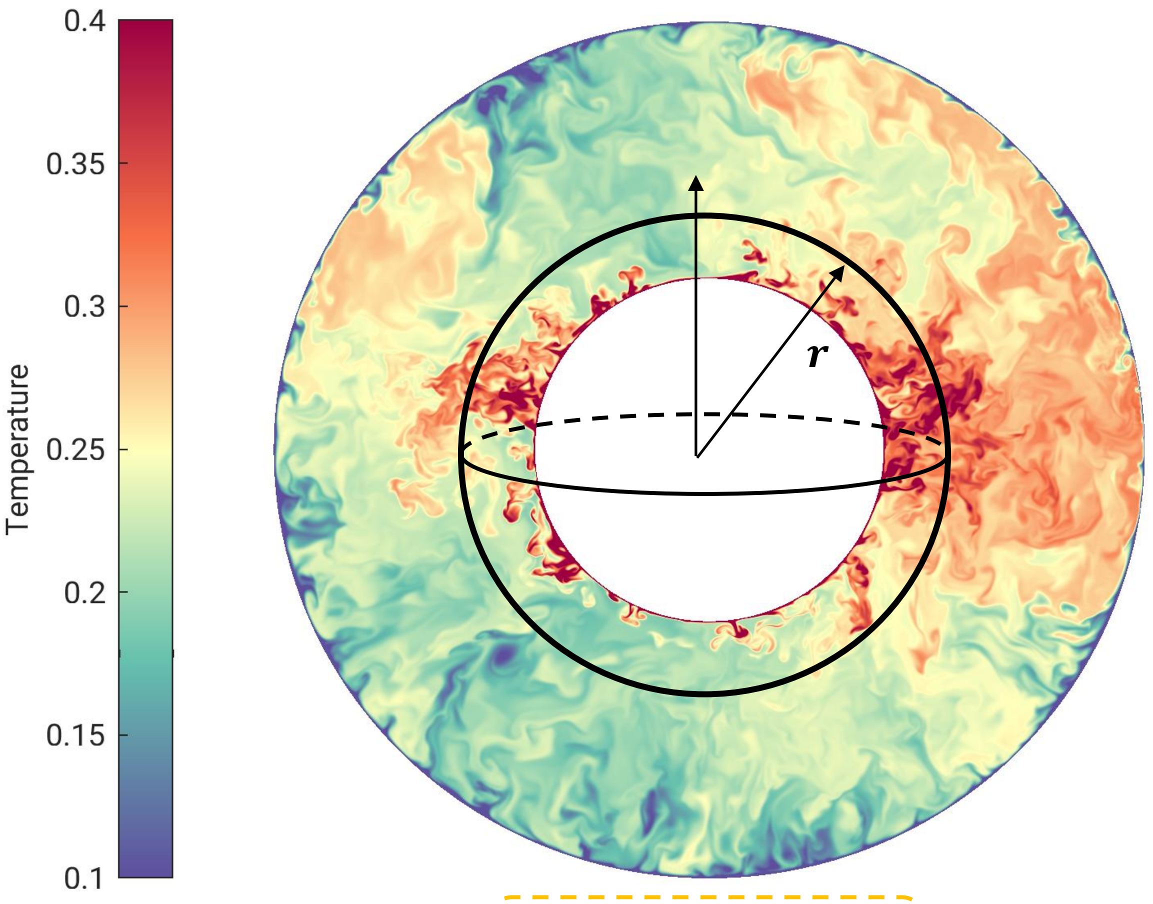
$r = r_{mid}$

$r = r_o - \lambda_\vartheta^o$



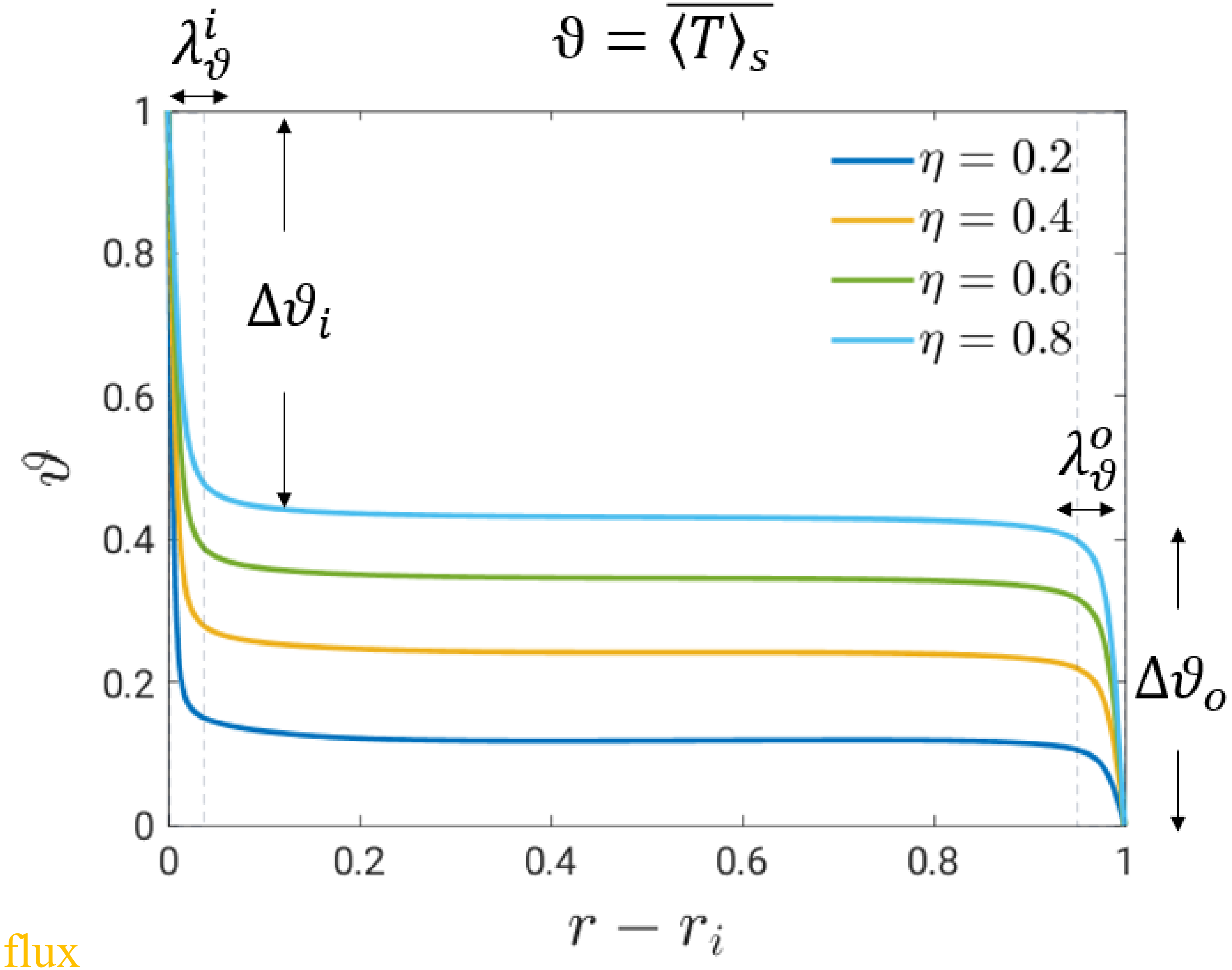
Temperature fluctuation $T' = T - \overline{\langle T \rangle_s}$

Conduction dominate BL and turbulent dominate bulk



$$Nu = \frac{\overline{\langle u_r T\rangle_s} - \frac{1}{Pr}\frac{d\overline{\langle T\rangle_s}}{dr}}{-\frac{1}{Pr}\frac{dT_c}{dr}}$$

