An Introduction to Dynamos

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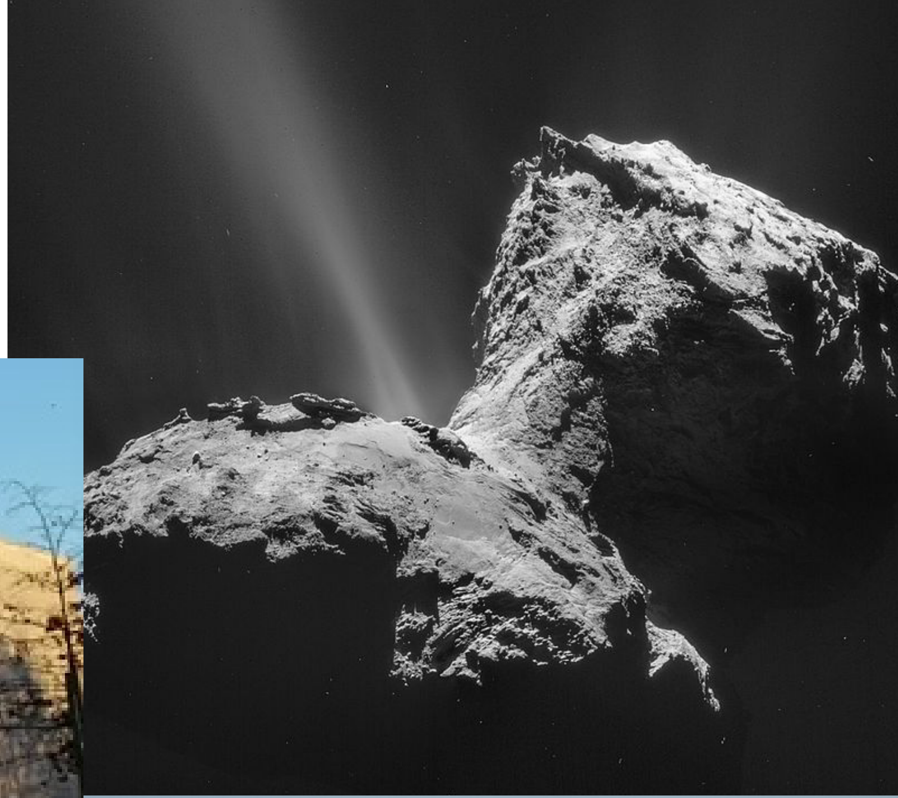


- on the North Campus in Göttingen since 2014
- departments: planetary science, sun, stellar interiors
- fundamental solar system science with space missions, laboratory work, computer simulations, theory

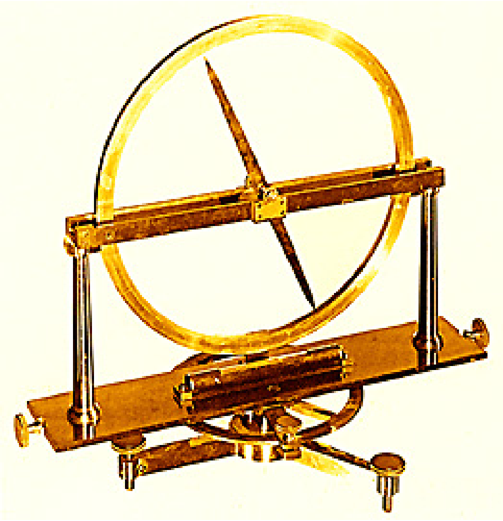
Overview

1.1 Magnetic Fields are Useful

- Magnetic fields have been of interest for a long time already!