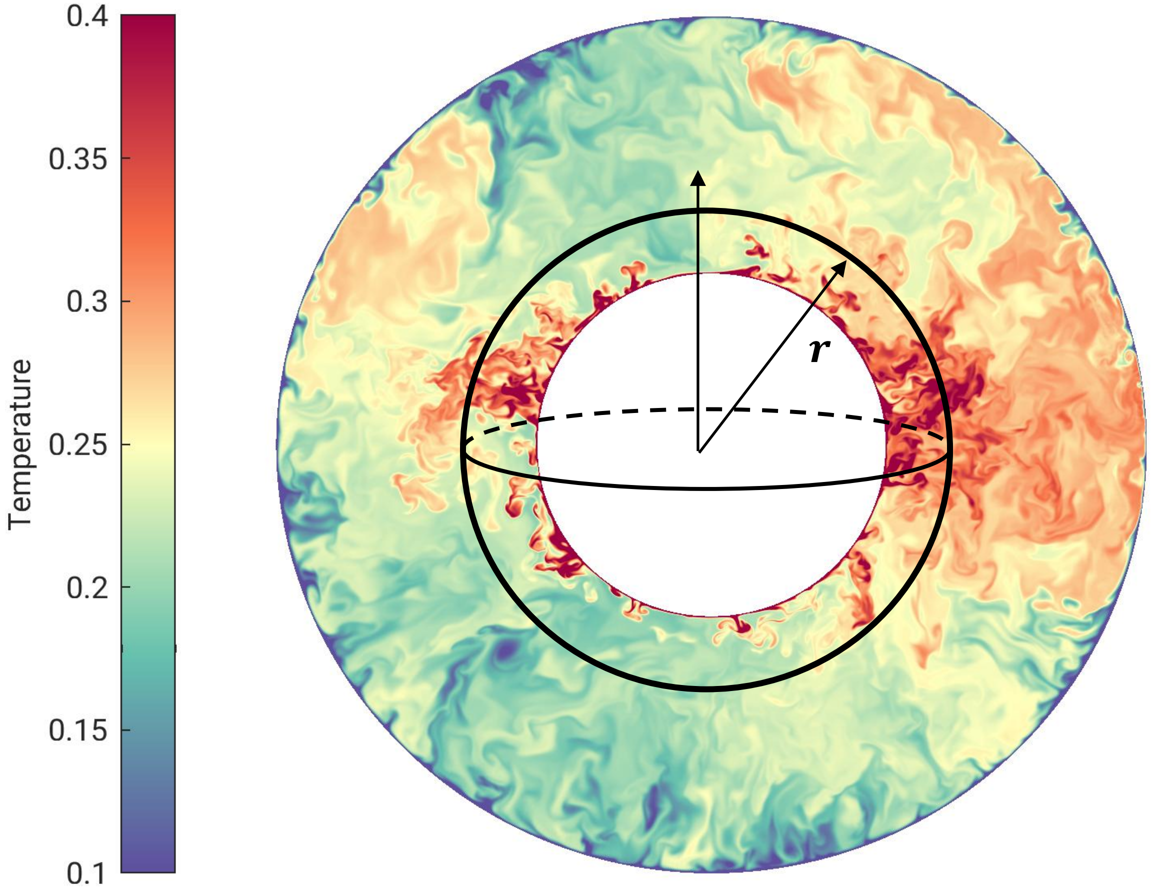
Total heat flux

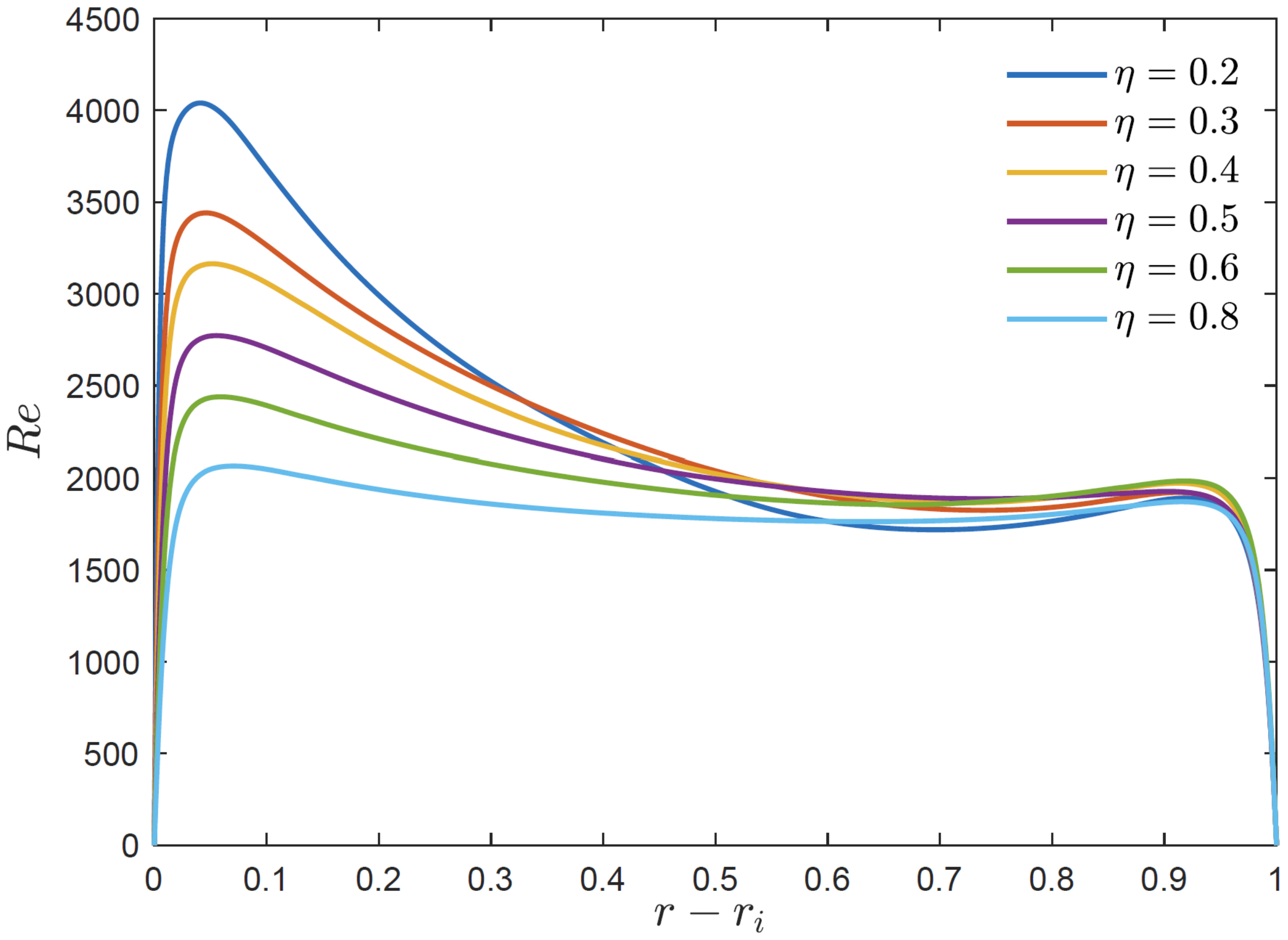


$$\bar{f} = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} f \, \mathrm{d}t \qquad \langle f\rangle_s = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(r,\theta,\phi)\sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

Low Re BL and highly turbulent bulk

$$Re(r) = \sqrt{\overline{\langle u_r^2 + u_\theta^2 + u_\phi^2 \rangle_s}}$$





Use BL models (e.g. Prandtl–Blasius BL) to describe near wall region, turbulence model (e.g. K41) for bulk region.