1.2 Göttingen Contribution

Carl Friedrich Gauß, 1777-1855



Inklinatorium

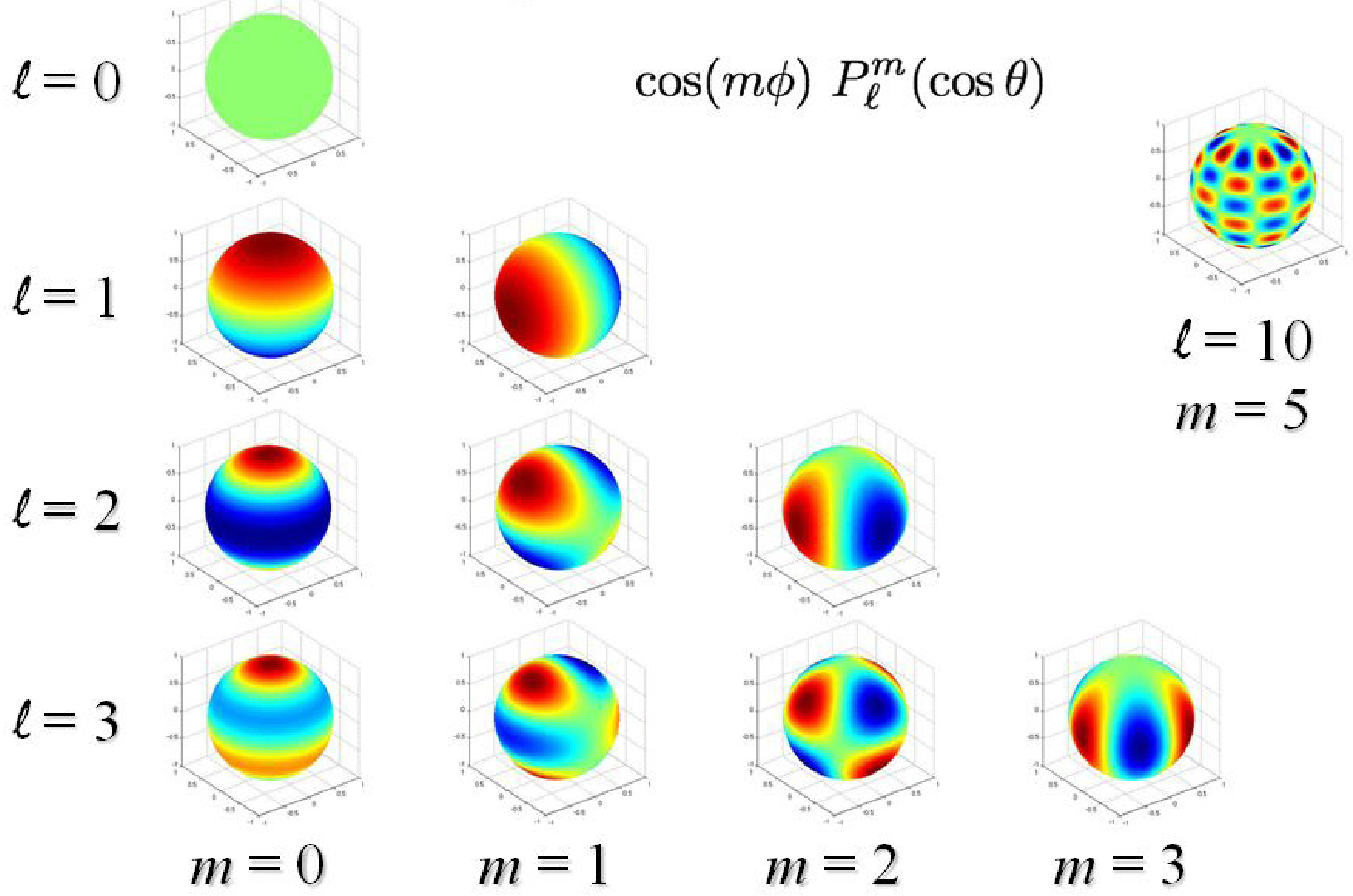
Gaußhaus

Determination of absolute value

- Fundamental steps in measuring and understanding Earth's magnetic field

1.3 Spherical Surface Harmonics

surface harmonics of degree l and order m
Legendre Polynomials P



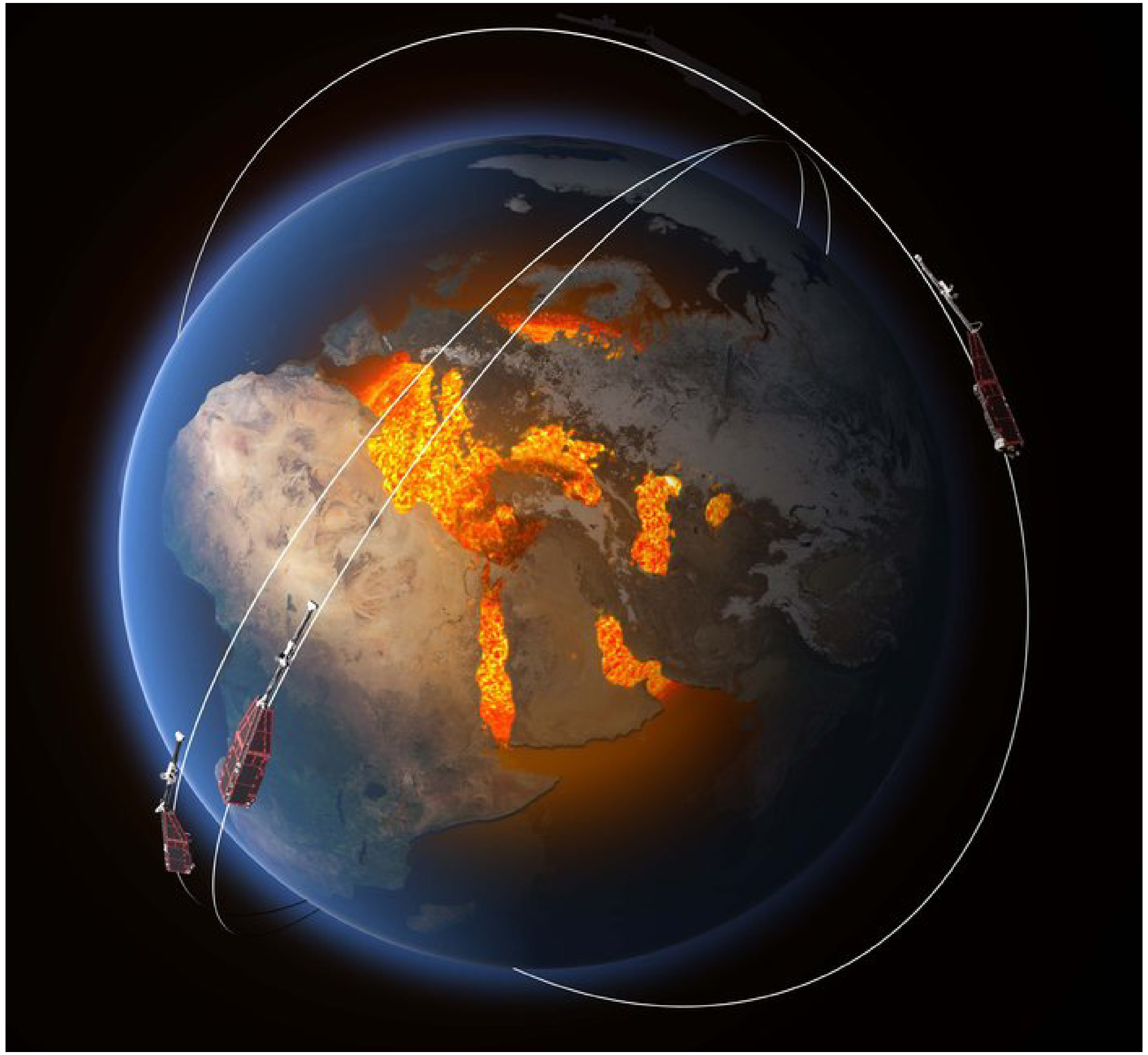
$$\vec{B} = -\nabla\, V$$

$$\nabla^2\, V = 0$$

$$V = r_s \sum_{l=1}^{\infty} \sum_{m=0}^{l} \left(\frac{r_s}{r}\right)^{l+1} \left(g_l^m\, cos(m\phi) + h_l^m\, \sin(m\phi) \right)\, \hat{P}_l^m(cos\theta)$$

- Describing magnetic field in a source free region.

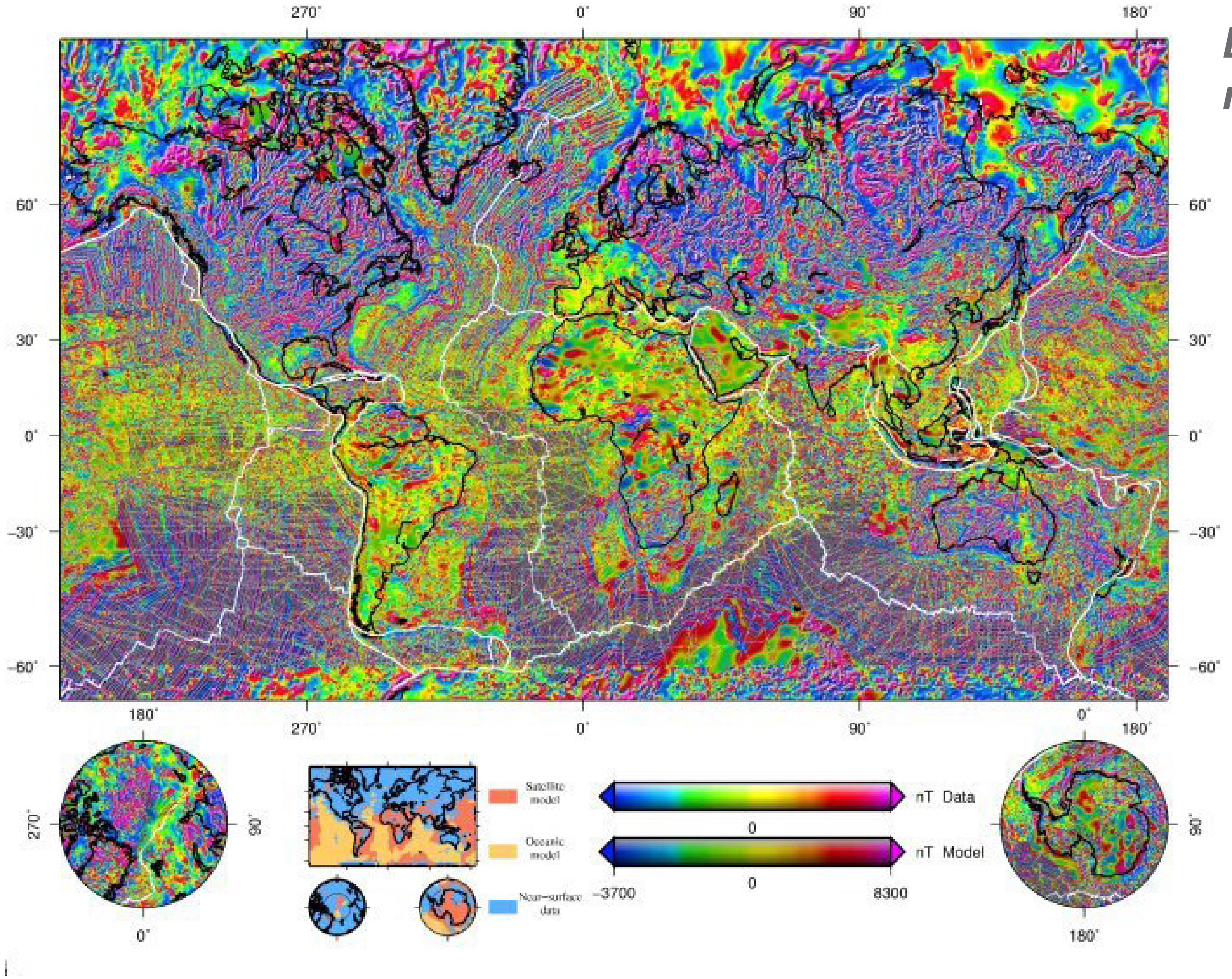
1.4 Measuring Earth’s Magnetic Field



ESA’s Swarm Mission

- Global coverage is key for a good spherical harmonic model.

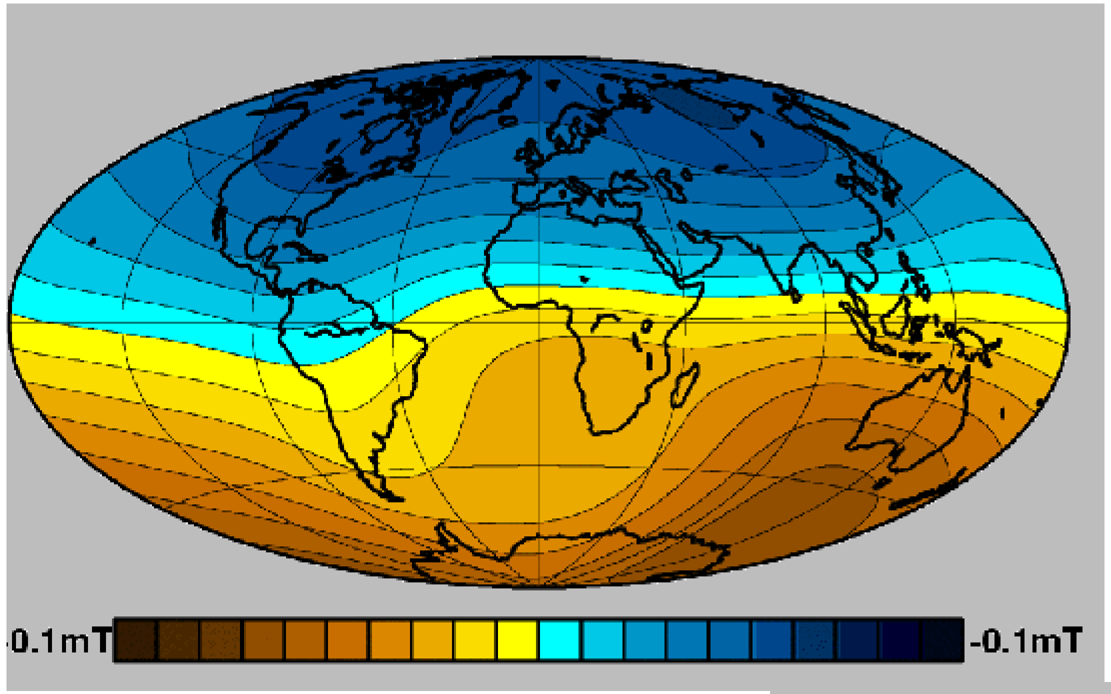
1.5 Crustal Field

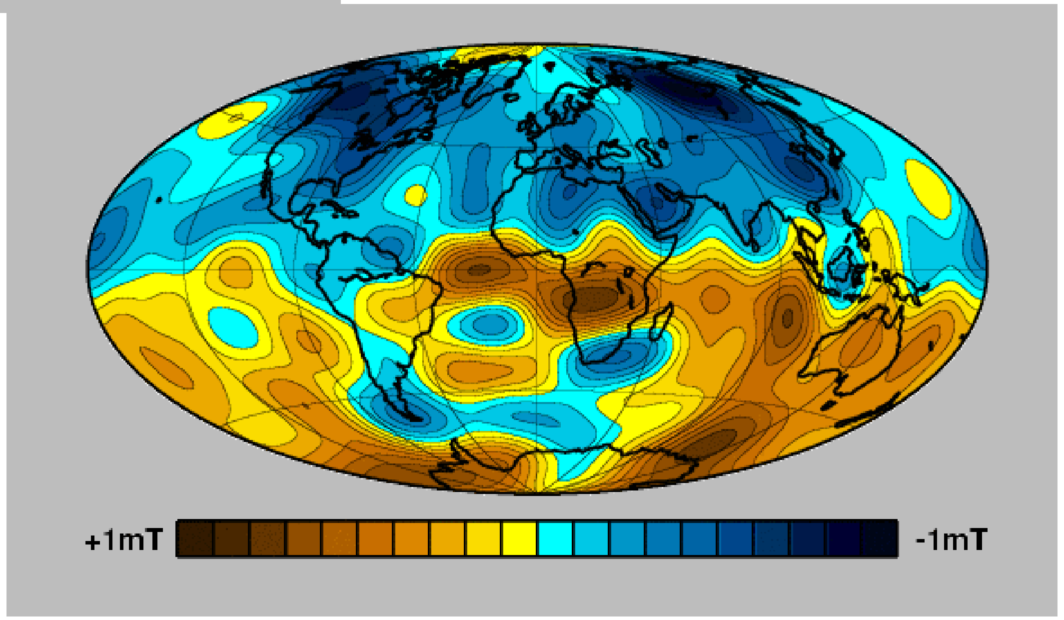


Earth's crustal magnetic field

- **Because of the crustal magnetisation we only know the interior field up to degree 14 (or so).**