$$\bar{f} = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} f \, \mathrm{d}t \qquad \langle f \rangle_s = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(r, \theta, \phi) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

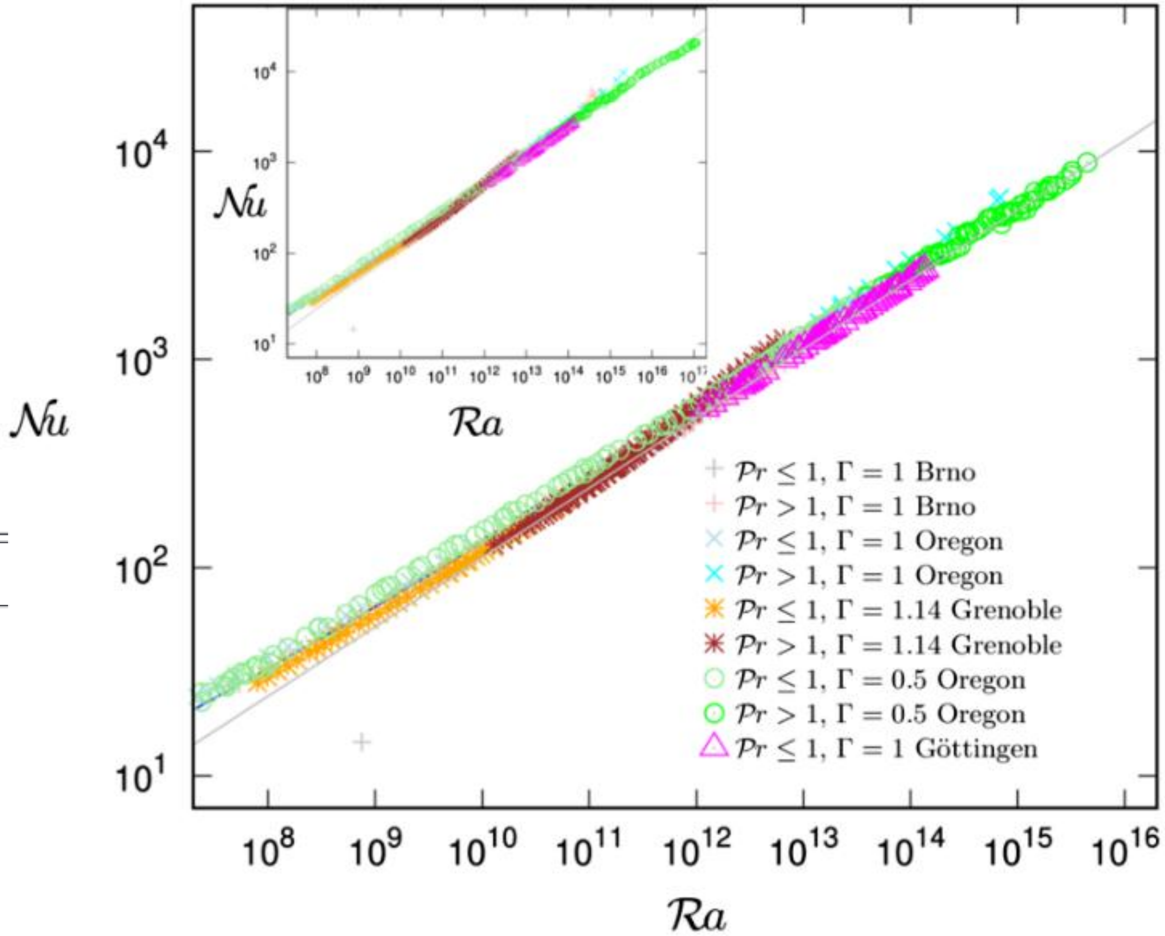
Nu, Re scaling in RBC

$$Nu \sim Ra^{\gamma_{Nu}}\, Pr^{\alpha_{Nu}}$$

$$Re \sim Ra^{\gamma_{Re}}\, Pr^{\alpha_{Re}}$$

Reference	Pr and Ra range	γ_{Nu}	α_{Nu}	γ_{Re}	$\gamma_{\mathrm{Re_{fluct}}}$	α_{Re}
Davis (1922a, 1922b)	Ra small	1/4				
Malkus (1954)		1/3				
Kraichnan (1962)	Ra ultimate, Pr<0.15	1/2	1/2	1/2		–1/2
	Ra ultimate, 0.15<Pr≤1	1/2	–1/4	1/2		–3/4
Spiegel (1971)	Ra ultimate	1/2	1/2	1/2		–1/2