1.6 Downward Continuation



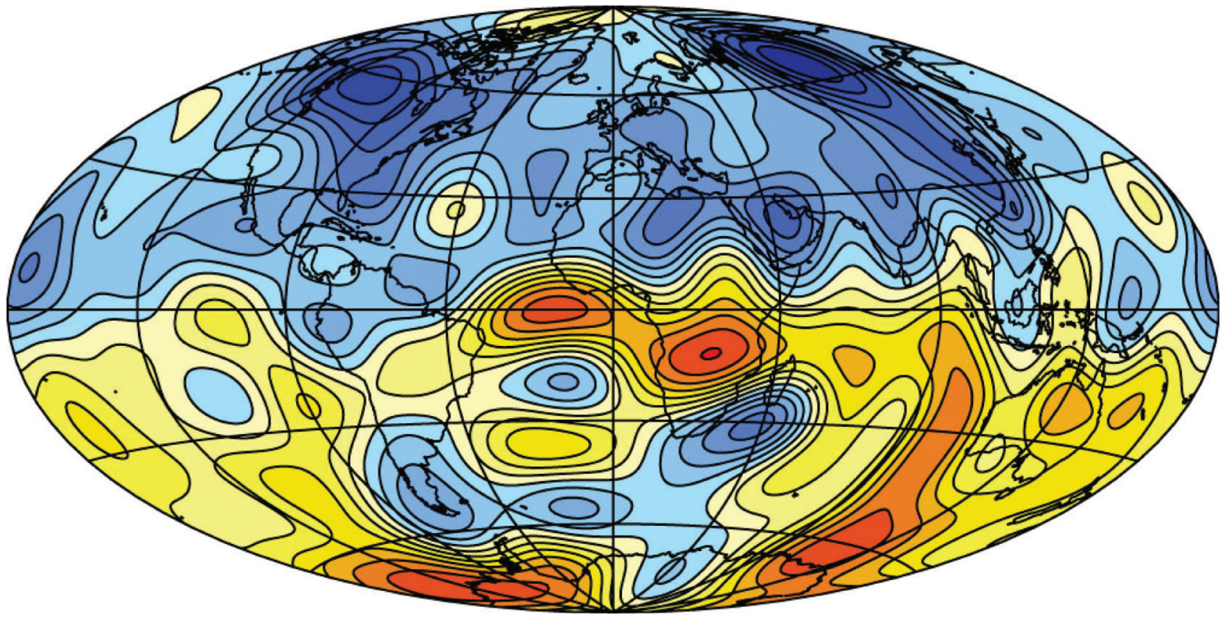


GUFM1 field for 1990 at Earth's surface and at the core-mantle boundary.

- Because of the crustal magnetisation we only know the interior field up to degree 14 (or so).

1.7 Comparison with Models

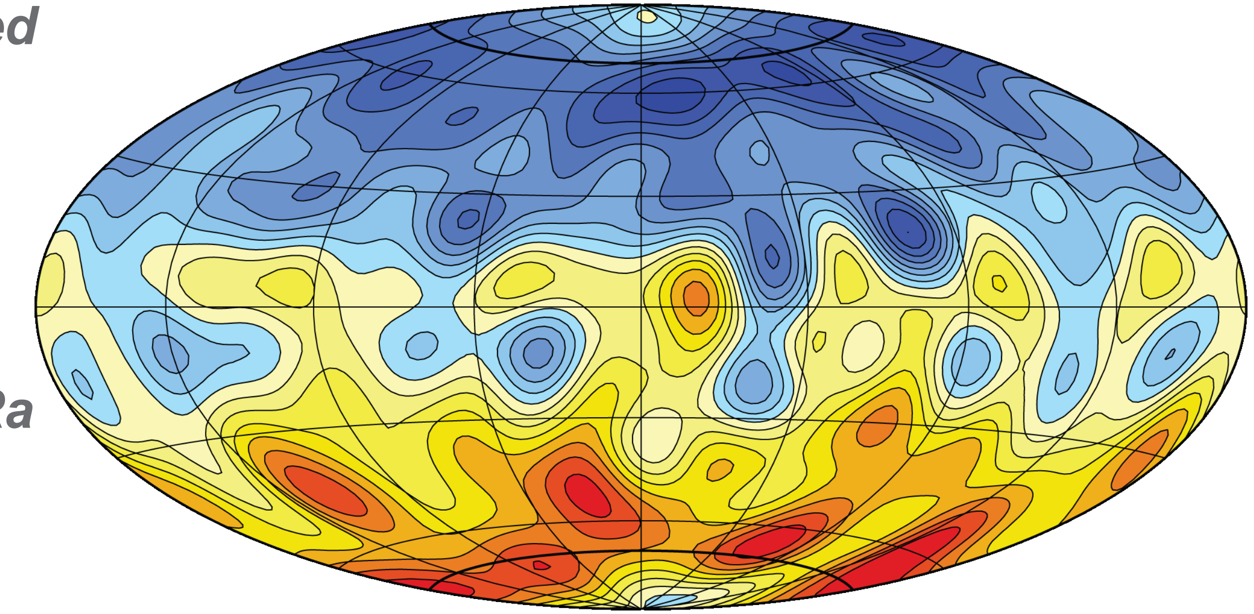
GUFM 1990



Best model, filtered

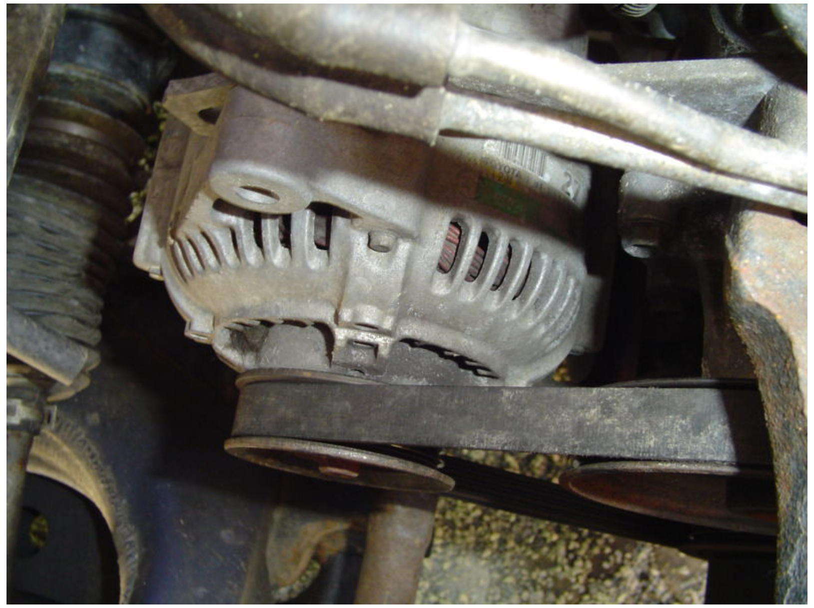
$E=3x10^{-5}$
Pm=1
$Ra=36\ Ra_c$

Low E and large Ra both required



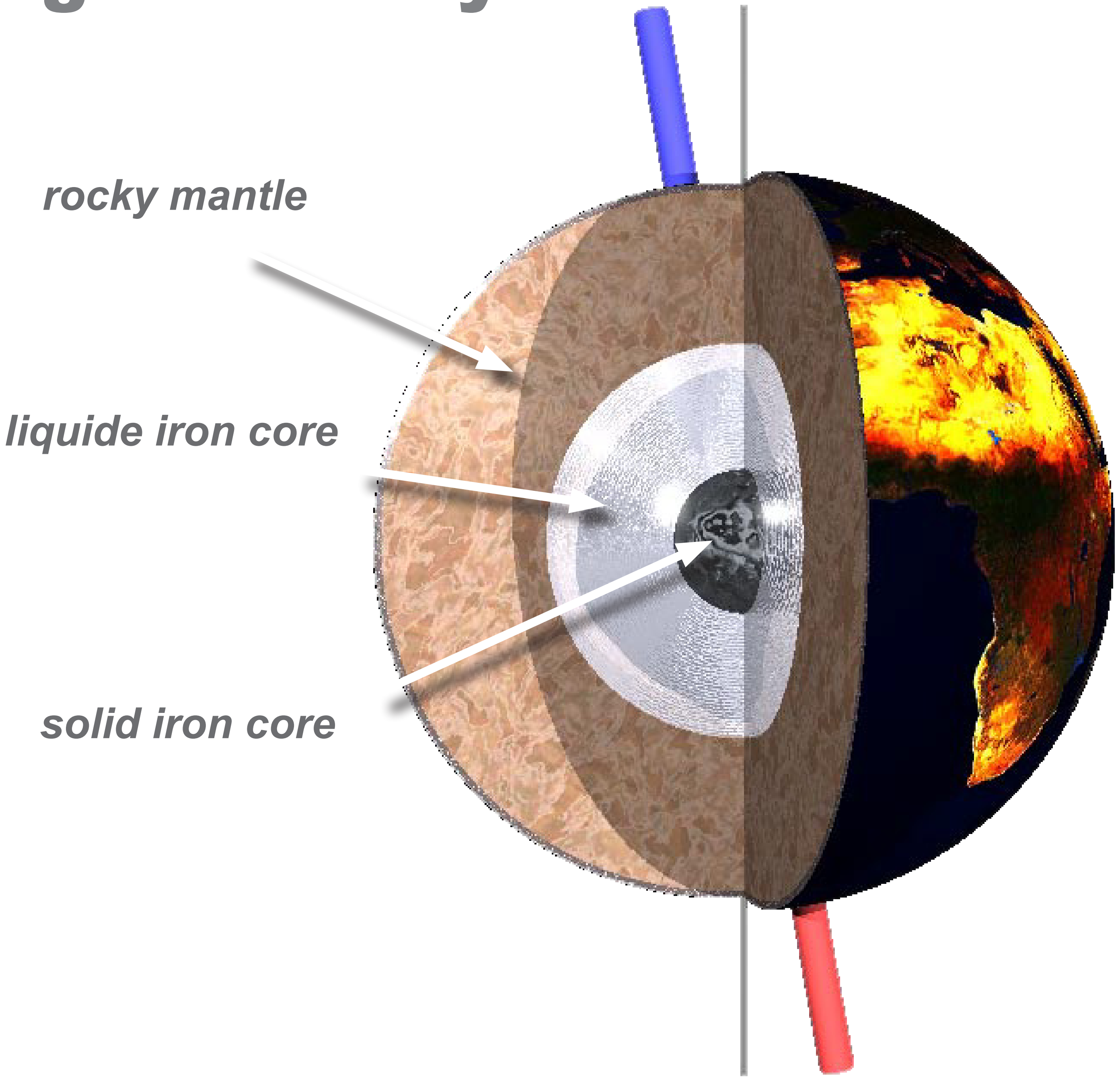
- Selected snapshot of my personal best model at degree 14.

2.1 Technical Dynamos



- Necessary ingredients for induction:
  1) electrically conducting material = copper cabels, coils, ..
  2) motion = rotation
  3) suitable geometry = complex combination of cabels, coils, ...

2.2 Earth’s Homogeneous Dynamo



- Earth’s magnetic field is created by a dynamo operating in the liquid core.
- Instead of a complex structure, this dynamo requires a complex flow!

2.3 Ohm's Law and Induction

- Dynamos are driven by magnetic induction.
- The essence of induction is formulated by Ohm's law:

$$\mathbf{j} = \sigma\left(\mathbf{U} \times \mathbf{B}_0 + \mathbf{E}\right)$$

with current density $\mathbf{j}$, electrical conductivity σ , flow $\mathbf{U}$, initial magnetic field $\mathbf{B}_0$, and electric field $\mathbf{E}$.