Castaing *et al.* (1989)		2/7		1/2	3/7	
Shraiman and Siggia (1990)	Pr>1	2/7	–1/7	3/7		–5/7
Yakhot (1992)		5/19			8/19	
Zaleski (1998)		2/7				
Cioni *et al.* (1997)	Pr<1	2/7	2/7		3/7	–4/7

Power-law exponents for Nu and Re as functions of Ra and Pr predicted by theories[1].



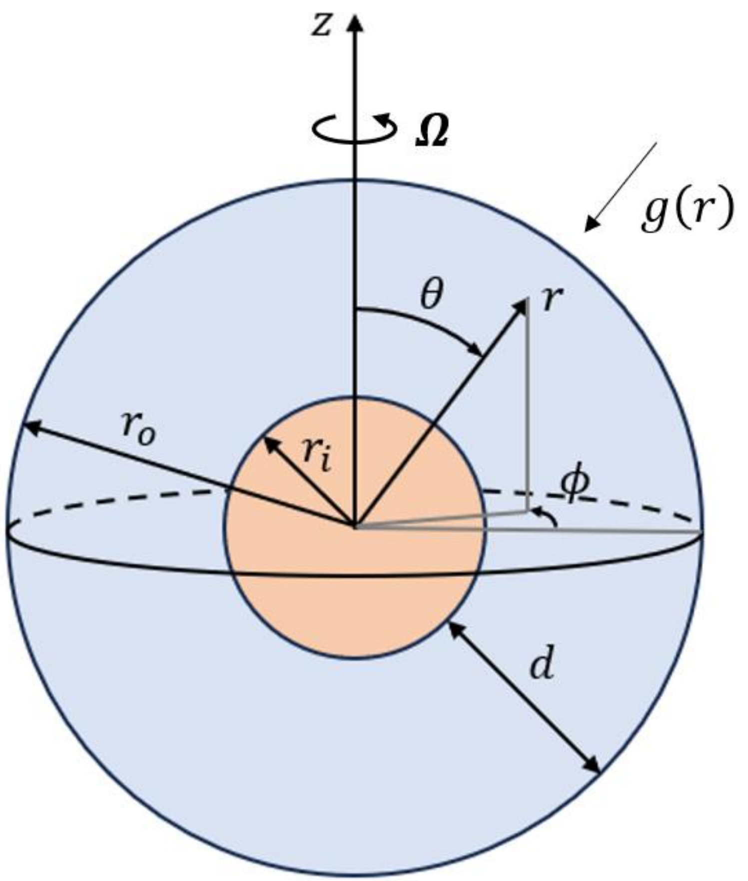
The ultimate regime: $Nu \propto Ra^{1/2}$ for $Ra > Ra_t (\sim 10^{13})$

The existence of the ultimate regime is debatable!

[1] Ahlers G, Grossmann S, Lohse D. Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection[J]. Reviews of modern physics, 2009, 81(2): 503-537.

[2] Detlef Lohse and Olga Shishkina: Ultimate Rayleigh-Benard Turbulence.