- Essential ingredients:

1) electrical conductor
2) motion
3) initial magnetic field

2.4 Deriving the Induction Equation

- Divide by σ and take the curl:

$$\nabla \times \left(\frac{1}{\sigma}\mathbf{j}\right) = \nabla \times (\mathbf{U} \times \mathbf{B}_0) + \nabla \times \mathbf{E}$$

- Use Ampere law and Maxwell-Faraday law of induction:

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} \qquad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

- Assume that the induced field can replace the initial field, use $\lambda = 1/(\sigma\mu_0)$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) - \nabla \times \lambda \nabla \times \mathbf{B}$$

- Assume homogeneous diffusivity λ and use $\nabla \cdot \mathbf{B} = 0$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}$$

2.5 Induction Equation

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}$$

- Ratio of induction to dissipation

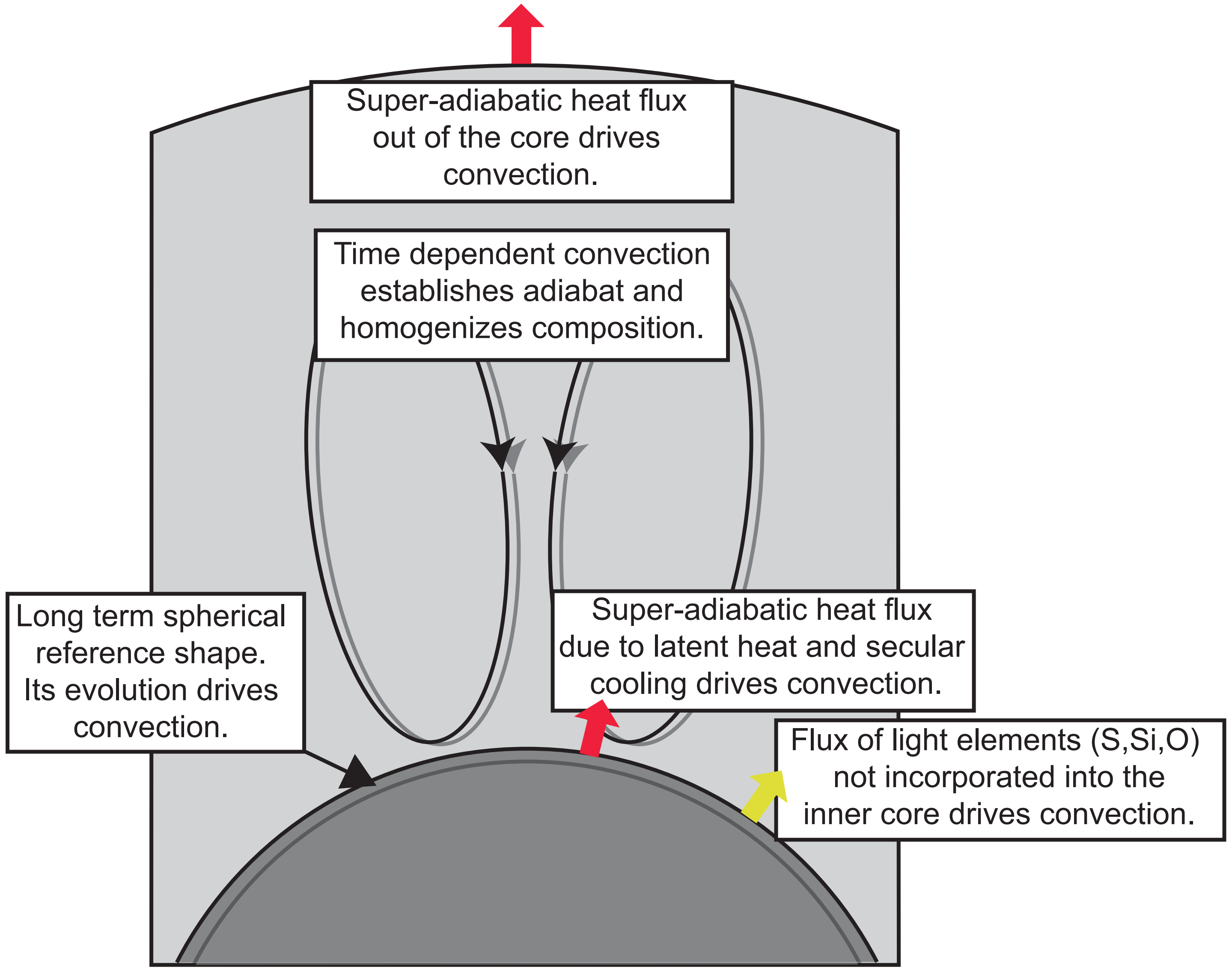
$$\frac{\text{induction}}{\text{dissipation}} = \frac{\nabla \times (\mathbf{U} \times \mathbf{B})}{\nabla \times \lambda \nabla \times \mathbf{B}} \approx \frac{UD}{\lambda}\frac{\ell}{D} = \mathrm{Rm}\frac{\ell}{D}$$

- Magnetic Reynolds number (VERY IMPORTANT!):

$$\mathrm{Rm} = \frac{UD}{\lambda}$$

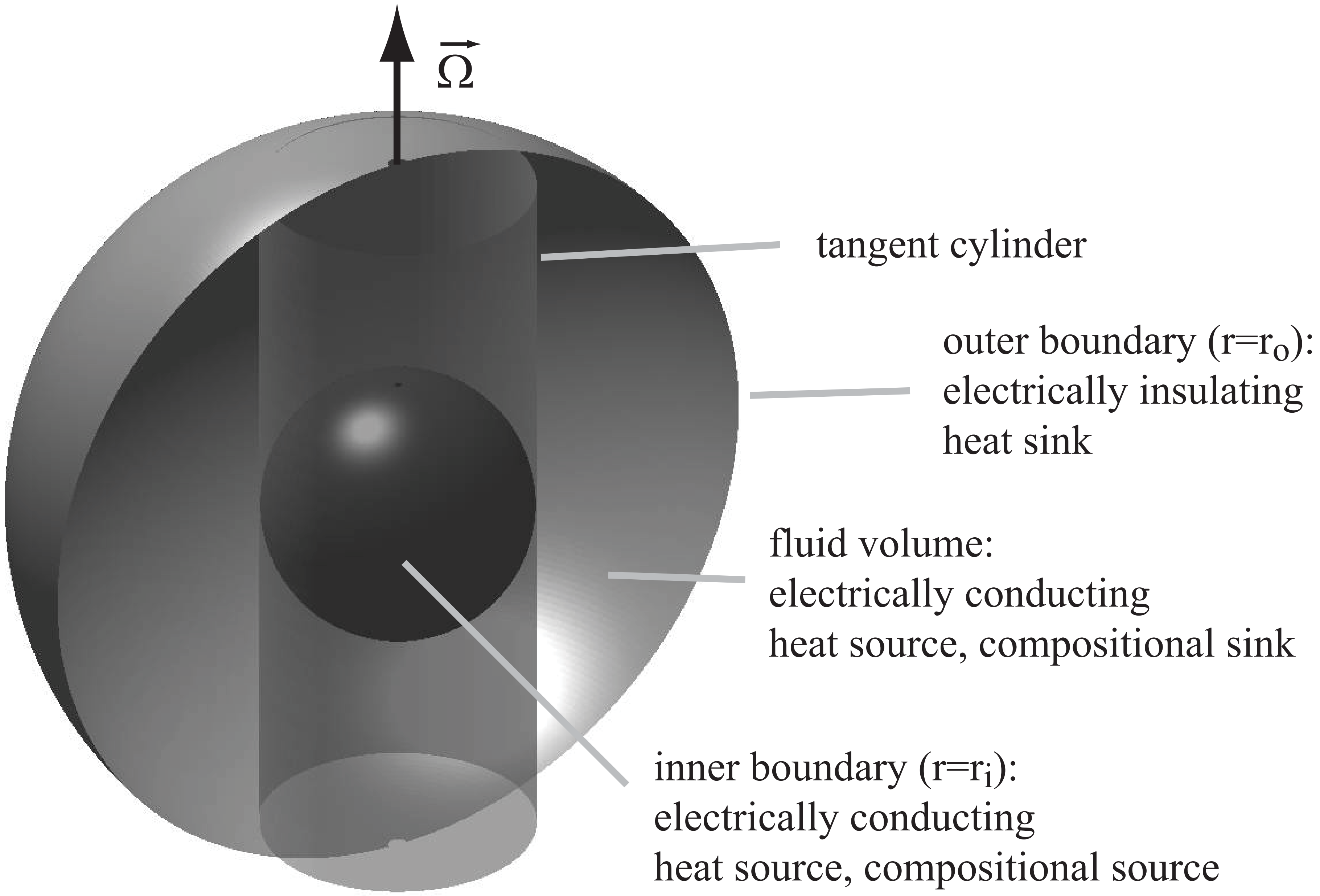
Earth	10^3
Jupiter	10^6
DNS	10^3
minimum	10^2

3.1 Driving the Dynamo



- Core convection is driven by the (super-adiabatic) heat flux and by light elements (O) emanating from the inner-core boundary.
- Earth's mantle imposes heat flux amplitude and pattern.

3.2 The Model Geometry



- Rotating spherical shell filled with viscous and conducting fluid.

3.2 Disturbances

- Solve for disturbance (prime) around a background state (tilde)

$$\epsilon = \frac{T'}{\widetilde{T}} \approx \frac{\rho'}{\widetilde{\rho}} \approx .... \ll 1$$

- The primary force balance in hydrostatic:

$$\frac{\partial \widetilde{p}}{\partial r} = \widetilde{\rho}\widetilde{g}$$

- Gradients given by material properties:

$$\frac{1}{\widetilde{T}} \frac{\partial \widetilde{T}}{\partial r} = \left(\frac{\partial T}{\partial p} \right)_S \frac{1}{\widetilde{T}} \frac{\partial \widetilde{p}}{\partial r} = \frac{\alpha}{c_p} \widetilde{g} \qquad \frac{1}{\widetilde{\rho}} \frac{\partial \widetilde{\rho}}{\partial r} = \beta_S \frac{\partial \widetilde{p}}{\partial r} = \beta_S \widetilde{\rho}\widetilde{g}$$

- Characterized by Dissipation number of Compressibility parameter:

$$Di = \frac{\alpha d}{c_p} \tilde{g} \qquad Co = d\beta\tilde{\rho}\tilde{g}$$

- Both vanish in the Boussinesq limit.

3.3 Navier-Stokes Equation

$$\frac{\partial \mathbf{U}}{\partial t} = -\nabla p - \mathbf{U}\cdot\nabla\mathbf{U} + 2\mathbf{\Omega}\times\mathbf{U} + \alpha_C g_o\,(r/r_o)\;\rho\,\hat{\mathbf{r}} + \frac{1}{\widetilde{\rho}\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\,\mathbf{U}$$

- Solve in corotating reference frame.
- Assume Boussinesq limit, which yields simplified continuity equation:

$$\nabla\cdot\mathbf{U} = 0$$

- Simplified equation of state with thermal expansivity α.

$$\rho = \alpha\,\widetilde{\rho}\,T$$

- Assume linear gravity (homogeneous density):

$$\mathbf{g} = (r/r_o)\,g_o\,\hat{\mathbf{r}}$$

- Assume Newtonian viscosity.

3.4 Navier-Stokes Equation

$$\frac{\partial \mathbf{U}}{\partial t} = -\nabla p - \mathbf{U}\cdot\nabla\mathbf{U} + 2\boldsymbol{\Omega}\times\mathbf{U} + \alpha_C g_o\,(r/r_o)\;\rho\,\hat{\mathbf{r}} + \frac{1}{\widetilde{\rho}\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\,\mathbf{U}$$

- Relative viscous force

$$\frac{\text{viscous}}{\text{Coriolis}} = \frac{\nu\nabla^2\mathbf{U}}{2\boldsymbol{\Omega}\times\mathbf{U}} \approx \mathrm{E}\frac{D^2}{\ell^2}$$

- Ekman number

$$\mathrm{E} = \frac{\nu}{\Omega D^2}$$

Earth	10^{-15}
Jupiter	10^{-18}
DNS	10^{-7}

- Depends on length scale, beware for example boundary layers

3.5 Relative Advective Force

$$\frac{\partial \mathbf{U}}{\partial t} = -\nabla p - \mathbf{U}\cdot\nabla\mathbf{U} + 2\boldsymbol{\Omega}\times\mathbf{U} + \alpha_C g_o (r/r_o)\ \rho\,\hat{\mathbf{r}} + \frac{1}{\tilde{\rho}\mu}(\nabla \times \mathbf{B})\times\mathbf{B} + \nu\nabla^2\,\mathbf{U}$$

- Relative advective force

$$\frac{\text{advection}}{\text{Coriolis}} = \frac{\mathbf{U}\cdot\nabla\mathbf{U}}{2\boldsymbol{\Omega}\times\mathbf{U}} \approx \frac{U}{\ell'\Omega} = \mathrm{Ro}\frac{D}{\ell'}$$

- Rossby number

$$\mathrm{Ro} = \frac{U}{D\Omega} = \mathrm{Re}\ \mathrm{E}$$

Earth	10^{-6}
Jupiter	10^{-6} (10^{-2} for jets)
DNS	10^{-3} (10^{-2} for jets)

- Transfers energy between different length scales

3.6 Relative Lorentz Force

$$\frac{\partial \mathbf{U}}{\partial t} = -\nabla p - \mathbf{U}\cdot\nabla\mathbf{U} + 2\boldsymbol{\Omega}\times\mathbf{U} + \alpha_C g_o (r/r_o)\ \rho\,\hat{\mathbf{r}} + \underbrace{\frac{1}{\tilde{\rho}\mu}(\nabla\times\mathbf{B})\times\mathbf{B}} + \nu\nabla^2\,\mathbf{U}$$