Rotating thermal convection



- Non-dimensionalized governing equation (MagIC):

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \boxed{\frac{2}{E_k} \boldsymbol{e_z} \times \boldsymbol{u}} = -\nabla p + \frac{Ra}{Pr} g T \boldsymbol{e_r} + \nabla^2 \boldsymbol{u},$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \frac{1}{Pr} \nabla^2 T.$$

- Length scale: $d = r_o - r_i$,
- Velocity scale: ν / d,
- Time scale: d^2/ν,
- Pressure scale: $\rho_0 (\nu/d)^2$,

- Input parameters:

$$E_k = \frac{viscous\ force}{Coriolis\ force} = \frac{\nu}{\Omega d^2}, \quad Ra = \frac{\alpha g_o \Delta T d^3}{\nu \kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad \eta = \frac{r_i}{r_o}.$$

$$Ro_c = \frac{buoyancy\ force}{Coriolis\ force} = \sqrt{\frac{\alpha g_o \Delta T}{\Omega^2 d}} = \sqrt{\frac{Ra}{Pr}} E_k, \quad \boxed{Ro = \frac{U}{2\Omega d}}$$

- Boundary conditions:

Thermal B.C.: $T_i = 1, \quad T_o = 0.$

$$\frac{\partial T}{\partial r}(r_i) = \frac{\partial T}{\partial r}(r_o) = Const$$

Mechanical B.C.:

No-slip: $u_i = u_o = 0$

Stress-free: $\sigma_{r\theta} = \sigma_{r\phi} = 0$

$$\sigma_{\varphi r} = \mu \left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} \right)$$

$$\sigma_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right),$$

Constraint of Rotation on Convection

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \alpha g (T - T_{ref})\boldsymbol{e_r} + \nu \nabla^2 \boldsymbol{u},$$

Consider $|(\vec{u} \cdot \nabla)\vec{u}| \ll |\vec{u}|$, $\frac{\partial \vec{u}}{\partial t} = 0$, and inviscid flow $\nu = 0$

$$\Downarrow$$

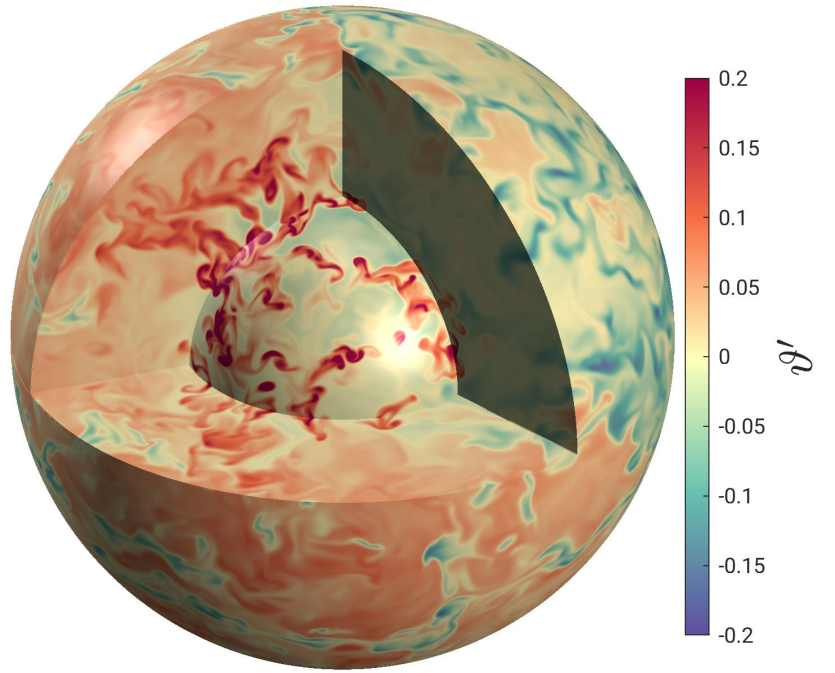
$$2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \alpha g \vartheta \boldsymbol{e_r}$$

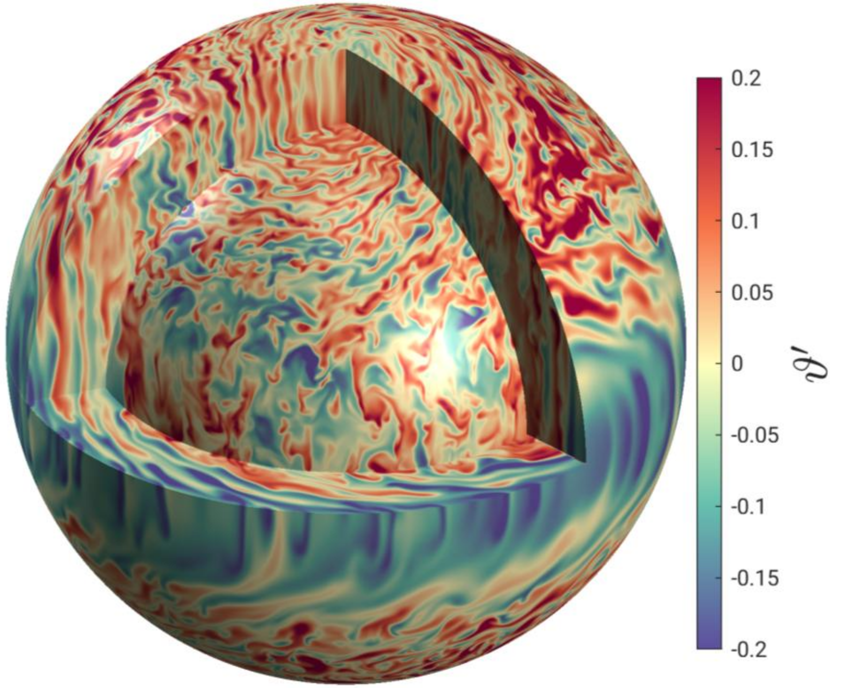
$$e_z \cdot \nabla \times \quad \Downarrow$$

$$2\Omega \frac{\partial}{\partial z}(\boldsymbol{e_z} \cdot \boldsymbol{u}) = 0$$

- Taylor-Proudman theorem:

$$\frac{\partial}{\partial z}(\boldsymbol{u}) = 0$$





Important scalings in Rotating thermal convection

- The critical Ra_c in rotating thermal convection[1]:

For moderate and large Pr (e.g $Pr > 10^{-2}$):

$$Ra_c = \frac{\alpha g_o \Delta T d^3}{\nu\kappa} \sim \left(\frac{1}{Ek}\right)^{4/3}, \qquad \frac{L_h}{d} \sim (E_k)^{1/3}$$

For small enough Pr (e.g $Pr < 10^{-2}$):

$$Ra_c = \frac{\alpha g_o \Delta T d^3}{\nu\kappa} \sim \left(\frac{1}{Ek}\right)^{1/2}, \qquad \frac{L_h}{d} \sim O(1)$$

- Ekman boundary layer

$$2\boldsymbol{\Omega} \times \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u}$$

$$\delta_E \approx \left(\frac{2\nu}{\Omega}\right)^{\frac{1}{2}}, \qquad \frac{\delta_E}{d} \approx \left(\frac{2\nu}{\Omega \mathrm{d}^2}\right)^{\frac{1}{2}} \sim (E_k)^{\frac{1}{2}}$$

$$\delta_\nu \approx \frac{ad}{Re^{1/2}}, \qquad Re = \frac{Ud}{\nu},$$

$$\Rightarrow \quad \frac{\delta_E}{\delta_\nu} \approx \frac{\sqrt{2\nu/\Omega}}{ad/\sqrt{Re}} \sim \sqrt{\frac{U}{2\Omega d}} = \sqrt{Ro}$$

[1] Zhang K, Liao X. Theory and modeling of rotating fluids: convection, inertial waves and precession[M]. Cambridge University Press, 2017.

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THANKS

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