- Relative Lorentz force

$$\frac{\text{Lorentz}}{\text{Coriolis}} = \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{2\rho\mu\boldsymbol{\Omega}\times\mathbf{U}} \approx \frac{B^2}{\tilde{\rho}\mu\Omega\ell U} = \Lambda_T\frac{\lambda}{\ell U} = \frac{\Lambda_T}{\text{Rm}}\frac{D}{\ell} = \Lambda$$

- Traditional Elsasser number

$$\Lambda_T = \frac{B^2}{\tilde{\rho}\mu\lambda\Omega}$$

Earth	1
Jupiter	10^2
DNS	10^{-1}-10^2

- Hard to estimate for planets!
- See for example Soderlund et al. 2012, 2015; Calkins 2018

3.7 Proxy for Lorentz Force

- Elsasser number of order 1 reasonable to achieve magnetic field saturation!
- Traditional Elsasser number often not a good estimate

$$\Lambda_T = \frac{B^2}{\tilde{\rho}\mu\lambda\Omega}$$

- Dynamic Elsasser number works better (in simulations)

$$\Lambda = \frac{\Lambda_T}{\mathrm{Rm}}\frac{D}{\ell}$$

- Alfven Mach number <1

$$M_A = \frac{\mathrm{U}}{\mathrm{U}_A} \qquad \mathbf{U}_A = \frac{\mathbf{B}}{\sqrt{\tilde{\rho}\mu}}$$

- Energy ratio >>1

$$M = \frac{\mathrm{U}_A^2}{\mathrm{U}^2}$$

3.8 Buoyancy

$$\frac{\partial \mathbf{U}}{\partial t} = -\nabla p' - \mathbf{U}\cdot\nabla\mathbf{U} + 2\mathbf{\Omega}\times\mathbf{U} + \boxed{\alpha_C g_o\,(r/r_o)\ \ \rho\,\hat{\mathbf{r}}} + \frac{1}{\tilde{\rho}\mu}\,(\nabla\times\ \mathbf{B})\times\mathbf{B} + \boxed{\nu\nabla^2\ \mathbf{U}}$$

- Rayleigh number

$$\mathrm{Ra} = \frac{\alpha g_o D^4 \Delta C}{\nu\kappa}$$

- Hard to estimate for planets!

3.8 Heat Equation

- In the Boussinesq limit the heat equation simplifies to

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla \, T = \kappa \nabla^2 \, T \, + \, q$$

with the thermal diffusivity

$$\kappa = \frac{k}{c_p \widetilde{\rho}}$$

- Drivers are heat through the boundaries and internal sources/sinks q .

4.1 Go Dimensionless

- Chose appropriate scales:

$$[\tilde{\rho}] = \tilde{\rho}(r = r_o); \quad [\tilde{T}] = \tilde{T}(r = r_o); \quad [r] = r_o - r_i;$$

$$[t] = \frac{d^2}{\nu}; \quad [u] = \frac{\nu}{d}; \quad [B] = \sqrt{\mu_0 \lambda \tilde{\rho} \Omega}; \quad [p'] = \tilde{\rho}(r = r_o)\frac{\nu^2}{d^2}$$

- Outer boundary reference density and temperature
- Shell thickness as length scale
- Viscous diffusion time as time scale
- Elsasser number scaling for magnetic field
- Velocity expressed as Reynolds number

4.1 Why Going Dimensionless?

- Reduces the number of parameters.
- Highlights the essential non-dimensional parameters.
- Highlights the essential physics.
- Highlights the essential regimes.
- Improves understanding.
- Allows for wider applicability.
- Scientific rather than engineer approach.

Dimensionless NS-Equation

$$\tilde{\rho}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\ \boldsymbol{u}\right) = -\boldsymbol{\nabla} p' - 2\rho\boldsymbol{\Omega}\times\boldsymbol{u} + \alpha g_o T'\frac{\boldsymbol{r}}{r_o}$$

$$+ \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} + \tilde{\rho}\nu\Delta\boldsymbol{u}$$

- Rescaling and reordering:

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\ \boldsymbol{u}\right) = -\boldsymbol{\nabla} p' - \frac{2}{E}\boldsymbol{e_z}\times\boldsymbol{u} + \frac{Ra}{Pr}T'\frac{\boldsymbol{r}}{r_o}$$

$$+ \frac{1}{EPm}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} + \Delta\boldsymbol{u}$$

- Dimensionless NS-equation with dimensionless parameters

Other Dimensionless Equations

- Induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \frac{1}{Pm}\Delta \boldsymbol{B}.$$

- Temperature equation

$$\frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T = \frac{1}{\mathrm{Pr}}\nabla^2 T + q$$

- Continuity equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

Dimensionless Parameter

symbol	definition	standard	high end	Earth
E	$\nu/(\Omega\ell^2)$	3×10^{-5}	10^{-7}	10^{-15}
Pm	ν/λ	1	0.1	10^{-6}
Pr	ν/κ	0.1 – 1	1	1??
$\mathcal{P}$	$P/(\ell^2\Omega^3)$	10^{-5}	10^{-7}	10^{-12}
Λ	$\mathrm{B}^2\sigma/(\rho\Omega)$	1 – 10	10	10
Le	$\mathrm{B}/(\rho^{1/2}\mu^{1/2}\Omega\ell)$	$5\times10^{-3} - 3\times10^{-2}$	3×10^{-3}	10^{-4}
Rm	$\mathrm{U}\ell/\lambda$	$10^2 - 10^3$	10^3	10^3
Re	$\mathrm{U}\ell/\nu$	$10^2 - 10^3$	10^4	10^9
Ro	$\mathrm{U}/(\ell\Omega)$	$3\times10^{-3} - 3\times10^{-2}$	10^{-3}	10^{-6}
M_A	$\mathrm{U}(\rho\mu)^{1/2}/\mathrm{B}$	0.2 – 2	10^{-1}	10^{-2}

- All simulations are too viscous!
- Rayleigh number chosen to yield the desired regime.

Take-Home Messages

- Study of the dynamo generated geomagnetic field has a long history.
- Dynamos rely on magnetic induction.
- They need fluid that is conducting enough and moves fast enough in a complex enough fashion.
- The mathematical formulations describes disturbances around an adiabatic, hydrostatic background state.
- For Earth we can use the Boussinesq approximation where the background temperature and density are assumed to be homogeneous.
- A dimensionless formulation is used where all physical properties ar collapsed into a few dimensionless parameters.
- Not all parameters can have realistic values in numerical simulations.